Uniform Convergence of Adaptive Multigrid Methods for Elliptic Problems and Maxwell's Equations

Ralf Hiptmair[1], Haijun Wu[2] and Weiying Zheng[3,*]

[1] *SAM, ETH Zürich, CH-8092 Zürich, Swizerland.*
[2] *Department of Mathematics, Nanjing University, Nanjing, Jiangsu, 210093, China.*
[3] *LSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China.*

Abstract. We consider the convergence theory of adaptive multigrid methods for second-order elliptic problems and Maxwell's equations. The multigrid algorithm only performs pointwise Gauss-Seidel relaxations on new degrees of freedom and their "immediate" neighbors. In the context of lowest order conforming finite element approximations, we present a unified proof for the convergence of adaptive multigrid V-cycle algorithms. The theory applies to any hierarchical tetrahedral meshes with uniformly bounded shape-regularity measures. The convergence rates for both problems are uniform with respect to the number of mesh levels and the number of degrees of freedom. We demonstrate our convergence theory by two numerical experiments.

1. Introduction

In this paper, we study the uniform convergence theory of the adaptive multigrid method for two model problems

$$-\Delta u + u = f \quad \text{in } \Omega, \tag{1.1}$$
$$u = 0 \quad \text{on } \Gamma, \tag{1.2}$$

and

$$\mathbf{curl\,curl\,u} + \mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \tag{1.3}$$
$$\mathbf{u} \times \boldsymbol{n} = 0 \quad \text{on } \Gamma, \tag{1.4}$$

*Corresponding author. *Email addresses:* hiptmair@sam.math.ethz.ch (R. Hiptmair), hjw@nju.edu.cn (H. Wu), zwy@lsec.cc.ac.cn (W. Zheng).

where $\Omega \subset \mathbb{R}^3$ is a Lipschitz polyhedron with boundary $\Gamma = \partial \Omega$, $\boldsymbol{n}$ is the unit outer normal of Γ, and $f \in L^2(\Omega)$, $\mathbf{f} \in (L^2(\Omega))^3$. Problem (1.1)–(1.2) and (1.3)–(1.4) are key model problems for the study of numerical methods for second-order elliptic boundary value problems and quasi-magnetostatic boundary value problems, respectively.

Linear $H_0^1(\Omega)$-conforming finite elements and lowest-order $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$-conforming edge elements provide natural finite element trial spaces for the Galerkin discretizations of (1.1)–(1.2) and (1.3)–(1.4), respectively. Here we study optimal iterative solvers for the resulting discrete problems. We remark that optimal approximation entails the use of adaptive finite element methods based on a posteriori error estimates, see [6, 10, 29, 31] for $H^1(\Omega)$-elliptic problems and [5, 11, 26, 39] for $\boldsymbol{H}(\mathbf{curl}, \Omega)$-elliptic problems. In this case we can expect the optimal asymptotic convergence rate

$$\left\| u - u_h \right\|_{H^1(\Omega)} \le C N_h^{-1/3}, \quad \left\| \mathbf{u} - \mathbf{u}_h \right\|_{H(\mathbf{curl}, \Omega)} \le C N_h^{-1/3}, \tag{1.5}$$

on families of finite element meshes arising from adaptive refinement. Here, u_h and $\mathbf{u}_h$ are the finite element solutions approximating u and $\mathbf{u}$ respectively, and N_h is the number of elements. An optimal solver delivers a satisfactory approximation of the discrete solution with a number of operations proportional to N_h. In finite element settings, this objective is usually achieved by using geometric multigrid methods, whose convergence theory and optimality on family of uniformly refined meshes have been well established for both $H^1(\Omega)$-elliptic problems [33, 34, 36, 37] and $\boldsymbol{H}(\mathbf{curl}, \Omega)$-elliptic problems [1, 15, 17].

To keep the optimal computational cost on locally refined meshes, one must adopt the local multigrid policy [3, 22, 32], which confines relaxations to degrees of freedom on new elements of each mesh level. Clearly this policy makes the computational cost of the local multigrid method proportional to the number of all elements appearing in the local refinement process, and thus proportional to the number of degrees of freedom on the finest mesh. The local multigrid policy with hybrid relaxations for Maxwell's equations are studied in [4, 11, 19, 28]. They show that the local multgird method is very efficient and robust for low-frequency problems on various non-convex domains, and is a good preconditioner for time-harmonic Maxwell's equations [11].

Suppose one seeks for the discrete solution u_h of (1.1)–(1.2) or (1.3)–(1.4) in finite dimensional Hilbert space V_h. For a given partition $\mathscr{T}_h$ of Ω, V_h is usually taken as the finite element space defined over $\mathscr{T}_h$. The multigrid method for solving u_h is designed upon some multilevel decomposition of V_h over a sequence of conforming meshes

$$\mathscr{T}_0 \prec \mathscr{T}_1 \prec \cdots \prec \mathscr{T}_L := \mathscr{T}_h.$$

Here $\mathscr{T}_0$ is a quasi-uniform mesh with small number of elements and "$\mathscr{T}_{l-1} \prec \mathscr{T}_l$" means that $\mathscr{T}_l$ is obtained by refining some or all elements in $\mathscr{T}_{l-1}$. The sequence of meshes $\{\mathscr{T}_l\}_{l=0}^L$ can be constructed either by adaptive refinement strategies starting from the initial mesh $\mathscr{T}_0$ (see, e.g., [11, 32]), or by some coarsening strategies starting from the final mesh $\mathscr{T}_L$ (see, e.g., [19, 35]). Recently, Xu, Chen, and Nochetto [35] present a unified framework for the uniform convergence of multilevel methods for $H^1(\Omega)$–, $\boldsymbol{H}(\mathbf{curl}, \Omega)$–, $\boldsymbol{H}(\mathrm{div}, \Omega)$– elliptic problems. In [19], Hiptmair and Zheng presented the uniform convergence of the

multigrid method for $H(\mathbf{curl}, \Omega)$–elliptic problems on meshes with and without hanging nodes.

In [35], Xu and Chen and Nochetto proposed a space decomposition via successive coarsening of compatible patches of $\mathscr{T}_l$, such that

- every mesh $\mathscr{T}_l$ is decomposed into compatible patches, each of which consists of the elements sharing one common refinement edge,

- all compatible patches of $\mathscr{T}_l$ are coarsened by removing their refinement edges to generate a coarse mesh $\mathscr{T}_{l-1}$,

where $l = L, L-1, \cdots, 1$. The finite element space is decomposed as follows

$$V_L = V_0 + \sum_{l=1}^{L} \sum_{\substack{b \in V_l \\ b \notin V_{l-1}}} \mathrm{Span}\{b\},$$

where V_0 is the finite element space on $\mathscr{T}_0$ and b is the nodal basis function of V_l. They also present coarsening algorithm and local multigrid algorithm for implementations. In [19], Hiptmair and Zheng proposed a different coarsening strategy to construct the mesh hierarchy $\{\mathscr{T}_l\}_{l=0}^{L}$. Given $\mathscr{T}_0$ and $\mathscr{T}_L$, $\mathscr{T}_l$ is so defined that the elements in $\mathscr{T}_l \setminus \mathscr{T}_{l-1}$ are obtained by the same number of subdivisions of elements in $\mathscr{T}_0$, $0 < l < L$. Similarly the finite element space is split into

$$V_L = V_0 + \sum_{l=1}^{L} \sum_{\sigma \in \mathscr{D}(\mathscr{T}_l \setminus \mathscr{T}_{l-1})} \mathrm{Span}\{b_\sigma\},$$

where $\mathscr{D}(\mathscr{T}_l \setminus \mathscr{T}_{l-1})$ is the set of degrees of freedom on $\mathscr{T}_l \setminus \mathscr{T}_{l-1}$ and b_σ is the nodal basis function of V_l belonging to σ. Numerical experiments in [19,35] show that their multigrid algorithms are very efficient and converge uniformly with respect to L and N_h. But some assumptions on the initial mesh $\mathscr{T}_0$ and the fine mesh $\mathscr{T}_h$ are required to guarantee that compatible patches of $\mathscr{T}_l$ do exist (for [35]) or that $\mathscr{T}_l$ is conforming (for [19]).

The multigrid algorithm based on adaptively refined meshes (adaptive MG) is easy to implement, since the mesh hierarchy $\{\mathscr{T}_l\}_{l=0}^{L}$ is readily obtained and the stiffness, prolongation, and restriction matrices have been computed on all previous levels. It is more preferable when solving problems with local singularities by adaptive finite element methods. In [32], Wu and Chen proved the uniform convergence of adaptive MG for two-dimensional $H^1(\Omega)$-elliptic problems. To the best of our knowledge, the adaptive MG — *the multilevel meshes for MG are just the adaptively refined meshes* — is still absent for Maxwell's equations in literatures.

The purpose of this paper is to present a unified proof for the uniform convergence of adaptive multigrid methods for (1.1)–(1.2) and (1.3)–(1.4). Here we would like to emphasize the novelty of our paper compared with [19,35] as follows:

1. our theory applies to the local MG on any hierarchy of conforming meshes, particularly, *"the adaptive multigrid method"* which uses the sequence of meshes generated by adaptive finite element method with a posteriori error estimates; while the references [19, 35] utilize "*the coarsening process to produce new multilevel meshes*", and the coarsening process is sometimes restrictive and uneasy in three dimensions;

2. the proofs in [19,35] mainly make use of a close relationship between MG-smoothing patches and uniform meshes; while we propose a new *scale-separation technique* for finite element basis functions in the proofs.

The layout of the paper is as follows. In section 2 we introduce the weak formulations of (1.1)–(1.2), (1.3)–(1.4) and their finite element approximations. Some useful results on finite element spaces are also presented for the study of multilevel decompositions of finite element functions. In section 3 we introduce local multigrid from the perspective of multilevel successive subspace decomposition. In section 4 we study the uniform convergence of local multigrid method in $H^1(\Omega)$. A key step is to study the multilevel decomposition of the linear Lagrangian finite element space. In section 5 we study the uniform convergence of local multigrid method in $\boldsymbol{H}(\mathbf{curl}, \Omega)$. The key tools are a discrete Helmholtz-type decomposition of the edge element space and local multigrid theories for $H^1(\Omega)$-elliptic problems. In section 6, we establish the so-called strengthened Cauchy-Schwartz equalities for both $H^1(\Omega)$-elliptic problems and $\boldsymbol{H}(\mathbf{curl}, \Omega)$-elliptic problems. In section 7 we present two numerical experiments to demonstrate our theories and the competitive performance of adaptive multigrid methods.

2. Finite element spaces

We start by introducing some notation and Hilbert spaces used in this paper. Let $L^2(\Omega)$ be the usual Hilbert space of square integrable functions equipped with the following inner product and norm:

$$(u, v) := \int_{\Omega} u(\boldsymbol{x})v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \quad \text{and} \quad \|u\|_{L^2(\Omega)} := (u, u)^{1/2}.$$

All through this paper, we use boldfaced notations for vectors, such as $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^3$ and so on. Define $H^1(\Omega) := \{v \in L^2(\Omega) : \nabla v \in \boldsymbol{L}^2(\Omega)\}$ which is equipped with the following semi-norm and norm

$$|u|_{H^1(\Omega)} := \|\nabla u\|_{L^2(\Omega)} \quad \text{and} \quad \|u\|_{H^1(\Omega)} := \left(\|u\|_{L^2(\Omega)}^2 + |u|_{H^1(\Omega)}^2 \right)^{1/2},$$

and let $H_0^1(\Omega)$ be the subspace of $H^1(\Omega)$ whose functions have zero traces on $\partial\Omega$. The following Hilbert spaces are used in the paper

$$\boldsymbol{H}(\mathbf{curl}, \Omega) := \left\{ \mathbf{v} \in \boldsymbol{L}^2(\Omega) : \ \mathbf{curl}\,\mathbf{v} \in \boldsymbol{L}^2(\Omega) \right\},$$

$$\boldsymbol{H}_0(\mathbf{curl}, \Omega) := \left\{ \mathbf{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega) : \mathbf{v} \times \boldsymbol{n} = 0 \ \text{on} \ \partial\Omega \right\},$$

which are equipped with the following norm:

$$\|\mathbf{v}\|_{H(\mathbf{curl},\Omega)} := \left(\|\mathbf{v}\|^2_{L^2(\Omega)} + \|\mathbf{curl}\,\mathbf{v}\|^2_{L^2(\Omega)} \right)^{1/2}.$$

A weak formulation of (1.1)–(1.2) reads: Find $u \in H^1_0(\Omega)$, such that

$$a_s(u,v) = (f,v), \quad \forall\, v \in H^1_0(\Omega), \tag{2.1}$$

and a weak formulation of (1.3)–(1.4) reads: Find $\mathbf{u} \in H_0(\mathbf{curl},\Omega)$, such that

$$a_v(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall\, \mathbf{v} \in H_0(\mathbf{curl},\Omega), \tag{2.2}$$

where the bilinear forms $a_s\colon H^1(\Omega) \times H^1(\Omega) \mapsto \mathbb{R}$ and $a_v\colon H(\mathbf{curl},\Omega) \times H(\mathbf{curl},\Omega) \mapsto \mathbb{R}$ are defined as follows

$$\begin{aligned} a_s(u,v) &:= (\nabla u, \nabla v) + (u,v), & \forall\, u,v \in H^1(\Omega), \\ a_v(\mathbf{u},\mathbf{v}) &:= (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{u},\mathbf{v}), & \forall\, \mathbf{u},\mathbf{v} \in H(\mathbf{curl},\Omega). \end{aligned}$$

Recall the operators $\mathbf{curl}$ and ∇ are closely connected in the deRham complex [2]. The results about (2.1) prove instrumental in the multigrid analysis for discretized versions of (2.2).

Let $\mathscr{T}_h$ be a conforming tetrahedral mesh of Ω, that is, each face of a tetrahedron is either a face of another tetrahedron or contained in $\partial\Omega$. We write $h \in L^\infty(\Omega)$ for the piecewise constant function, which assumes value $h_K := |K|^{-1/3}$ in each element $K \in \mathscr{T}_h$. The ratio of $\mathrm{diam}(K)$ to the radius of the largest ball contained in K is called the shape-regularity measure ρ_K. The shape-regularity measure of $\mathscr{T}_h$ is defined by

$$\rho(\mathscr{T}_h) := \max\left\{\rho_K : \ \forall\, K \in \mathscr{T}_h\right\}.$$

We introduce the Lagrangian finite element space of piecewise linear continuous functions on $\mathscr{T}_h$

$$V(\mathscr{T}_h) := \left\{ u_h \in H^1_0(\Omega) : \ u_h|_K \in P_1(K), \ \forall\, K \in \mathscr{T}_h \right\}, \tag{2.3}$$

where $P_m(K)$ is the space of 3-variable polynomials of degree $\leq m$ on K. The space of lowest order $H_0(\mathbf{curl},\Omega)$-conforming edge finite elements is defined as follows

$$\mathbf{U}(\mathscr{T}_h) := \left\{ \mathbf{v}_h \in H_0(\mathbf{curl},\Omega) : \ (\mathbf{v}_h|_K)(\mathbf{x}) = \mathbf{a} + \mathbf{b} \times \mathbf{x}, \ \forall\, \mathbf{a},\mathbf{b} \in \mathbb{R}^3, \ \forall\, K \in \mathscr{T}_h \right\}.$$

The Galerkin approximation to (2.1) reads: Find $u_h \in V(\mathscr{T}_h)$ such that

$$a_s(u_h, v_h) = (f, v_h), \quad \forall\, v_h \in V(\mathscr{T}_h), \tag{2.4}$$

and the Galerkin approximation to (2.2) reads: Find $\mathbf{u}_h \in \mathbf{U}(\mathscr{T}_h)$ such that

$$a_v(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathbf{U}(\mathscr{T}_h). \tag{2.5}$$

Appropriate global degrees of freedom (d.o.f.) for $V(\mathscr{T}_h)$ and $\mathbf{U}(\mathscr{T}_h)$ are respectively given by

$$v_h \quad \rightarrow \quad v_h(\boldsymbol{p})\,, \qquad \forall\, \boldsymbol{p} \in \mathscr{N}(\mathscr{T}_h),\ v_h \in V(\mathscr{T}_h), \tag{2.6}$$

$$\mathbf{v}_h \quad \rightarrow \quad \int_E \mathbf{v}_h \cdot \mathrm{d}\vec{s}\,, \quad \forall\, E \in \mathscr{E}(\mathscr{T}_h),\ \mathbf{v}_h \in \mathbf{U}(\mathscr{T}_h), \tag{2.7}$$

where $\mathscr{N}(\mathscr{T}_h)$ is the set of interior vertices of $\mathscr{T}_h$ and $\mathscr{E}(\mathscr{T}_h)$ is the set of interior edges of $\mathscr{T}_h$. Respectively, we denote by $b^{\boldsymbol{p}}$ the nodal basis function of $V(\mathscr{T}_h)$ belonging to $\boldsymbol{p} \in \mathscr{N}(\mathscr{T}_h)$ and by $\boldsymbol{b}^E$ the edge basis function of $\mathbf{U}(\mathscr{T}_h)$ belonging to $E \in \mathscr{E}(\mathscr{T}_h)$.

Now we introduce the nodal interpolation operators $\mathscr{I}_h : \mathrm{dom}(\mathscr{I}_h) \subset H_0^1(\Omega) \mapsto V(\mathscr{T}_h)$ and $\boldsymbol{\Pi}_h : \mathrm{dom}(\boldsymbol{\Pi}_h) \subset \boldsymbol{H}_0(\mathbf{curl},\Omega) \mapsto \mathbf{U}(\mathscr{T}_h)$ induced by d.o.f. in (2.6) and (2.7) respectively:

$$\mathscr{I}_h v = \sum_{\boldsymbol{p} \in \mathscr{N}(\mathscr{T}_h)} v(\boldsymbol{p}) \cdot b^{\boldsymbol{p}}\,, \qquad \boldsymbol{\Pi}_h \mathbf{v} = \sum_{E \in \mathscr{E}(\mathscr{T}_h)} \left(\int_E \mathbf{v} \cdot \mathrm{d}\vec{s} \right) \cdot \mathbf{b}^E\,, \tag{2.8}$$

where $\mathrm{dom}(\mathscr{I}_h), \mathrm{dom}(\boldsymbol{\Pi}_h)$ are the domains of $\mathscr{I}_h, \boldsymbol{\Pi}_h$ respectively. Obviously, both $\mathscr{I}_h$ and $\boldsymbol{\Pi}_h$ are local projections and respect the well-known *commuting diagram property* (cf. e.g., [16, Page 263])

$$\boldsymbol{\Pi}_h \circ \nabla = \nabla \circ \mathscr{I}_h \quad \text{on } \mathrm{dom}(\mathscr{I}_h)\,. \tag{2.9}$$

To end this section, we introduce the decomposition of $\left(V(\mathscr{T}_h) \right)^3$ from [19] which reveals one important relationship between the linear Lagrangian finite element space of vector functions and the lowest-order edge element space.

Lemma 2.1. [19, *Lemma* 2.2] *Let* $V_2(\mathscr{T}_h) := \{ u_h \in H_0^1(\Omega) : u_h|_K \in P_2(K),\ \forall K \in \mathscr{T}_h \}$ *be the quadratic Lagrangian finite element space and define*

$$\widetilde{V}_2(\mathscr{T}_h) = \left\{ v_h \in V_2(\mathscr{T}_h) : \mathscr{I}_h v_h = 0 \right\}.$$

For all $\boldsymbol{\Psi}_h \in (V(\mathscr{T}_h))^3$ *we can find* $\widetilde{v}_h \in \widetilde{V}_2(\mathscr{T}_h)$ *such that*

$$\boldsymbol{\Psi}_h = \boldsymbol{\Pi}_h \boldsymbol{\Psi}_h + \nabla \widetilde{v}_h\,,$$
$$C^{-1} \left\| \boldsymbol{\Psi}_h \right\|_{L^2(\Omega)}^2 \leq \left\| \boldsymbol{\Pi}_h \boldsymbol{\Psi}_h \right\|_{L^2(\Omega)}^2 + \left\| \nabla \widetilde{v}_h \right\|_{L^2(\Omega)}^2 \leq C \left\| \boldsymbol{\Psi}_h \right\|_{L^2(\Omega)}^2\,,$$

where the constant C *only depends on the shape-regularity* $\rho(\mathscr{T}_h)$.

3. Local multigrid methods

In this section we are going to study the local multigrid algorithms for (2.4) and (2.5) using the abstract multigrid framework. To focus on the main theme, we provide the abstract framework in Appendix 7. According to Algotithm A.1 and A.2, the multigrid V-cycle algorithms are completely defined by specifying the multilevel decompositions of the finite element spaces on the fine mesh.

3.1. Multigrid V-cycle algorithms in $H^1(\Omega)$ and $H(\mathbf{curl}, \Omega)$

Let $\mathcal{T}_0 \prec \mathcal{T}_1 \prec \cdots \prec \mathcal{T}_L$ be a sequence of nested tetrahedral meshes. For convenience we simply assume that $\{\mathcal{T}_l\}_{l=0}^L$ are conforming meshes, namely, each $\mathcal{T}_l$ has no hanging nodes. Clearly $\{\mathcal{T}_l\}_{l=0}^L$ can be viewed as successive local refinements of a quasi-uniform mesh $\mathcal{T}_0$. Let $V(\mathcal{T}_l) \subset H_0^1(\Omega)$ be the linear Lagrangian finite element space on $\mathcal{T}_l$ and denote by b_l^p be the nodal basis function of $V(\mathcal{T}_l)$ belonging to vertex $\boldsymbol{p} \in \mathcal{N}(\mathcal{T}_l)$. Let $\mathbf{U}(\mathcal{T}_l) \subset \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ be the lowest order edge element space on $\mathcal{T}_l$ and denote by $\mathbf{b}_l^E$ the nodal basis function of $\mathbf{U}(\mathcal{T}_l)$ belonging to E. Now we have two sequences of nested finite element spaces

$$V(\mathcal{T}_0) \subset V(\mathcal{T}_1) \subset \cdots \subset V(\mathcal{T}_L), \qquad \mathbf{U}(\mathcal{T}_0) \subset \mathbf{U}(\mathcal{T}_1) \subset \cdots \subset \mathbf{U}(\mathcal{T}_L).$$

We define the sets of vertices and the sets of edges on which Gauss-Seidel relaxations are carried out as follows: for $0 \le l \le L$ and $\mathcal{T}_{-1} = \emptyset$,

$$\mathcal{N}_l := \left\{ \boldsymbol{p} \in \mathcal{N}(\mathcal{T}_l) : \boldsymbol{p} \notin \mathcal{N}(\mathcal{T}_{l-1}) \text{ or } \boldsymbol{p} \in \mathcal{N}(\mathcal{T}_{l-1}) \text{ but } b_l^p \ne b_{l-1}^p \right\}, \qquad (3.1)$$

$$\mathcal{E}_l := \left\{ E \in \mathcal{E}(\mathcal{T}_l) : E \notin \mathcal{E}(\mathcal{T}_{l-1}) \text{ or } E \in \mathcal{E}(\mathcal{T}_{l-1}) \text{ but } \mathbf{b}_l^E \ne \mathbf{b}_{l-1}^E \right\}. \qquad (3.2)$$

It is easy to see that $\mathcal{N}_l$ is a subset of $\mathcal{N}(\mathcal{T}_l \cap \mathcal{T}_{l-1})$, the set of all vertices of $\mathcal{T}_l \setminus \mathcal{T}_{l-1}$, and $\mathcal{E}_l$ is a subset of $\mathcal{E}(\mathcal{T}_l \cap \mathcal{T}_{l-1})$, the set of all edges of $\mathcal{T}_l \setminus \mathcal{T}_{l-1}$.

Remark 3.1. If we use the bisection algorithm (cf. e.g., [22, 24]) for mesh refinements, $\mathcal{N}_l$ is the set of new vertices and their immediately neighboring vertices (cf. [32]) and $\mathcal{E}_l$ is the set of new edges and their immediately neighboring edges (see Fig. 1 (right), the smoothed vertices are labeled with black balls and the smoothed edges are labeled with thick lines).
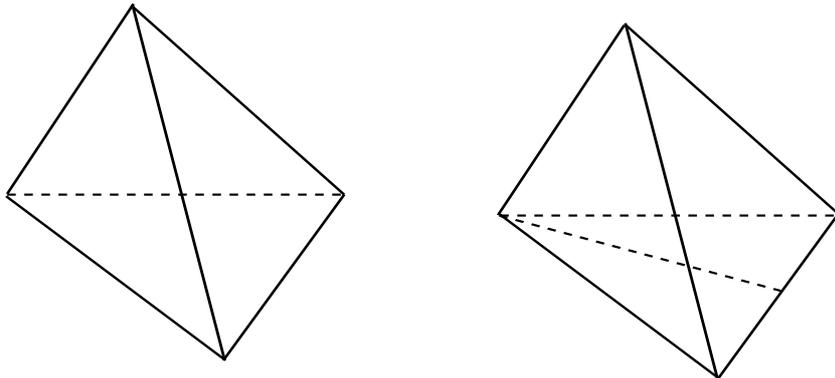

Figure 1: Left: A tetrahedron to be refined. Right: The tetrahedron is bisected into two tetrahedrons. The smoothed vertices are the new vertex and the two endpoints of the refinement edge (two "immediately neighboring vertices"). The smoothing edges are the four new edges and the old edges of the two faces which share the refinement edge ("immediate neighboring edges" of the centerlines).

First we study the local multigrid algorithm for (2.4). To fit the multigrid framework of (A.1), we set

$$a(\cdot,\cdot) = a_s(\cdot,\cdot), \qquad f = f, \qquad H_l := V(\mathcal{T}_l), \quad 0 \leq l \leq L.$$

The multilevel decomposition of $V(\mathcal{T}_L)$ is defined by:

$$V(\mathcal{T}_L) = V(\mathcal{T}_0) + \sum_{l=1}^{L} \sum_{\boldsymbol{p} \in \mathcal{N}_l} \mathrm{Span}\left\{b_l^{\boldsymbol{p}}\right\}. \tag{3.3}$$

The decomposition agrees with (A.4) if we define $H_l^i := \mathrm{Span}\left\{b_l^{\boldsymbol{p}}\right\}$ for $l > 0$ as the one-dimensional spaces spanned by nodal basis functions. Thus Algorithm A.2 is actually doing Gauss-Seidel relaxations on nodes in $\mathcal{N}_l$. The convergence theory of local multigrid for (2.4) boils down to the estimation of the error propagation operator

$$E_{sL} = I - B_{sL}A_{sL}, \tag{3.4}$$

where $B_{sL} = B_L$ is the multigrid operator defined in Algorithm A.1 and $A_{sL}: V(\mathcal{T}_L) \to V(\mathcal{T}_L)$ is the discrete differential operator defined by

$$(A_{sL}v, w) = a_s(v, w), \qquad \forall\, v, w \in V(\mathcal{T}_L).$$

To study the local multigrid algorithm for (2.5), we adapt the problem to the multigrid framework of (A.1) by setting

$$a(\cdot,\cdot) = a_v(\cdot,\cdot), \qquad f = \mathbf{f}, \qquad H_l := \mathbf{U}(\mathcal{T}_l), \quad 0 \leq l \leq L.$$

Motivated by [4, 11], the multilevel decomposition of $\mathbf{U}(\mathcal{T}_L)$ incorporates an appropriate local multilevel decomposition of $V(\mathcal{T}_L)$:

$$\mathbf{U}(\mathcal{T}_L) = \mathbf{U}(\mathcal{T}_0) + \sum_{l=1}^{L} \sum_{\boldsymbol{p} \in \mathcal{N}_l} \mathrm{Span}\left\{\nabla b_l^{\boldsymbol{p}}\right\} + \sum_{l=1}^{L} \sum_{E \in \mathcal{E}_l} \mathrm{Span}\left\{\mathbf{b}_l^{E}\right\}. \tag{3.5}$$

The decomposition agrees with (A.4) if we define $H_l^i := \mathrm{Span}\left\{\nabla b_l^{\boldsymbol{p}}\right\}$, $H_l^j := \mathrm{Span}\left\{\mathbf{b}_l^{E}\right\}$ for $l > 0$. At this stage, Algorithm A.2 performs hybrid local relaxations at nodes in $\mathcal{N}_l$ and edges in $\mathcal{E}_l$. Similarly, the convergence theory of local multigrid for (2.5) boils down to the estimation of the error propagation operator

$$E_{vL} = I - B_{vL}A_{vL}, \tag{3.6}$$

where $B_{vL} = B_L$ is the multigrid operator in Algorithm A.1 and $A_{vL}: \mathbf{U}(\mathcal{T}_L) \to \mathbf{U}(\mathcal{T}_L)$ is defined by

$$(A_{vL}\mathbf{v}, \mathbf{w}) = a_v(\mathbf{v}, \mathbf{w}), \qquad \forall\, \mathbf{v}, \mathbf{w} \in \mathbf{U}(\mathcal{T}_L).$$

3.2. Convergence

The multigrid V-cycle algorithms solving (2.4)–(2.5) are presented in Algorithm A.1 with local smoothing process described by Algorithm A.2. Furthermore, Algorithm A.2 is induced by the multilevel decomposition (3.3) for (2.4) and by the multilevel decomposition (3.5) for (2.5). The optimality of multigrid methods means that, one multigrid iteration uses $O(N_L)$ computations and reduces the error of the approximate solution by a factor which is bounded away from 1 and independent of N_L and L. Here N_L is the number of d.o.f. on $\mathcal{T}_L$. In view of Theorem A.1, it is sufficient to prove that both constants C_{stab} and C_{orth} are independent of L, N_L. This is the challenge of asymptotic multigrid analysis and will be postponed to the following sections of this article.

Before stating the main theorem of this paper, we make the following assumptions on the meshes:

(H1) There exists a constant $\rho_{\max} > 0$ independent of $\{\mathcal{T}_l\}_{l=0}^{L}$ such that $\rho(\mathcal{T}_l) \leq \rho_{\max}$, $0 \leq l \leq L$.

(H2) There exist two constants $C > 0$ and $0 < \theta < 1$ independent of l such that

$$C^{-1}\theta^m \leq h_K \leq C\theta^m, \quad m = \mathcal{G}(K), \quad \forall K \in \bigcup_{l=0}^{L} \mathcal{T}_l, \tag{3.7}$$

where $\mathcal{G}(K)$ is called *the generation of K* and is defined by the number of subdivisions for generating K from one element $K_0 \in \mathcal{T}_0$. For easy understanding, we restrict our analysis to bisection strategies of the mesh [20]. In this case, $\mathcal{G}(K)$ is defined by the number of bisections for generating K from $K_0 \in \mathcal{T}_0$. The concept "generation" is also used in [35].

(H3) There exists a constant $C > 0$ only depending on $\theta, \rho_{\max}$ such that, for any $K \in \mathcal{T}_{l-1}$ and $0 < l \leq L$,

$$h_K \leq Ch_{K'}, \quad \forall K' \subset K, \, K' \in \mathcal{T}_l \, .$$

We remark that (H1)–(H3) are rather mild in practice. In fact, (H1) is a common assumption in traditional finite element analysis, (H2) estimates the reduction rate of the diameter of each tetrahedron under successive subdivisions, and (H3) indicates that each element in $\mathcal{T}_l$ is obtained by subdividing one element in $\mathcal{T}_{l-1}$ a few times. It is clear that $\theta = 2^{-1/3}$ for the popular bisection strategy [20,21]. For other refinement strategies, if we can define "subdivision of an element" properly in (H2), the extension of the convergence theory is straightforward. We do not get to the details here.

For any operators $O_s \colon H_0^1(\Omega) \mapsto H_0^1(\Omega)$ and $O_v \colon \boldsymbol{H}_0(\mathbf{curl}, \Omega) \mapsto \boldsymbol{H}_0(\mathbf{curl}, \Omega)$, we define the following norms

$$\left\| O_s \right\|_{a_s} := \sup_{\phi \in H_0^1(\Omega)} \frac{\left\| O_s(\phi) \right\|_{H^1(\Omega)}}{\left\| \phi \right\|_{H^1(\Omega)}}, \qquad \left\| O_v \right\|_{a_v} := \sup_{\phi \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)} \frac{\left\| O_v(\phi) \right\|_{H(\mathbf{curl}, \Omega)}}{\left\| \phi \right\|_{H(\mathbf{curl}, \Omega)}}.$$

Theorem 3.1 (Uniform convergence of local multigrid methods). *Let* (H1)–(H2) *be satisfied. Then there exist two constants* $\delta_s < 1$, $\delta_v < 1$ *which depend on* $\rho_{\max}$ *and* θ*, but are independent of the meshes, such that*

$$\left\| I - B_{sL} A_{sL} \right\|_{a_s} < \delta_s \,, \qquad \left\| I - B_{vL} A_{vL} \right\|_{a_v} < \delta_v \,.$$

In the following sections we shall establish the stability estimate and the Strengthened Cauchy-Schwartz inequality in Theorem A.1 for the multilevel decompositions (3.3) and (3.5). The key ingredient is to prove that the two constants C_{stab}, C_{orth} are independent of the meshes. Therefore, Theorem 3.1 is concluded from Theorem A.1 and the estimates for C_{stab}, C_{orth}.

4. Multilevel decomposition of $V(\mathcal{T}_L)$

This section is devoted to the stability estimate for the local multilevel decomposition (3.3). It also plays a key role in the stability estimate for (3.5) which will be studied in the next section.

4.1. Local quasi-interpolation operator

Quasi-interpolation operators are projectors onto finite element spaces that have been devised to accommodate two conflicting goals: locality and boundedness in weak norms [12, 23, 25, 27]. It is a key tool for the multilevel decomposition of $V(\mathcal{T}_h)$. Here we resort to a Clément-type quasi-interpolation taking into account Dirichlet boundary conditions [12, 25].

Let $\bar{\mathcal{N}}(\mathcal{T}_h)$ and $\bar{\mathcal{E}}(\mathcal{T}_h)$ be the sets of vertices and edges in $\bar{\Omega}$. Through this paper, we shall use notions and operators with an overbar for finite element spaces oblivious of boundary conditions. For example, $\bar{\mathbf{U}}(\mathcal{T}_h) \subset \mathbf{H}(\mathbf{curl}, \Omega)$, $\bar{V}(\mathcal{T}_h) \subset H^1(\Omega)$ are finite element spaces without boundary conditions, and the same convention for $\bar{\mathbf{\Pi}}_h$, $\bar{\mathcal{I}}_h$, etc.

For any $p \in \bar{\mathcal{N}}(\mathcal{T}_h)$, denote by $\Omega^p := \text{supp}(b^p)$ and define $\mathcal{T}_h^p = \{T \in \mathcal{T}_h : T \subset \Omega^p\}$. Let $\psi^p \in \bar{V}(\mathcal{T}_h^p)$ be a piecewise linear function defined as follows:

$$\int_{\Omega^p} \psi^p(\boldsymbol{x}) v(\boldsymbol{x}) \, d\boldsymbol{x} = v(\boldsymbol{p}), \quad \forall v \in \bar{V}(\mathcal{T}_h^p). \tag{4.1}$$

Direct calculations show that

$$\psi^p = \frac{1}{|\Omega^p|} (20 b^p - 4). \tag{4.2}$$

It is obvious that

$$C^{-1} \leq |\Omega^p| \left\| \psi^p \right\|_{L^2(\Omega^p)}^2 \leq C, \quad C^{-1} \leq \left\| \psi^p \right\|_{L^1(\Omega^p)} \leq C \,. \tag{4.3}$$

Definition 4.1. *The quasi-interpolation operators $\mathcal{Q}_h$: $L^2(\Omega) \mapsto V(\mathcal{T}_h)$ and $\bar{\mathcal{Q}}_h$: $L^2(\Omega) \mapsto \bar{V}(\mathcal{T}_h)$ are defined as follows:*

$$\mathcal{Q}_h u = \sum_{\boldsymbol{p} \in \mathcal{N}(\mathcal{T}_h)} \int_{\Omega^p} \psi^{\boldsymbol{p}}(\boldsymbol{x}) u(\boldsymbol{x}) \, d\boldsymbol{x} \cdot b^{\boldsymbol{p}} \,, \tag{4.4}$$

$$\bar{\mathcal{Q}}_h u = \sum_{\boldsymbol{p} \in \bar{\mathcal{N}}(\mathcal{T}_h)} \int_{\Omega^p} \psi^{\boldsymbol{p}}(\boldsymbol{x}) u(\boldsymbol{x}) \, d\boldsymbol{x} \cdot b^{\boldsymbol{p}} \,. \tag{4.5}$$

Clearly (4.4) indicates that $\mathcal{Q}_h u = 0$ on Γ. From (4.1) we know that Q_h and $\bar{Q}_h$ are projections onto $V(\mathcal{T}_h)$ and $\bar{V}(\mathcal{T}_h)$ respectively:

$$\mathcal{Q}_h v = v, \quad \forall v \in V(\mathcal{T}_h), \quad \text{and} \quad \bar{\mathcal{Q}}_h w = w, \quad \forall w \in \bar{V}(\mathcal{T}_h). \tag{4.6}$$

Moreover, they satisfy the following local stabilities and approximation properties.

Lemma 4.1. *There exists a constant C only depending on Ω and the shape-regularity $\rho(\mathcal{T}_h)$ such that, for any tetrahedron $K \in \mathcal{T}_h$ and face $F \subset \partial K$,*

$$\left\| \mathcal{Q}_h u \right\|_{0,K} \le C \left\| u \right\|_{0,\Omega_K}, \qquad\qquad \forall u \in L^2(\Omega), \tag{4.7}$$

$$\left| \mathcal{Q}_h u \right|_{1,K} \le C \left| u \right|_{1,\Omega_K}, \qquad\qquad \forall u \in H_0^1(\Omega), \tag{4.8}$$

$$\left\| u - \mathcal{Q}_h u \right\|_{0,K} \le C h_K^m \left| u \right|_{m,\Omega_K}, \qquad\qquad \forall u \in H^m(\Omega) \cap H_0^1(\Omega), \tag{4.9}$$

$$\left\| u - \mathcal{Q}_h u \right\|_{0,F} \le C h_K^{m-1/2} \left| u \right|_{m,\Omega_K}, \qquad\qquad \forall u \in H^m(\Omega) \cap H_0^1(\Omega), \tag{4.10}$$

where $m = 1, 2$ and $\Omega_K := \bigcup \{\bar{K}' : K' \in \mathcal{T}_h, \ \bar{K}' \cap \bar{K} \ne \emptyset\}$. The above estimates also hold for $\bar{\mathcal{Q}}_h$ by replacing $H_0^1(\Omega)$ with $H^1(\Omega)$.

Proof. We only prove the stabilities and error estimates for $\mathcal{Q}_h$. The proofs for $\bar{\mathcal{Q}}_h$ are similar and easier.

Pick $K \in \mathcal{T}_h$ with four vertices $\boldsymbol{p}_i$, $1 \le i \le 4$. Note that $\bigcup_{i=1}^4 \mathcal{T}_h^{\boldsymbol{p}_i}$ is quasi-uniform and $|K| \le |\Omega_K| \le C|K|$. Using (4.1) and (4.3), (4.7) is easily proved:

$$\left\| \mathcal{Q}_h u \right\|_{0,K}^2 \le C \sum_{i=1}^4 |\mathcal{Q}_h u(\boldsymbol{p}_i)|^2 \left\| b^{\boldsymbol{p}_i} \right\|_{0,K}^2 \le C|K| \sum_{i=1}^4 \left\| \psi^{\boldsymbol{p}_i} \right\|_{0,\Omega^{p_i}}^2 \left\| u \right\|_{0,\Omega^{p_i}}^2 \le C \left\| u \right\|_{0,\Omega_K}^2.$$

In order to tackle the H^1-continuity of $\mathcal{Q}_h$, we use the fact that $\nabla V_h \subset \boldsymbol{U}_h$. Then

$$\left\| \nabla(\mathcal{Q}_h u) \right\|_{0,K}^2 \le C \sum_{1 \le i < j \le 4} \left| \int_{\boldsymbol{p}_i}^{\boldsymbol{p}_j} \nabla(\mathcal{Q}_h u) \cdot \tau \right|^2 \int_K \left| \lambda_i \nabla \lambda_j - \lambda_i \nabla \lambda_j \right|^2$$

$$\le C h_K \sum_{1 \le i < j \le 4} \left| \mathcal{Q}_h u(\boldsymbol{p}_j) - \mathcal{Q}_h u(\boldsymbol{p}_i) \right|^2,$$

where λ_i is the barycentric coordinate of K associated with $\boldsymbol{p}_i$, $1 \le i \le 4$.

(I) Suppose $\boldsymbol{p}_i, \boldsymbol{p}_j \notin \partial\Omega$. The definition of $\mathcal{Q}_h$ indicates that

$$
\begin{aligned}
&\left| (\mathcal{Q}_h u)(\boldsymbol{p}_j) - (\mathcal{Q}_h u)(\boldsymbol{p}_i) \right| \\
&= \left| \int_{\Omega^{\boldsymbol{p}_j}} \int_{\Omega^{\boldsymbol{p}_i}} \psi^{\boldsymbol{p}_i}(\boldsymbol{x}) \psi^{\boldsymbol{p}_j}(\boldsymbol{y}) [u(\boldsymbol{x}) - u(\boldsymbol{y})] \, \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{x} \right| \\
&= \left| \int_{\Omega^{\boldsymbol{p}_j}} \int_{\Omega^{\boldsymbol{p}_i}} \psi^{\boldsymbol{p}_i}(\boldsymbol{x}) \psi^{\boldsymbol{p}_j}(\boldsymbol{y}) \int_0^1 \nabla u\Big(\boldsymbol{y} + \tau(\boldsymbol{x} - \boldsymbol{y})\Big) \cdot (\boldsymbol{x} - \boldsymbol{y}) \, \mathrm{d}\tau \, \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{x} \right| \\
&\leq \operatorname{diag}(\Omega^{\boldsymbol{p}_i} \cup \Omega^{\boldsymbol{p}_j}) \left\| \psi^{\boldsymbol{p}_i} \right\|_{L^1(\Omega^{\boldsymbol{p}_i})} \left\| \psi^{\boldsymbol{p}_j} \right\|_{L^2(\Omega^{\boldsymbol{p}_j})} |u|_{H^1(\Omega^{\boldsymbol{p}_i} \cup \Omega^{\boldsymbol{p}_j})} \\
&\leq C h_K^{-1/2} |u|_{H^1(\Omega^{\boldsymbol{p}_i} \cup \Omega^{\boldsymbol{p}_j})}.
\end{aligned}
$$

(II) Suppose $\boldsymbol{p}_i \in \partial\Omega$ and $\boldsymbol{p}_j \notin \partial\Omega$. Then $\partial\Omega_K \cap \partial\Omega$ has positive two-dimensional measure. By $(\mathcal{Q}_h u)(\boldsymbol{p}_i) = 0$ and (4.3), we have

$$
\begin{aligned}
&|(\mathcal{Q}_h u)(\boldsymbol{p}_j) - (\mathcal{Q}_h u)(\boldsymbol{p}_i)| \\
&= |(\mathcal{Q}_h u)(\boldsymbol{p}_j)| = \left| \int_{\Omega^{\boldsymbol{p}_j}} \psi^{\boldsymbol{p}_j}(\boldsymbol{x}) u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right| \\
&\leq C \| \psi^{\boldsymbol{p}_j} \|_{L^2(\Omega^{\boldsymbol{p}_j})} \| u \|_{L^2(\Omega^{\boldsymbol{p}_j})} \leq C \, |\Omega^{\boldsymbol{p}_j}|^{-1/2} \operatorname{diam}(\Omega_K) \, |u|_{H^1(\Omega_K)} \\
&\leq C h_K^{-1/2} |u|_{H^1(\Omega_K)},
\end{aligned}
$$

where in the second inequality we have used scaling arguments and Poincáre's inequality due to $u = 0$ on $\partial\Omega_K \cap \partial\Omega$.

This leads to (4.8).

The quasi-interpolation error estimate (4.9) results from scaling arguments. Pick $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and let $\mathcal{I}_h u \in V(\mathcal{T}_h)$ be the nodal interpolation of u. From (4.6) and the L^2-stability of $\mathcal{Q}_h$, we have

$$
\begin{aligned}
&\left\| (\mathrm{Id} - \mathcal{Q}_h) u \right\|_{L^2(K)} \\
&= \left\| (\mathrm{Id} - \mathcal{Q}_h)(u - \mathcal{I}_h u) \right\|_{L^2(K)} \leq C \left\| u - \mathcal{I}_h u \right\|_{L^2(\Omega_K)} \leq C h_K^2 |u|_{H^2(\Omega_K)}.
\end{aligned}
$$

Estimate (4.9) for $m = 1$ follows by scaling arguments and interpolation between the Sobolev spaces $H^2(\Omega_K)$ and $L^2(\Omega_K)$.

The last estimate can be proved similarly. $\square$

4.2. Local multilevel decomposition

We start by the multilevel splitting of $\bar{V}(\mathcal{T}_L)$ — the finite element space without boundary condition. The multilevel splitting of $V(\mathcal{T}_L)$ utilizes the splitting of $\bar{V}(\mathcal{T}_L)$ by removing the contributions from boundary d.o.f.. We denote by $\bar{\mathcal{Q}}_l : L^2(\Omega) \mapsto \bar{V}(\mathcal{T}_l)$ and

$\mathcal{Q}_l : L^2(\Omega) \mapsto V(\mathcal{T}_l)$ the interpolation operators in Definition 4.1 on $\mathcal{T}_l$. We examine the candidate multilevel decompositions

$$\bar{u}_h = \sum_{l=0}^{L} \bar{u}_l, \quad \bar{u}_0 := \bar{\mathcal{Q}}_0 \bar{u}_h, \quad \bar{u}_l := (\bar{\mathcal{Q}}_l - \bar{\mathcal{Q}}_{l-1})\bar{u}_h, \quad \bar{u}_h \in \bar{V}(\mathcal{T}_L), \tag{4.11}$$

$$u_h = \sum_{l=0}^{L} u_l, \quad u_0 := \mathcal{Q}_0 u_h, \quad u_l := (\mathcal{Q}_l - \mathcal{Q}_{l-1})u_h, \quad u_h \in V(\mathcal{T}_L). \tag{4.12}$$

Lemma 4.2. *Let $\bar{u}_l$, u_l be the splitting components in (4.11) and (4.12) respectively. Then*

$$\bar{u}_l(\boldsymbol{p}) = u_l(\boldsymbol{p}) = 0, \qquad \forall \, \boldsymbol{p} \in \bar{\mathcal{N}}(\mathcal{T}_l) \cap \bar{\mathcal{N}}(\mathcal{T}_{l-1}) \text{ satisfying } b_l^{\boldsymbol{p}} = b_{l-1}^{\boldsymbol{p}}.$$

Proof. Since u_l is defined by removing from $\bar{u}_l$ those basis functions which belong to boundary vertices, it suffices to prove the lemma only for $\bar{u}_l$.

Pick any $\boldsymbol{p} \in \bar{\mathcal{N}}(\mathcal{T}_l) \cap \bar{\mathcal{N}}(\mathcal{T}_{l-1})$ satisfying $b_l^{\boldsymbol{p}} = b_{l-1}^{\boldsymbol{p}}$, it is equivalent to prove $(\bar{\mathcal{Q}}_l u_h)(\boldsymbol{p}) = (\bar{\mathcal{Q}}_{l-1} u_h)(\boldsymbol{p})$ from (4.11). By Definition 4.1, we need only prove

$$\int_{\mathrm{supp}(b_l^{\boldsymbol{p}})} \psi_l^{\boldsymbol{p}}(\boldsymbol{x}) u_h(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_{\mathrm{supp}(b_{l-1}^{\boldsymbol{p}})} \psi_{l-1}^{\boldsymbol{p}}(\boldsymbol{x}) u_h(\boldsymbol{x}) \mathrm{d}\boldsymbol{x},$$

where $\psi_j^{\boldsymbol{p}}$ is the piecewise linear function defined in (4.2) with respect to $b_j^{\boldsymbol{p}}$, $j = l-1, l$. This equality holds clearly due to $b_l^{\boldsymbol{p}} = b_{l-1}^{\boldsymbol{p}}$. $\quad\blacksquare$

Now we introduce the so-called K-functional

$$K(t,v)^2 := \inf_{w \in H^2(\Omega)} \left\{ \|v - w\|_{L^2(\Omega)}^2 + t^2 |w|_{H^2(\Omega)}^2 \right\}, \quad \forall \, t \in \mathbb{R}^1, \; v \in L^2(\Omega). \tag{4.13}$$

It will play the key role in our proof for the $H^1(\Omega)$-stability of decomposition (4.11). We refer to [7] and [30, Appendix A.1, Page 339] for the following lemma.

Lemma 4.3. *Let $\Omega \in \mathbb{R}^3$ be a Lipschitz domain and $0 \leq t < 1$. There exists a constant C only depending on Ω and t such that*

$$\sum_{m=1}^{\infty} t^{-m} K(t^m, v)^2 \leq C |v|_{H^1(\Omega)}^2, \quad v \in H^1(\Omega).$$

The proof for the stabilities of (4.11) and (4.12) depends on a scale-separation of tetrahedra in $\mathcal{T}_{\mathrm{all}} = \bigcup_{l=0}^{L} \mathcal{T}_l$. We define the following sets of tetrahedra according to their generations:

$$\widehat{\mathcal{T}_i} := \left\{ K \in \mathcal{T}_{\mathrm{all}} : \mathcal{G}(K) = i \right\}, \quad i \geq 0, \tag{4.14}$$

where $\mathcal{G}(K)$ is the generation of K defined in Assumption (H2). Clearly we have $\bigcup_{i=0}^{\infty} \widehat{\mathcal{T}_i} = \mathcal{T}_{\mathrm{all}}$. Since the elements in $\widehat{\mathcal{T}_i}$ are generated by i subdivisions of some tetrahedra in $\mathcal{T}_0$, they are mutually nonintersecting and form a subset of Ω, namely,

$$\bigcup \left\{ \bar{K} : K \in \widehat{\mathcal{T}_i} \right\} \subset \bar{\Omega}, \qquad \forall \, i \geq 0. \tag{4.15}$$

Furthermore, $\mathcal{T}_l \setminus \mathcal{T}_{l-1}$, $0 \le l \le L$ ($\mathcal{T}_{-1} = \emptyset$) are nonintersecting sets and

$$\bigcup_{l=0}^{L} (\mathcal{T}_l \setminus \mathcal{T}_{l-1}) = \mathcal{T}_{\text{all}}. \tag{4.16}$$

Definition 4.2. *"Element→Level"–mapping:*

$$\begin{cases} \mathcal{T}_{\text{all}} \mapsto \{0, 1, \cdots, L\}, \\ K \to l(K) \quad \text{satisfying} \quad K \in \mathcal{T}_{l(K)} \setminus \mathcal{T}_{l(K)-1}. \end{cases} \tag{4.17}$$

From (4.16) we know that $l(K)$ is uniquely defined for any $K \in \mathcal{T}_{\text{all}}$.

Lemma 4.4. *Let* (H1)–(H2) *be satisfied. There exists a constant $C > 0$ only depending on Ω, the uniform bound $\rho_{\max}$ of shape-regularity measures, and the mesh-size reduction factor θ such that*

$$\left\| \bar{u}_0 \right\|_{H^1(\Omega)}^2 + \sum_{l=1}^{L} \left\| h^{-1} \bar{u}_l \right\|_{L^2(\Omega)}^2 \le C \left\| \bar{u}_h \right\|_{H^1(\Omega)}^2, \quad \forall \bar{u}_h \in \bar{V}(\mathcal{T}_L), \tag{4.18}$$

where $\bar{u}_h = \sum_{l=0}^{L} \bar{u}_l$ is the multilevel decomposition defined in (4.11).

Proof. Define

$$\bar{\mathcal{N}}_l := \left\{ \boldsymbol{p} \in \bar{\mathcal{N}}(\mathcal{T}_l) : \boldsymbol{p} \notin \bar{\mathcal{N}}(\mathcal{T}_{l-1}) \text{ or } \boldsymbol{p} \in \bar{\mathcal{N}}(\mathcal{T}_{l-1}) \text{ but } b_l^{\boldsymbol{p}} \ne b_{l-1}^{\boldsymbol{p}} \right\}. \tag{4.19}$$

Let $\mathcal{N}(K)$ be the set of four vertices of any tetrahedron K. It is clear that $\bar{\mathcal{N}}_l \subset \{ \boldsymbol{p} \in \mathcal{N}(K) : K \in \mathcal{T}_l \setminus \mathcal{T}_{l-1} \}$. Denote by $\Omega_l^{\boldsymbol{p}} = \text{supp}(b_l^{\boldsymbol{p}})$. From Lemma 4.2 and the local overlapping of $\{ \Omega_l^{\boldsymbol{p}} : \boldsymbol{p} \in \bar{\mathcal{N}}(\mathcal{T}_l) \}$, we have

$$\sum_{l=1}^{L} \left\| h^{-1} \bar{u}_l \right\|_{L^2(\Omega)}^2$$

$$= \sum_{l=1}^{L} \left\| h^{-1} \sum_{\boldsymbol{p} \in \bar{\mathcal{N}}_l} \bar{u}_l(\boldsymbol{p}) b_l^{\boldsymbol{p}} \right\|_{L^2(\Omega)}^2 \le C \sum_{l=1}^{L} \sum_{\boldsymbol{p} \in \bar{\mathcal{N}}_l} \text{diam}(\Omega_l^{\boldsymbol{p}}) \left| \bar{u}_l(\boldsymbol{p}) \right|^2$$

$$\le C \sum_{l=1}^{L} \sum_{K \in \mathcal{T}_l \setminus \mathcal{T}_{l-1}} \sum_{\boldsymbol{p} \in \mathcal{N}(K)} h_K \left| \bar{u}_l(\boldsymbol{p}) \right|^2 \le C \sum_{l=1}^{L} \sum_{K \in \mathcal{T}_l \setminus \mathcal{T}_{l-1}} h_K^{-2} \left\| \bar{u}_l \right\|_{L^2(K)}^2. \tag{4.20}$$

For any $K \in \mathcal{T}_l \setminus \mathcal{T}_{l-1}$, let $T_K \in \mathcal{T}_{l-1}$ satisfy $K \subset T_K$. Then assumption (H3) indicates $h_{T_K} \le C h_K$ with C independent of K. Define

$$D_K := \bigcup \left\{ \bar{T}' : T' \in \mathcal{T}_{l-1}, \, \bar{T}' \cap \bar{T}_K \ne \emptyset \right\}.$$

By Definition 4.2 we know that $l = l(K)$ and thus the definition of D_K only depends on K. We also notice that D_K is just the patch Ω_{T_K} defined in Lemma 4.1 and satisfies $\mathrm{diam}(D_K) \leq Ch_{T_K} \leq Ch_K$. From Lemma 4.1 we know that,

$$
\begin{aligned}
\left\| \bar{u}_l \right\|_{L^2(K)} &\leq \inf_{w \in H^2(\Omega)} \left\{ \left\| (\bar{\mathcal{Q}}_l - \bar{\mathcal{Q}}_{l-1})(\bar{u}_h - w) \right\|_{L^2(K)} + \left\| (\bar{\mathcal{Q}}_l - \bar{\mathcal{Q}}_{l-1}) w \right\|_{L^2(K)} \right\} \\
&\leq C \inf_{w \in H^2(\Omega)} \left\{ \left\| \bar{u}_h - w \right\|_{L^2(D_K)} + h_K^2 \, |w|_{H^2(D_K)} \right\} \\
&\leq C \inf_{w \in H^2(\Omega)} \left\{ \left\| \bar{u}_h - w \right\|_{L^2(D_K)} + \theta^{2m} \, |w|_{H^2(D_K)} \right\},
\end{aligned} \tag{4.21}
$$

where $m = \mathcal{G}(K)$. Then inserting (4.21) into (4.20) yields

$$
\begin{aligned}
&\sum_{l=1}^{L} \left\| h^{-1} \bar{u}_l \right\|_{L^2(\Omega)}^2 \\
&\leq \sum_{l=1}^{L} \sum_{m=1}^{\infty} \sum_{\substack{K \in \mathscr{T}_l \setminus \mathscr{T}_{l-1} \\ \mathcal{G}(K)=m}} h_K^{-2} \| \bar{u}_l \|_{L^2(K)}^2 \leq C \sum_{m=1}^{\infty} \theta^{-2m} \sum_{l=1}^{L} \sum_{\substack{K \in \mathscr{T}_l \setminus \mathscr{T}_{l-1} \\ \mathcal{G}(K)=m}} \| \bar{u}_l \|_{L^2(K)}^2 \\
&\leq C \sum_{m=0}^{\infty} \theta^{-2m} \inf_{w \in H^2(\Omega)} \sum_{K \in \hat{\mathscr{T}}_m} \left[\left\| \bar{u}_h - w \right\|_{L^2(D_K)}^2 + \theta^{4m} \, |w|_{H^2(D_K)}^2 \right] \\
&\leq C \sum_{m=0}^{\infty} \theta^{-2m} \inf_{w \in H^2(\Omega)} \left[\left\| \bar{u}_h - w \right\|_{L^2(\Omega)}^2 + \theta^{4m} \, |w|_{H^2(\Omega)}^2 \right].
\end{aligned}
$$

From Lemma 4.3 we conclude that

$$
\sum_{l=1}^{L} \left\| h^{-1} \bar{u}_l \right\|_{L^2(\Omega)}^2 \leq C \sum_{m=0}^{\infty} \theta^{-2m} K(\theta^{2m}, \bar{u}_h)^2 \leq C \left\| \bar{u}_h \right\|_{H^1(\Omega)}^2.
$$

The leading term of the decomposition is a direct consequence of Lemma 4.1:

$$
\left\| \bar{u}_0 \right\|_{H^1(\Omega)} = \left\| \bar{\mathcal{Q}}_0 \bar{u}_h \right\|_{H^1(\Omega)} \leq C \left\| \bar{u}_h \right\|_{H^1(\Omega)}.
$$

The proof is completed. $\square$

Lemma 4.5. *Let* (H1)–(H2) *be satisfied. There exists a constant $C > 0$ only depending on Ω, $\rho_{\max}$, and θ, such that*

$$
\left\| u_0 \right\|_{H^1(\Omega)}^2 + \sum_{l=1}^{L} \left\| h^{-1} u_l \right\|_{L^2(\Omega)}^2 \leq C \left\| u_h \right\|_{H^1(\Omega)}^2, \quad \forall \, u_h \in V(\mathscr{T}_L), \tag{4.22}
$$

where $u_h = \sum_{l=0}^{L} u_l$ is the multilevel decomposition defined in (4.12).

Proof. First we regard u_h as a function in $\bar{V}(\mathcal{T}_L)$. Then it admits the multilevel decomposition given by (4.11), that is,

$$u_h = \sum_{l=0}^{L} \bar{u}_l, \quad \bar{u}_0 = \bar{\mathcal{Q}}_0 u_h, \quad \bar{u}_l := (\bar{\mathcal{Q}}_l - \bar{\mathcal{Q}}_{l-1}) u_h \text{ for } l \geq 1. \tag{4.23}$$

The stability follows from Lemma 4.4

$$\left\|\bar{u}_0\right\|_{H^1(\Omega)}^2 + \sum_{l=1}^{L} \left\|h^{-1}\bar{u}_l\right\|_{L^2(\Omega)}^2 \leq C \left\|u_h\right\|_{H^1(\Omega)}^2. \tag{4.24}$$

Notice that $\mathcal{Q}_l u_h$ is defined by removing from $\bar{\mathcal{Q}}_l u_h$ those basis functions belonging to boundary vertices. It is easy to see

$$\bar{u}_l = u_l + v_l, \quad v_l(\boldsymbol{x}) := \sum_{\boldsymbol{p} \in \bar{\mathcal{N}}(\mathcal{T}_l) \backslash \mathcal{N}(\mathcal{T}_l)} \bar{u}_l(\boldsymbol{p}) b_l^p(\boldsymbol{x}). \tag{4.25}$$

Here v_l stands for boundary terms and is only supported in the layer of tetrahedra attached to $\partial\Omega$. Clearly we have

$$\left\|h^{-1}v_l\right\|_{L^2(\Omega)}^2 = \sum_{\substack{K \in \mathcal{T}_l \\ \partial K \cap \partial\Omega \neq \emptyset}} h_K^{-2} \left\|v_l\right\|_{L^2(K)}^2 \leq C \sum_{\substack{K \in \mathcal{T}_l \\ \partial K \cap \partial\Omega \neq \emptyset}} h_K \sum_{\boldsymbol{p} \in \mathcal{N}(K)} \left|\bar{u}_l(\boldsymbol{p})\right|^2$$

$$\leq C \sum_{\substack{K \in \mathcal{T}_l \\ \partial K \cap \partial\Omega \neq \emptyset}} h_K^{-2} \left\|\bar{u}_l\right\|_{L^2(K)}^2 \leq C \left\|h^{-1}\bar{u}_l\right\|_{L^2(\Omega)}^2, \tag{4.26}$$

where the constant C only depends on the shape-regularity measure $\rho(\mathcal{T}_l)$, but is independent of $\mathcal{T}_l$ and l.

Now using (4.24)–(4.26), we deduce that

$$\sum_{l=1}^{L} \left\|h^{-1}u_l\right\|_{L^2(\Omega)}^2 \leq 2 \sum_{l=1}^{L} \left\{ \left\|h^{-1}\bar{u}_l\right\|_{L^2(\Omega)}^2 + \left\|h^{-1}v_l\right\|_{L^2(\Omega)}^2 \right\} \leq C \left\|u_h\right\|_{H^1(\Omega)}^2.$$

The proof is completed by using the fact $\left\|u_0\right\|_{H^1(\Omega)} = \left\|\mathcal{Q}_0 u_h\right\|_{H^1(\Omega)} \leq C \left\|u_h\right\|_{H^1(\Omega)}$. $\qquad\square$

Theorem 4.1. *Let* (H1)–(H2) *be satisfied. For any* $u_h \in V(\mathcal{T}_L)$ *and* $\bar{u}_h \in \bar{V}(\mathcal{T}_L)$*, there exist* $u_0 \in V(\mathcal{T}_0)$*,* $\bar{u}_0 \in \bar{V}(\mathcal{T}_0)$ *and* $u_l^p, \bar{u}_l^p \in \mathrm{Span}\left\{b_l^p\right\}$ *such that*

$$u_h = u_0 + \sum_{l=1}^{L} \sum_{\boldsymbol{p} \in \mathcal{N}_l} u_l^p, \qquad \bar{u}_h = \bar{u}_0 + \sum_{l=1}^{L} \sum_{\boldsymbol{p} \in \bar{\mathcal{N}}_l} \bar{u}_l^p, \tag{4.27}$$

$$\left\|u_0\right\|_{H^1(\Omega)}^2 + \sum_{l=1}^{L} \sum_{\boldsymbol{p} \in \mathcal{N}_l} \left\|u_l^p\right\|_{H^1(\Omega)}^2 \leq C \left\|u_h\right\|_{H^1(\Omega)}^2, \tag{4.28}$$

$$\left\|\bar{u}_0\right\|_{H^1(\Omega)}^2 + \sum_{l=1}^{L} \sum_{\boldsymbol{p} \in \bar{\mathcal{N}}_l} \left\|\bar{u}_l^p\right\|_{H^1(\Omega)}^2 \leq C \left\|\bar{u}_h\right\|_{H^1(\Omega)}^2, \tag{4.29}$$

where $\mathscr{N}_l$, $\bar{\mathscr{N}}_l$ are respectively defined in (3.1), (4.19) and the constant $C > 0$ only depends on Ω, $\rho_{\max}$, θ.

Proof. From (4.12) we know that $u_h = u_0 + \sum_{l=1}^{L} u_l$. Define $u_l^p = u_l(\boldsymbol{p}) b_l^p$. The decomposition (4.27) follows clearly from Lemma 4.2. The local norm equivalence indicates that

$$
\sum_{\boldsymbol{p} \in \mathscr{N}_l} \left\| u_l^p \right\|_{H^1(\Omega)}^2
$$

$$
\leq C \sum_{K \in \mathscr{T}_l \setminus \mathscr{T}_{l-1}} h_K \sum_{\boldsymbol{p} \in \mathscr{N}(K)} |u_l(\boldsymbol{p})|^2 \leq C \sum_{K \in \mathscr{T}_l \setminus \mathscr{T}_{l-1}} \left\| h^{-1} u_l \right\|_{L^2(K)}^2 \leq C \left\| h^{-1} u_l \right\|_{L^2(\Omega)}^2,
$$

where $\mathscr{N}(K)$ is the set of four vertices of K. Summing up the above inequality in $1 \leq l \leq L$, (4.28) follows from Lemma 4.5.

Similarly, we can prove the multilevel decomposition of $\bar{u}_h$ and the stability estimate (4.29). The proof is completed. $\qquad\square$

5. Multilevel decomposition of $\mathbf{U}(\mathscr{T}_L)$

The purpose of this section is to tackle (3.5) — the decomposition of $\mathbf{U}(\mathscr{T}_L)$ into the sum of edge element space on the initial mesh and one-dimensional subspaces on fine meshes. First we state the multilevel decomposition of $\mathbf{U}(\mathscr{T}_L)$.

Theorem 5.1. *Let* (H1)–(H2) *be satisfied. For any* $\mathbf{v}_h \in \mathbf{U}(\mathscr{T}_L)$, *there exist* $\mathbf{v}_0 \in \mathbf{U}(\mathscr{T}_0)$ *and* $\mathbf{v}_l \in \mathrm{Span}\left\{ \mathbf{b}_l^E : E \in \mathscr{E}_l \right\}$, $v_l \in \mathrm{Span}\left\{ b_l^p : \boldsymbol{p} \in \mathscr{N}_l \right\}$, $1 \leq l \leq L$ *such that*

$$
\mathbf{v}_h = \mathbf{v}_0 + \sum_{l=1}^{L} (\mathbf{v}_l + \nabla v_l), \tag{5.1}
$$

$$
\left\| \mathbf{v}_0 \right\|_{H(\mathbf{curl},\Omega)}^2 + \sum_{l=1}^{L} \left(\left\| h^{-1} \mathbf{v}_l \right\|_{L^2(\Omega)}^2 + \left\| h^{-1} v_l \right\|_{L^2(\Omega)}^2 \right) \leq C \left\| \mathbf{v}_h \right\|_{H(\mathbf{curl},\Omega)}^2, \tag{5.2}
$$

where the constant C only depends on Ω, θ, and $\rho_{\max}$.

The proof of Theorem 5.1 will be postponed to the end of this section. By splitting the components in (5.1) to local contributions of basis functions, we arrive at the main result of this section.

Theorem 5.2. *Let* (H1)–(H2) *be satisfied. For any* $\mathbf{v}_h \in \mathbf{U}(\mathscr{T}_L)$, *there exist* $\mathbf{v}_0 \in \mathbf{U}(\mathscr{T}_0)$, $\mathbf{v}_l^E \in \mathrm{Span}\left\{ \mathbf{b}_l^E \right\}$, $v_l^p \in \mathrm{Span}\left\{ b_l^p \right\}$ *with* $E \in \mathscr{E}_l$, $\boldsymbol{p} \in \mathscr{N}_l$, $1 \leq l \leq L$ *such that*

$$
\mathbf{v}_h = \mathbf{v}_0 + \sum_{l=1}^{L} \left(\sum_{E \in \mathscr{E}_l} \mathbf{v}_l^E + \sum_{\boldsymbol{p} \in \mathscr{N}_l} \nabla v_l^p \right), \tag{5.3}
$$

$$
\left\| \mathbf{v}_0 \right\|_{H(\mathbf{curl},\Omega)}^2 + \sum_{l=1}^{L} \left(\sum_{E \in \mathscr{E}_l} \left\| \mathbf{v}_l^E \right\|_{H(\mathbf{curl},\Omega)}^2 + \sum_{\boldsymbol{p} \in \mathscr{N}_l} \left| v_l^p \right|_{H^1(\Omega)}^2 \right) \leq C \left\| \mathbf{v}_h \right\|_{H(\mathbf{curl},\Omega)}^2, \tag{5.4}
$$

where the constant C only depends on Ω, θ, and $\rho_{\max}$.

Proof. From (5.1) we have the multilevel decomposition $\mathbf{v}_h = \mathbf{v}_0 + \sum_{l=1}^{L}(\mathbf{v}_l + \nabla v_l)$. Define

$$\mathbf{v}_l^E = \left(\int_E \mathbf{v}_l \cdot \mathrm{d}\vec{s} \right) \mathbf{b}_l^E$$

and $v_l^{\boldsymbol{p}} = v_l(\boldsymbol{p}) b_l^{\boldsymbol{p}}$ for any $1 \le l \le L$. Then (5.3) is a direct consequence of Theorem 5.1. Furthermore, the local norm equivalence indicates that

$$\sum_{E \in \mathscr{E}_l} \left\| \mathbf{v}_l^E \right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}^2 + \sum_{\boldsymbol{p} \in \mathscr{N}_l} \left| v_l^{\boldsymbol{p}} \right|_{H^1(\Omega)}^2$$

$$\le C \sum_{K \in \mathscr{T}_l \backslash \mathscr{T}_{l-1}} \left(\sum_{E \in \mathscr{E}(K)} h_K^{-1} \left| \int_E \mathbf{v}_l \cdot \mathrm{d}\vec{s} \right|^2 + h_K \sum_{\boldsymbol{p} \in \mathscr{N}(K)} |v_l(\boldsymbol{p})|^2 \right)$$

$$\le C \sum_{K \in \mathscr{T}_l \backslash \mathscr{T}_{l-1}} \left(\left\| h^{-1} \mathbf{v}_l \right\|_{L^2(K)}^2 + \left\| h^{-1} v_l \right\|_{L^2(K)}^2 \right).$$

The proof is completed by summing up the inequality in $1 \le l \le L$ and using (5.2). $\qquad\square$

5.1. Discrete Helmholtz decomposition

The technique that we shall use to prove Theorem 5.1 is the discrete Helmholtz decomposition of the edge element space. It builds a close connection between the linear Lagrangian finite element space and the lowest-order Nédélec's edge element space. We refer to [18, Lemma 5.1] for the proof of the following lemma.

Lemma 5.1. *For any $\mathbf{v}_h \in \mathbf{U}(\mathscr{T}_L)$, there exist $\boldsymbol{\Psi}_h \in (V(\mathscr{T}_L))^3$, $p_h \in V(\mathscr{T}_L)$, and $\widetilde{\mathbf{v}}_h \in \mathbf{U}(\mathscr{T}_L)$ such that $\mathbf{v}_h = \widetilde{\mathbf{v}}_h + \boldsymbol{\Pi}_L \boldsymbol{\Psi}_h + \nabla p_h$ and*

$$\left\| h^{-1} \widetilde{\mathbf{v}}_h \right\|_{L^2(\Omega)} + \left\| \boldsymbol{\Psi}_h \right\|_{H^1(\Omega)} + \left| p_h \right|_{H^1(\Omega)} \le C \left\| \mathbf{curl}\, \mathbf{v}_h \right\|_{L^2(\Omega)}, \tag{5.5}$$

where $\boldsymbol{\Pi}_L$ is the nodal edge interpolation operator onto $\mathbf{U}(\mathscr{T}_L)$, and the constant C only depends on Ω and $\rho(\mathscr{T}_L)$.

According to Lemma 5.1, we are going to consider the multilevel splitting of each term in the decomposition $\mathbf{v}_h = \widetilde{\mathbf{v}}_h + \boldsymbol{\Pi}_L \boldsymbol{\Psi}_h + \nabla p_h$ respectively. We study $\boldsymbol{\Pi}_L \boldsymbol{\Psi}_h$ first. From Lemma 4.5, there exist $\boldsymbol{\Psi}_l = (\mathscr{Q}_l - \mathscr{Q}_{l-1})\boldsymbol{\Psi}_h \in \mathrm{Span}\left\{ b_l^{\boldsymbol{p}} : \boldsymbol{p} \in \mathscr{N}_l \right\}^3$, $0 \le l \le L$ such that

$$\boldsymbol{\Psi}_h = \sum_{l=0}^{L} \boldsymbol{\Psi}_l, \qquad \left| \boldsymbol{\Psi}_0 \right|_{H^1(\Omega)}^2 + \sum_{l=1}^{L} \left\| h^{-1} \boldsymbol{\Psi}_l \right\|_{L^2(\Omega)}^2 \le C \left| \boldsymbol{\Psi}_h \right|_{H^1(\Omega)}^2. \tag{5.6}$$

Observe that the function $\boldsymbol{\Psi}_l$ does not belong to $\mathbf{U}(\mathscr{M}_l)$. We target it with edge element interpolation operator $\boldsymbol{\Pi}_l$ onto $\mathbf{U}(\mathscr{M}_l)$, see (2.8), and obtain the splitting described in Lemma 2.1:

$$\boldsymbol{\Psi}_l = \boldsymbol{\Pi}_l \boldsymbol{\Psi}_l + \nabla w_l, \quad w_l \in \widetilde{V}_2(\mathscr{M}_l). \tag{5.7}$$

Using norm-equivalence, the first term of (5.7) is well-controlled locally

$$\left\|\mathbf{\Pi}_l\mathbf{\Psi}_l\right\|_{L^2(K)} \le C\left\|\mathbf{\Psi}_l\right\|_{L^2(K)}, \quad \forall K \in \mathscr{T}_l, \ 0 \le l \le L. \tag{5.8}$$

Because of $\mathbf{curl}\,\mathbf{\Pi}_0\mathbf{\Psi}_0 = \mathbf{curl}\,\mathbf{\Psi}_0$, we infer from (5.6) that

$$\left\|\mathbf{\Pi}_0\mathbf{\Psi}_0\right\|^2_{H(\mathbf{curl},\Omega)} + \sum_{l=1}^{L}\left\|h^{-1}\mathbf{\Pi}_l\mathbf{\Psi}_l\right\|^2_{L^2(\Omega)} \le C\left|\mathbf{\Psi}_h\right|^2_{H^1(\Omega)}. \tag{5.9}$$

Summing up (5.7) in l, we arrive at $\mathbf{\Psi}_h = \sum_{l=0}^{L}\mathbf{\Pi}_l\mathbf{\Psi}_l + \nabla s_h$ where $s_h := \sum_{l=0}^{L}w_l$. By $\mathbf{\Pi}_L\mathbf{\Pi}_l\mathbf{\Psi}_l = \mathbf{\Pi}_l\mathbf{\Psi}_l$ and the commuting diagram $\mathbf{\Pi}_L(\nabla s_h) = \nabla(\mathscr{I}_L s_h)$ (see (2.9)), we have

$$\mathbf{\Pi}_L\mathbf{\Psi}_h = \mathbf{\Pi}_0\mathbf{\Psi}_0 + \sum_{l=1}^{L}\mathbf{\Pi}_l\mathbf{\Psi}_l + \nabla\psi_h, \qquad \psi_h := \mathscr{I}_L s_h \in V(\mathscr{T}_L). \tag{5.10}$$

5.2. Multilevel decomposition of $\sum_{l=1}^{L}\mathbf{\Pi}_l\mathbf{\Psi}_l$

Notice that $\mathbf{\Pi}_l\mathbf{\Psi}_l \notin \mathrm{Span}\left\{\mathbf{b}_l^E : E \in \mathscr{E}_l\right\}$ for $l \ge 1$. We are going to tackle this issue by the following lemma.

Lemma 5.2. *There exist* $\mathbf{u}_0 \in \mathbf{U}(\mathscr{T}_0)$, $\mathbf{u}_l \in \mathrm{Span}\left\{\mathbf{b}_l^E : E \in \mathscr{E}_l\right\}$, $l \ge 1$ *and a constant* C *depending only on* Ω, θ, $\rho_{\max}$ *such that*

$$\sum_{l=1}^{L}\mathbf{\Pi}_l\mathbf{\Psi}_l = \sum_{i=0}^{L}\mathbf{u}_i, \quad \left\|\mathbf{u}_0\right\|^2_{H(\mathbf{curl},\Omega)} + \sum_{i=1}^{L}\left\|h^{-1}\mathbf{u}_i\right\|^2_{L^2(\Omega)} \le C\left|\mathbf{\Psi}_h\right|^2_{H^1(\Omega)}. \tag{5.11}$$

Proof. We start with denoting

$$\mathbf{\Psi}_l^E := \left(\int_E \mathbf{\Pi}_l\mathbf{\Psi}_l \cdot \mathrm{d}\vec{s}\right)\mathbf{b}_l^E = \left(\int_E \mathbf{\Psi}_l \cdot \mathrm{d}\vec{s}\right)\mathbf{b}_l^E, \qquad \forall E \in \mathscr{E}(\mathscr{T}_l), \ 1 \le l \le L.$$

By the definition of $\{\mathscr{E}_0, \cdots, \mathscr{E}_L\}$, it is obvious that

$$\mathscr{E}(\mathscr{T}_l) = \bigcup_{i=0}^{l}\left\{E \in \mathscr{E}_i : \mathbf{b}_i^E = \mathbf{b}_l^E\right\}, \qquad \forall l \ge 0. \tag{5.12}$$

Since $\mathbf{\Psi}_l \in \mathrm{Span}\left\{b_l^p : p \in \mathscr{N}_l\right\}^3$, we know that $\mathbf{\Pi}_l\mathbf{\Psi}_l \in \mathrm{Span}\left\{\mathbf{b}_l^E : E \in \check{\mathscr{E}}_l\right\}$ where

$$\check{\mathscr{E}}_l = \left\{E \in \mathscr{E}(\mathscr{T}_l) : E \text{ has one endpoint in } \mathscr{N}_l\right\}. \tag{5.13}$$

Clearly $\mathscr{E}_l \subset \check{\mathscr{E}}_l$. Using (5.12) we have

$$\sum_{l=1}^{L}\mathbf{\Pi}_l\mathbf{\Psi}_l = \sum_{l=1}^{L}\sum_{E \in \check{\mathscr{E}}_l}\mathbf{\Psi}_l^E = \sum_{l=1}^{L}\sum_{i=0}^{l}\sum_{\substack{E \in \check{\mathscr{E}}_l \cap \mathscr{E}_i \\ \mathbf{b}_i^E = \mathbf{b}_l^E}}\mathbf{\Psi}_l^E = \sum_{i=0}^{L}\mathbf{u}_i, \tag{5.14}$$

where $\mathbf{u}_i \in \mathrm{Span}\left\{\mathbf{b}_i^E : E \in \mathscr{E}_i\right\}$ are defined as follows

$$\mathbf{u}_i := \sum_{l=i,l\neq 0}^{L} \sum_{E\in\breve{\mathscr{E}}_l\cap\mathscr{E}_i, \mathbf{b}_l^E=\mathbf{b}_i^E} \mathbf{\Psi}_l^E, \qquad i \geq 0.$$

Now we take an $E \in \mathscr{E}_i$ and denote $\mathscr{T}_i^E = \bigcup\{T : T \in \mathscr{T}_i, \partial T \cap E \neq \emptyset\}$. For any $l > i$ satisfying $E \in \breve{\mathscr{E}}_l \cap \mathscr{E}_i$, E has one endpoint $\boldsymbol{p} \in \mathscr{N}_l$. From (3.1), there exists a new element $K \in \mathscr{T}_l \backslash \mathscr{T}_{l-1}$ such that

$$E \cup K \subset \Omega_l^p \subset \bigcup\{\bar{T} : T \in \mathscr{T}_i^E\}.$$

Thus each mesh in the set $\{\mathscr{T}_l : E \in \breve{\mathscr{E}}_l \cap \mathscr{E}_i, i < l \leq L\}$ shares edge E and refines $\mathscr{T}_i^E$ successively. By the shape regularity of the meshes, the total number of refinements for $\mathscr{T}_i^E$ must be bounded and independent of L. Thus we conclude

$$\#\{\mathscr{T}_l : E \in \breve{\mathscr{E}}_l \cap \mathscr{E}_i, i \leq l \leq L\} \leq C, \qquad \forall E \in \mathscr{E}_i, \; 0 \leq i \leq L, \tag{5.15}$$

where $\#A$ stands for the cardinality of set A and the constant C is independent of L.

By (5.15), the localness of basis functions, and (5.12), we have

$$\sum_{i=0}^{L} \left\|h^{-1}\mathbf{u}_i\right\|_{L^2(\Omega)}^2 \leq C \sum_{i=0}^{L} \sum_{\substack{l=i\\l\neq 0}}^{L} \sum_{\substack{E\in\breve{\mathscr{E}}_l\cap\mathscr{E}_i\\\mathbf{b}_l^E=\mathbf{b}_i^E}} \left\|h^{-1}\mathbf{\Psi}_l^E\right\|_{L^2(\Omega)}^2$$

$$\leq \sum_{l=1}^{L} \sum_{E\in\mathscr{E}(\mathscr{T}_l)} \left\|h^{-1}\mathbf{\Psi}_l^E\right\|_{L^2(\Omega)}^2 \leq C \sum_{l=1}^{L} \left\|h^{-1}\mathbf{\Psi}_l\right\|_{L^2(\Omega)}^2.$$

Now the stability estimate follows (5.6) and the inverse estimate on $\mathbf{u}_0$. $\qquad\square$

5.3. Multilevel decomposition of $\Pi_L\mathbf{\Psi}_h$

In view of (5.10) and Lemma 5.2, it is left to treat the gradient term $\nabla\psi_h$ in (5.10). We shall use the technique of scale separation as done in the proof of Lemma 4.4.

Lemma 5.3. *There exists a multilevel decomposition of ψ_h satisfying*

$$\psi_h = \sum_{l=0}^{L} \psi_l, \qquad \psi_l \in \mathrm{Span}\left\{b_l^p : \boldsymbol{p} \in \mathscr{N}_l\right\}, \tag{5.16}$$

$$\left|\psi_0\right|_{H^1(\Omega)}^2 + \sum_{l=1}^{L} \left\|h^{-1}\psi_l\right\|_{L^2(\Omega)}^2 \leq C\left|\psi_h\right|_{H^1(\Omega)}^2 \leq C\left|\mathbf{\Psi}_h\right|_{H^1(\Omega)}^2, \tag{5.17}$$

where the constant C only depends on Ω, θ, and $\rho_{\max}$.

Proof. From (5.10) we know that $\psi_h := \mathscr{I}_L s_h \in V(\mathscr{T}_L)$. A direct application of Lemma 4.5 shows that $\psi_h = \sum_{l=0}^{L} \psi_l$ with $\psi_l \in \mathrm{Span}\left\{b_l^{\boldsymbol{p}} : \boldsymbol{p} \in \mathscr{N}_l\right\}$ and

$$\left|\psi_0\right|_{H^1(\Omega)}^2 + \sum_{l=1}^{L} \left\|h^{-1}\psi_l\right\|_{L^2(\Omega)}^2 \leq C\left|\psi_h\right|_{H^1(\Omega)}^2.$$

By $\psi_h = \mathscr{I}_L s_h$ and inverse estimates, it is clear that

$$\left|\psi_h\right|_{H^1(\Omega)} \leq \left|s_h\right|_{H^1(\Omega)} + C\left\|h^{-1}(s_h - \mathscr{I}_L s_h)\right\|_{L^2(\Omega)} \leq C\left|s_h\right|_{H^1(\Omega)}.$$

The proof is completed if we can prove $\left|s_h\right|_{H^1(\Omega)} \leq C\left|\Psi_h\right|_{H^1(\Omega)}$, which is the objective of the rest of the proof.

For any $E \in \mathscr{E}(\mathscr{T}_l)$, we let $\boldsymbol{p}_E$ be the middle point of E and let $b_l^E \in \widetilde{V}_2(\mathscr{T}_l)$ be the basis function belonging to E. For any $K \in \mathscr{T}_l$ satisfying $E \subset \partial K$, let $\boldsymbol{q}_i, \boldsymbol{q}_j \in \mathscr{N}(K)$ be the endpoints of E. Then

$$b_l^E := 4\lambda_i^K \lambda_j^K \qquad \text{in } K,$$

where λ_i^K, λ_j^K are the barycentric coordinates in K belonging to $\boldsymbol{q}_i, \boldsymbol{q}_j$ respectively. From (5.7) we know that $s_h := \sum_{l=0}^{L} w_l$, $w_l \in \widetilde{V}_2(\mathscr{T}_l)$. Recalling from (5.6) that

$$\Psi_l \in \mathrm{Span}\left\{b_l^{\boldsymbol{p}} : \boldsymbol{p} \in \mathscr{N}_l\right\}^3, \qquad \Pi_l \Psi_l \in \mathrm{Span}\left\{\mathbf{b}_l^E : E \in \breve{\mathscr{E}}_l\right\},$$

where $\breve{\mathscr{E}}_l$ is defined in (5.13). We find that $w_l \in \mathrm{Span}\left\{b_l^E : E \in \breve{\mathscr{E}}_l\right\}$. Then

$$s_h = \sum_{l=0}^{L} \sum_{E \in \breve{\mathscr{E}}_l} w_l(\boldsymbol{p}_E) b_l^E = \sum_{l=0}^{L} \sum_{K \in \mathscr{T}_l \setminus \mathscr{T}_{l-1}} \sum_{E \in \mathscr{E}_l^K} \frac{w_l(\boldsymbol{p}_E)}{\mathsf{N}_l(E)} b_l^E,$$

where $\mathscr{E}_l^K := \left\{E \in \mathscr{E}(\mathscr{T}_l) : E \cap \bar{K} \neq \emptyset\right\}$ and $\mathsf{N}_l(E)$ is the multiplicity of E appearing in the sum $\sum_{K \in \mathscr{T}_l \setminus \mathscr{T}_{l-1}} \sum_{E \in \mathscr{E}_l^K}$, namely,

$$\mathsf{N}_l(E) = \#\left\{K \in \mathscr{T}_l \setminus \mathscr{T}_{l-1} : E \in \mathscr{E}_l^K\right\}.$$

The shape-regularity of the meshes indicates that $1 \leq \mathsf{N}_l(E) \leq C$ for any E and l.

Let $l(\cdot)$ be the "Element→Level"–mapping in (4.17). Then we know that $l(K) = l$ for any $K \in \mathscr{T}_l \setminus \mathscr{T}_{l-1}$ and $0 \leq l \leq L$. To replace the index l with K, we define

$$w_K^E := \frac{w_{l(K)}(\boldsymbol{p}_E)}{\mathsf{N}_{l(K)}(E)} b_{l(K)}^E = \frac{w_l(\boldsymbol{p}_E)}{\mathsf{N}_l(E)} b_l^E, \qquad \forall E \in \mathscr{E}_l^K, K \in \mathscr{T}_l \setminus \mathscr{T}_{l-1}.$$

It follows that

$$s_h = \sum_{l=0}^{L} \sum_{K \in \mathscr{T}_l \setminus \mathscr{T}_{l-1}} \sum_{E \in \mathscr{E}_l^K} w_K^E = \sum_{m=0}^{\infty} \sum_{l=0}^{L} \sum_{\substack{K \in \mathscr{T}_l \setminus \mathscr{T}_{l-1} \\ \mathscr{G}(K)=m}} \sum_{E \in \mathscr{E}_l^K} w_K^E = \sum_{m=0}^{\infty} s_m, \qquad (5.18)$$

where we have used (4.14) and (4.16) and

$$s_m(\boldsymbol{x}) := \sum_{K \in \widehat{\mathcal{T}}_m} \sum_{E \in \mathcal{E}_{l(K)}^K} w_K^E(\boldsymbol{x}). \tag{5.19}$$

Notice that $\mathscr{I}_l w_l \equiv 0$ from Lemma 2.1. Using (5.6)–(5.8) and local norm-equivalence, we deduce that

$$
\begin{aligned}
\left| w_K^E \right|_{H^1(\Omega)}^2 &\leq C h_K \left| (w_l - \mathscr{I}_l w_l)(\boldsymbol{p}_E) \right|^2 \leq C h_K^{-2} \left\| w_l - \mathscr{I}_l w_l \right\|_{L^2(\Omega_K)}^2 \leq C \left| w_l \right|_{H^1(\Omega_K)}^2 \\
&\leq C \left\| \boldsymbol{\Psi}_l \right\|_{L^2(\Omega_K)}^2 = C \left\| (\mathcal{Q}_l - \mathcal{Q}_{l-1}) \boldsymbol{\Psi}_h \right\|_{L^2(\Omega_K)}^2 \\
&\leq C h_K^2 \sum_{\substack{T \in \mathcal{T}_{l-1} \\ T \cap \Omega_K \neq \emptyset}} \left| \boldsymbol{\Psi}_h \right|_{H^1(\Omega_T)}^2,
\end{aligned}
$$

where $l = l(K)$ and Ω_K, Ω_T are the patches defined in Lemma 4.1. Assumption (H3) yields $\mathrm{diam}(\Omega_T) \leq C \mathrm{diam}(\Omega_K) \leq C h_K$. From the reasoning of (4.15) and the local overlapping property of these Ω_T, Ω_K, each s_m can be estimated as follows

$$\left| s_m \right|_{H^1(\Omega)}^2 \leq \sum_{K \in \widehat{\mathcal{T}}_m} \sum_{E \in \mathcal{E}_{l(K)}^K} \left| w_K^E \right|_{H^1(\Omega)}^2 \leq C \theta^{2m} \left| \boldsymbol{\Psi}_h \right|_{H^1(\Omega)}^2.$$

The estimate for s_h now follows

$$\left| s_h \right|_{H^1(\Omega)} \leq \sum_{m=0}^{\infty} \left| s_m \right|_{H^1(\Omega)} \leq C \sum_{m=0}^{\infty} \theta^m \left| \boldsymbol{\Psi}_h \right|_{H^1(\Omega)} \leq C \left| \boldsymbol{\Psi}_h \right|_{H^1(\Omega)}.$$

This completes the proof. □

Lemma 5.4. *Let* (H1)–(H2) *be satisfied. For any* $\boldsymbol{\Psi}_h \in V(\mathcal{T}_L)^3$, *there exist* $\mathbf{w}_0 \in \mathbf{U}(\mathcal{T}_0)$ *and* $\mathbf{w}_l \in \mathrm{Span} \left\{ \mathbf{b}_l^E : E \in \mathcal{E}_l \right\}$, $\psi_l \in \mathrm{Span} \left\{ b_l^p : \boldsymbol{p} \in \mathcal{N}_l \right\}$, $1 \leq l \leq L$ *such that*

$$\boldsymbol{\Pi}_L \boldsymbol{\Psi}_h = \mathbf{w}_0 + \sum_{l=1}^{L} (\mathbf{w}_l + \nabla \psi_l), \tag{5.20}$$

$$\left\| \mathbf{w}_0 \right\|_{\mathbf{H}(\mathbf{curl}, \Omega)}^2 + \sum_{l=1}^{L} \left(\left\| h^{-1} \mathbf{w}_l \right\|_{L^2(\Omega)}^2 + \left\| h^{-1} \psi_l \right\|_{L^2(\Omega)}^2 \right) \leq C \left| \boldsymbol{\Psi}_h \right|_{H^1(\Omega)}^2. \tag{5.21}$$

Proof. Let $\sum_{l=1}^{L} \boldsymbol{\Pi}_l \boldsymbol{\Psi}_l = \sum_{l=0}^{L} \mathbf{u}_l$ and $\psi_h = \sum_{l=0}^{L} \psi_l$ be the decompositions in (5.11) and (5.16) respectively. Then (5.20) is obtained by setting $\mathbf{w}_0 = \boldsymbol{\Pi}_0 \boldsymbol{\Psi}_0 + \mathbf{u}_0 + \nabla \psi_0 \in \mathbf{U}(\mathcal{T}_0)$ and $\mathbf{w}_l = \mathbf{u}_l \in \mathrm{Span} \left\{ \mathbf{b}_l^E : E \in \mathcal{E}_l \right\}$ for $l \geq 1$. The stability estimate (5.21) is a direct consequence of (5.9), (5.11), and (5.17). □

5.4. Proof of Theorem 5.1

To end this section, we present the proof of Theorem 5.1.

Proof. of Theorem 5.1. We consider the discrete Helmholtz decomposition of $\mathbf{v}_h$:

$$\mathbf{v}_h = \widetilde{\mathbf{v}}_h + \mathbf{\Pi}_L \mathbf{\Psi}_h + \nabla p_h, \quad \mathbf{\Psi}_h \in V(\mathcal{T}_L)^3, \; p_h \in V(\mathcal{T}_L), \; \widetilde{\mathbf{v}}_h \in \mathbf{U}(\mathcal{T}_L), \quad (5.22)$$

$$\left\| h^{-1} \widetilde{\mathbf{v}}_h \right\|_{L^2(\Omega)} + \left\| \mathbf{\Psi}_h \right\|_{H^1(\Omega)} + \left\| p_h \right\|_{H^1(\Omega)} \leq C \left\| \mathbf{v}_h \right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}. \quad (5.23)$$

The second term of the splitting has already been treated in Lemma 5.4. According to Lemma 4.5, the local multilevel splitting of p_h is easy:

$$p_h = \sum_{l=0}^{L} p_l, \quad p_l \in \text{Span}\left\{ b_l^p : \boldsymbol{p} \in \mathcal{N}_l \right\}, \quad (5.24)$$

$$\left| p_0 \right|_{H^1(\Omega)}^2 + \sum_{l=1}^{L} \left\| h^{-1} p_l \right\|_{L^2(\Omega)}^2 \leq C \left| p_h \right|_{H^1(\Omega)}^2 \leq C \left\| \mathbf{v}_h \right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}^2. \quad (5.25)$$

Now it is left to attack $\widetilde{\mathbf{v}}_h$. The idea is to distribute $\widetilde{\mathbf{v}}_h$ to all refinement zones. To do this, we classify $\mathcal{E}(\mathcal{T}_L)$ according to different mesh levels: $\widetilde{\mathcal{E}}_l := \mathcal{E}(\mathcal{T}_L) \cap \mathcal{E}(\mathcal{T}_l)$, $0 \leq l \leq L$. It is easy to see $\widetilde{\mathcal{E}}_{l-1} \subset \widetilde{\mathcal{E}}_l$. We define

$$\widetilde{\mathbf{\Pi}}_l \widetilde{\mathbf{v}}_h = \sum_{E \in \widetilde{\mathcal{E}}_l} \left(\int_E \widetilde{\mathbf{v}}_h \cdot d\vec{s} \right) \cdot \mathbf{b}_l^E \in \mathbf{U}(\mathcal{T}_l). \quad (5.26)$$

Clearly $\widetilde{\mathbf{\Pi}}_l$ is well-defined since $\widetilde{\mathbf{v}}_h$ is linear on each $E \in \widetilde{\mathcal{E}}_l$, $0 \leq l \leq L$. Then $\widetilde{\mathbf{v}}_h = \widetilde{\mathbf{\Pi}}_L \widetilde{\mathbf{v}}_h$ admits the following multilevel decomposition

$$\widetilde{\mathbf{v}}_h = \sum_{l=0}^{L} \widetilde{\mathbf{v}}_l, \quad \widetilde{\mathbf{v}}_0 := \widetilde{\mathbf{\Pi}}_0 \widetilde{\mathbf{v}}_h, \quad \widetilde{\mathbf{v}}_l := \left(\widetilde{\mathbf{\Pi}}_l - \widetilde{\mathbf{\Pi}}_{l-1} \right) \widetilde{\mathbf{v}}_h, \; 1 \leq l \leq L. \quad (5.27)$$

For each $E \in \mathcal{E}(\mathcal{T}_l) \setminus \mathcal{E}_l$, we have $E \in \mathcal{E}(\mathcal{T}_l) \cap \mathcal{E}(\mathcal{T}_{l-1})$ and $\mathbf{b}_l^E = \mathbf{b}_{l-1}^E$. Then (5.26)–(5.27) show that

$$\int_E \widetilde{\mathbf{v}}_l \cdot d\vec{s} = \int_E \widetilde{\mathbf{\Pi}}_l \widetilde{\mathbf{v}}_h \cdot d\vec{s} - \int_E \widetilde{\mathbf{\Pi}}_{l-1} \widetilde{\mathbf{v}}_h \cdot d\vec{s} = 0.$$

This indicates that $\widetilde{\mathbf{v}}_l \in \text{Span}\left\{ \mathbf{b}_l^E : E \in \mathcal{E}_l \right\}$ for $0 \leq l \leq L$.

From (5.26), it is easy to see that

$$\widetilde{\mathbf{v}}_l = \sum_{E \in \widetilde{\mathcal{E}}_l \setminus \widetilde{\mathcal{E}}_{l-1}} \left(\int_E \widetilde{\mathbf{v}}_l \cdot d\vec{s} \right) \cdot \mathbf{b}_l^E, \quad \forall \, 1 \leq l \leq L.$$

From (5.26)–(5.27) we deduce that

$$\left| \int_E \widetilde{\mathbf{v}}_l \cdot d\vec{s} \right| \leq \left| \int_E \widetilde{\mathbf{v}}_h \cdot d\vec{s} \right| + \sum_{E' \in \widetilde{\mathcal{E}}_{l-1}, E \subset \text{supp}\left(\mathbf{b}_{l-1}^{E'} \right)} C \left| \int_{E'} \widetilde{\mathbf{v}}_h \cdot d\vec{s} \right| \leq C \sum_{\substack{K \in \mathcal{T}_{l-1}, \\ \bar{K} \cap E \neq \emptyset}} h_K^{-1/2} \left\| \widetilde{\mathbf{v}}_h \right\|_{L^2(K)}.$$

From Assumption (H3) we have $h_K \leq C|E|$ for any $K \in \mathcal{T}_{l-1}, E \cap \bar{K} \neq \emptyset$. Then

$$\left\| h^{-1} \widetilde{\mathbf{v}}_l \right\|_{L^2(\Omega)}^2 \leq C \sum_{E \in \widetilde{\mathcal{E}}_l \setminus \widetilde{\mathcal{E}}_{l-1}} \frac{1}{|E|} \left| \int_E \widetilde{\mathbf{v}}_l \cdot \mathrm{d}\vec{s} \right|^2 \leq C \sum_{E \in \widetilde{\mathcal{E}}_l \setminus \widetilde{\mathcal{E}}_{l-1}} \sum_{\substack{K \in \mathcal{T}_{l-1} \\ E \cap \bar{K} \neq \emptyset}} \left\| h^{-1} \widetilde{\mathbf{v}}_h \right\|_{L^2(K)}^2.$$

Notice that $\widetilde{\mathcal{E}}_l \setminus \widetilde{\mathcal{E}}_{l-1}$, $1 \leq l \leq L$ are nonintersecting sets and $\bigcup_{l=1}^{L} \left(\widetilde{\mathcal{E}}_l \setminus \widetilde{\mathcal{E}}_{l-1} \right) \subset \mathcal{E}(\mathcal{T}_L)$. Summing up the above inequalities in $1 \leq l \leq L$ and using (5.23), we get

$$\sum_{l=1}^{L} \left\| h^{-1} \widetilde{\mathbf{v}}_l \right\|_{L^2(\Omega)}^2 \leq C \left\| h^{-1} \widetilde{\mathbf{v}}_h \right\|_{L^2(\Omega)}^2 \leq C \left\| \mathbf{v}_h \right\|_{H(\mathbf{curl},\Omega)}^2. \tag{5.28}$$

The first term satisfies

$$\left\| \widetilde{\mathbf{v}}_0 \right\|_{H(\mathbf{curl},\Omega)}^2 \leq C \sum_{K \in \mathcal{T}_0 \cap \mathcal{T}_L} \left\| \widetilde{\mathbf{v}}_h \right\|_{H(\mathbf{curl},K)}^2 \leq C \left\| \mathbf{v}_h \right\|_{H(\mathbf{curl},\Omega)}^2. \tag{5.29}$$

Finally, inserting (5.20), (5.24), (5.27) into (5.22), we get (5.1) with $\mathbf{v}_0 := \mathbf{w}_0 + \widetilde{\mathbf{v}}_0 + \nabla p_0$ and $\mathbf{v}_l := \mathbf{w}_l + \widetilde{\mathbf{v}}_l$, $v_l := \psi_l + p_l$, $1 \leq l \leq L$. The stability estimate (5.2) is obtained by summing up (5.21), (5.25), and (5.28)–(5.29). $\qquad\square$

6. Strengthened Cauchy-Schwartz inequality

To this end, the proof of Theorem 3.1 only requires strengthened Cauchy-Schwartz inequalities for (3.3) and (3.5). The strengthened Cauchy-Schwartz inequality has been established in [33, 38] for linear Lagrangian finite element spaces, in [14, Sect. 6] for $\mathbf{H}(\mathrm{div})$-elliptic variational problems and so-called face elements, and in [19] for the lowest order edge elements. The proofs here are a little different from those in [14, 19] since $\mathcal{T}_l \setminus \mathcal{T}_{l-1}$, $1 \leq l \leq L$ are nonuniform.

We first prove the Cauchy-Schwarz Inequality on $H_0^1(\Omega)$-conforming finite element spaces. Recall that $\widehat{\mathcal{T}}_m$ is the set of elements in the m-th generation (see (4.14)) and that $l(\cdot)$ is the "Element→Level"–mapping defined in (4.17). Clearly $K \in \mathcal{T}_{l(K)} \setminus \mathcal{T}_{l(K)-1}$ for any $K \in \widehat{\mathcal{T}}_m$.

Lemma 6.1. *Let* $T \in \mathcal{T}_l$, $\mathbf{p} \in \mathcal{N}(T)$, $v \in \mathrm{Span}\left\{ b_l^p \right\}$, *and let* $m \geq \mathcal{G}(T)$. *There exists a constant C only depending on* $\theta, \rho_{\max}$ *such that*

$$\sum_{K \in \widehat{\mathcal{T}}_m} \sum_{\mathbf{q} \in \mathcal{N}(K)} a_s\left(v, w_{l(K)}^{\mathbf{q}} \right) \leq C \theta^{\frac{m - \mathcal{G}(T)}{2}} \| v \|_{H^1(\Omega)} \left(\sum_{K \in \widehat{\mathcal{T}}_m} \sum_{\mathbf{q} \in \mathcal{N}(K)} \left\| w_{l(K)}^{\mathbf{q}} \right\|_{H^1(\Omega_l^p)}^2 \right)^{\frac{1}{2}}, \tag{6.1}$$

where $\Omega_l^p = \mathrm{supp}(b_l^p)$ *and* $w_s^{\mathbf{q}} \in \mathrm{Span}\left\{ b_s^{\mathbf{q}} \right\}$.

Proof. The idea of this proof is drawn from [33, Lemma 6.1]. For convenience we denote $n = \mathscr{G}(T)$. Notice that the tetrahedra of $\mathscr{T}_l$ contained in $\Omega_l^p = \mathrm{supp}(v)$ are quasi-uniform and their diameters are order of θ^n. There exists a constant integer Z_0 only depending on $\rho_{\max}$ such that

$$\max_{T' \in \mathscr{T}_l, T' \subset \Omega_l^p} \mathscr{G}(T') \leq \min_{T' \in \mathscr{T}_l, T' \subset \Omega_l^p} \mathscr{G}(T') + Z_0. \tag{6.2}$$

We denote $w = \sum_{K \in \widehat{\mathscr{T}}_m} \sum_{\boldsymbol{q} \in \mathcal{N}(K)} w_{l(K)}^{\boldsymbol{q}}$. If $m - n \leq Z_0$, the Cauchy-Schwarz inequality and the localness of $w_{l(K)}^{\boldsymbol{q}}$ indicate that

$$a_s(v, w) \leq C \|v\|_{H^1(\Omega_l^p)} \|w\|_{H^1(\Omega_l^p)} \leq C \|v\|_{H^1(\Omega)} \Big(\sum_{K \in \widehat{\mathscr{T}}_m} \sum_{\boldsymbol{q} \in \mathcal{N}(K)} \Big\| w_{l(K)}^{\boldsymbol{q}} \Big\|_{H^1(\Omega_l^p)}^2 \Big)^{\frac{1}{2}}.$$

Then (6.1) follows from the observation that $\theta^{-|m-n|/2} \leq \theta^{-Z_0/2} \leq C$. The rest of the proof is devoted to the case $m > n + Z_0$. Since

$$\max_{T' \in \mathscr{T}_l, T' \subset \Omega_l^p} \mathscr{G}(T') \leq n + Z_0 < m,$$

we know that w is piecewise linear in any $T' \in \mathscr{T}_l$, $T' \subset \Omega_l^p$ and

$$w = \xi := \sum_{K \in \widehat{\mathscr{T}}_m} \sum_{\boldsymbol{q} \in \mathcal{N}(K) \cap \partial T'} w_{l(K)}^{\boldsymbol{q}} \qquad \text{on } \partial T'.$$

It is clear that $\mathrm{supp}(\xi) \cap T' \subset \Gamma_{T'}$ where

$$\Gamma_{T'} := \bigcup \Big\{ K \in \widehat{\mathscr{T}}_m : K \subset T' \text{ and } \partial K \cap \partial T' \neq \emptyset \Big\}$$

is a narrow strip along $\partial T'$. Since v is linear in T', using Green's formula we have

$$\int_{T'} \nabla v \cdot \nabla w = \int_{\partial T'} \frac{\partial v}{\partial \boldsymbol{n}} \xi = \int_{\Gamma_{T'}} \nabla v \cdot \nabla \xi \leq C \theta^{\frac{m-n}{2}} \|\nabla v\|_{L^2(T')} \|\nabla \xi\|_{L^2(T')}.$$

Summing up the above inequality over all $T' \subset \Omega_l^p$ yields

$$\int_{\Omega_l^p} \nabla v \cdot \nabla w \leq C \theta^{\frac{m-n}{2}} \|v\|_{H^1(\Omega_l^p)} \Big(\sum_{K \in \widehat{\mathscr{T}}_m} \sum_{\boldsymbol{q} \in \mathcal{N}(K)} \Big\| w_{l(K)}^{\boldsymbol{q}} \Big\|_{H^1(\Omega_l^p)}^2 \Big)^{\frac{1}{2}}. \tag{6.3}$$

The lower order term is estimated by using Poincáre's inequality:

$$\Big| \int_{\Omega_l^p} v w \Big| \leq C \|v\|_{L^2(\Omega_l^p)} \Big(\sum_{K \in \widehat{\mathscr{T}}_m} \sum_{\boldsymbol{q} \in \mathcal{N}(K)} \Big\| w_{l(K)}^{\boldsymbol{q}} \Big\|_{L^2(\Omega_l^p \cap \Omega_{l(K)}^{\boldsymbol{q}})}^2 \Big)^{\frac{1}{2}}$$

$$\leq C \theta^{m+n} |v|_{H^1(\Omega_l^p)} \Big(\sum_{K \in \widehat{\mathscr{T}}_m} \sum_{\boldsymbol{q} \in \mathcal{N}(K)} \Big\| w_{l(K)}^{\boldsymbol{q}} \Big\|_{H^1(\Omega_l^p)}^2 \Big)^{\frac{1}{2}}. \tag{6.4}$$

Adding up (6.3) and (6.4) yields (6.1). $\qquad\square$

For convenience in notation, we denote $\mathcal{N}_l = \{\boldsymbol{p}_l^i : \; 1 \le i \le N_l\}$ and define $V_l^i :=$ $\mathrm{Span}\left\{b_l^{\boldsymbol{p}_l^i}\right\}$ for $1 \le i \le N_l$ and $1 \le l \le L$, where $b_l^{\boldsymbol{p}} \in V(\mathcal{T}_l)$ is the nodal basis function belonging to $\boldsymbol{p}$. For the initial mesh, we define $N_0 = 1$ and $V_0^1 = V(\mathcal{T}_0)$.

Theorem 6.1. *Let* (H1)–(H2) *be satisfied. For any* $v_l^i, w_l^i \in V_l^i$ *with* $1 \le i \le N_l$ *and* $0 \le l \le L$, *the strengthened Cauchy-Schwartz inequality holds*

$$\sum_{l=0}^{L}\sum_{i=1}^{N_l}\sum_{(s,j)>(l,i)} a_s(v_l^i, w_s^j) \le C \Big(\sum_{l=0}^{L}\sum_{i=1}^{N_l}\big\|v_l^i\big\|_{H^1(\Omega)}^2\Big)^{1/2}\Big(\sum_{l=0}^{L}\sum_{i=1}^{N_l}\big\|w_l^i\big\|_{H^1(\Omega)}^2\Big)^{1/2},$$

where the constant C *only depends on* θ *and* $\rho_{\max}$.

Proof. It is obvious that

$$\sum_{l=0}^{L}\sum_{i=1}^{N_l}\sum_{(s,j)>(l,i)} a_s(v_l^i, w_s^j)$$
$$= \sum_{s=1}^{L}\sum_{j=1}^{n_s} a_s(v_0^1, w_s^j) + \sum_{l=1}^{L}\sum_{i=1}^{N_l}\Big\{\sum_{j=i+1}^{N_l} a_s(v_l^i, w_l^j) + \sum_{s=l+1}^{L}\sum_{j=1}^{N_s} a_s(v_l^i, w_s^j)\Big\}. \qquad (6.5)$$

From Lemma 6.1 we have

$$\Big\|\sum_{s=1}^{L}\sum_{j=1}^{n_s} w_s^j\Big\|_{H^1(\Omega)}^2 = 2\sum_{l=1}^{L}\sum_{i=1}^{n_l}\sum_{(s,j)>(i,l)} a_s(w_l^i, w_s^j) + \sum_{l=1}^{L}\sum_{i=1}^{n_l}\big\|w_l^i\big\|_{H^1(\Omega)}^2$$
$$\le C\sum_{l=1}^{L}\sum_{i=1}^{n_l}\big\|w_l^i\big\|_{H^1(\Omega)}^2.$$

The first term on the righthand side of (6.5) is a direct consequence of the Cauchy-Schwarz Inequality and the above estimate. Since v_l^i, w_l^i are locally supported, the Cauchy-Schwarz Inequality shows that

$$\sum_{l=1}^{L}\sum_{i=1}^{N_l}\sum_{j=i+1}^{N_l} a_s(v_l^i, w_l^j) \le C\Big(\sum_{l=1}^{L}\sum_{i=1}^{N_l}\big\|v_l^i\big\|_{H^1(\Omega)}^2\Big)^{1/2}\Big(\sum_{l=1}^{L}\sum_{i=1}^{N_l}\big\|w_l^i\big\|_{H^1(\Omega)}^2\Big)^{1/2}. \qquad (6.6)$$

It is left to estimate $I := \sum_{l=1}^{L}\sum_{i=1}^{N_l}\sum_{s=l+1}^{L}\sum_{j=1}^{N_s} a_s(v_l^i, w_s^j)$.

Since $\mathcal{N}_l \subset \bigcup\{\mathcal{N}(T) : T \in \mathcal{T}_l \backslash \mathcal{T}_{l-1}\}$, we let $\mathsf{N}_l(\boldsymbol{p}) \ge 1$ be the number of elements in $\mathcal{T}_l \backslash \mathcal{T}_{l-1}$ which share $\boldsymbol{p} \in \mathcal{N}_l$. We also denote $v_l^{\boldsymbol{p}} = v_l^i$, $w_l^{\boldsymbol{p}} = w_l^i$ for $\boldsymbol{p} = \boldsymbol{p}_l^i$. Then

$$I = \sum_{l=1}^{L}\sum_{s=l+1}^{L}\sum_{n,m=0}^{\infty}\sum_{\substack{T\in\mathcal{T}_l\backslash\mathcal{T}_{l-1}\\ \mathcal{G}(T)=n}}\sum_{\substack{K\in\mathcal{T}_s\backslash\mathcal{T}_{s-1}\\ \mathcal{G}(K)=m}}\sum_{\substack{\boldsymbol{p}\in\mathcal{N}(T)\\ \boldsymbol{q}\in\mathcal{N}(K)}} a_s(\tilde{v}_l^{\boldsymbol{p}}, \tilde{w}_s^{\boldsymbol{q}}), \qquad (6.7)$$

where for any $\boldsymbol{p} \in \mathcal{N}(T), \boldsymbol{q} \in \mathcal{N}(K)$, we define

$$\tilde{v}_l^{\boldsymbol{p}} := \begin{cases} v_l^{\boldsymbol{p}}/\mathrm{N}_l(\boldsymbol{p}), & \text{if } \boldsymbol{p} \in \mathcal{N}_l, \\ 0, & \text{if } \boldsymbol{p} \notin \mathcal{N}_l, \end{cases} \qquad \tilde{w}_s^{\boldsymbol{q}} := \begin{cases} w_s^{\boldsymbol{q}}/\mathrm{N}_l(\boldsymbol{q}), & \text{if } \boldsymbol{q} \in \mathcal{N}_s, \\ 0, & \text{if } \boldsymbol{q} \notin \mathcal{N}_s. \end{cases} \qquad (6.8)$$

We define two sets of elements which appear in the sum of (6.7) and are in the same generations

$$\widehat{\mathcal{T}}_m^{(l)} = \Big(\bigcup_{s=l+1}^{L} \mathcal{T}_s \backslash \mathcal{T}_{s-1} \Big) \bigcap \widehat{\mathcal{T}}_m, \qquad \widehat{\mathcal{M}}_n^{(s)} = \Big(\bigcup_{l=1}^{s-1} \mathcal{T}_l \backslash \mathcal{T}_{l-1} \Big) \bigcap \widehat{\mathcal{T}}_n.$$

By the definition of the "Element→Level"–mapping in (4.17), for any $K \in \widehat{\mathcal{T}}_m^{(l)}$, there exists a unique $s > l$ such that $K \in \mathcal{T}_s \backslash \mathcal{T}_{s-1}$ and $l(K) = s$. Thus

$$\sum_{\substack{s=l+1}}^{L} \sum_{\substack{K \in \mathcal{T}_s \backslash \mathcal{T}_{s-1} \\ \mathcal{G}(K)=m}} \sum_{\boldsymbol{q} \in \mathcal{N}(K)} a_s(\tilde{v}_l^{\boldsymbol{p}}, \tilde{w}_s^{\boldsymbol{q}}) = a_s\Big(\tilde{v}_l^{\boldsymbol{p}}, \sum_{K \in \widehat{\mathcal{T}}_m^{(l)}} \sum_{\boldsymbol{q} \in \mathcal{N}(K)} \tilde{w}_{l(K)}^{\boldsymbol{q}} \Big). \qquad (6.9)$$

Similarly we also have

$$\sum_{\substack{l=1}}^{s-1} \sum_{\substack{T \in \mathcal{T}_l \backslash \mathcal{T}_{l-1} \\ \mathcal{G}(T)=n}} \sum_{\boldsymbol{p} \in \mathcal{N}(T)} a_s(\tilde{v}_l^{\boldsymbol{p}}, \tilde{w}_s^{\boldsymbol{q}}) = a_s\Big(\sum_{T \in \widehat{\mathcal{M}}_n^{(s)}} \sum_{\boldsymbol{p} \in \mathcal{N}(T)} \tilde{v}_{l(T)}^{\boldsymbol{p}}, \tilde{w}_s^{\boldsymbol{q}} \Big). \qquad (6.10)$$

Splitting the sum in (6.7) according to $m \geq n$ and $m < n$ and using (6.9)–(6.10), we find that $I = I_1 + I_2$, where

$$I_1 = \sum_{l=1}^{L} \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \sum_{\substack{T \in \mathcal{T}_l \backslash \mathcal{T}_{l-1} \\ \mathcal{G}(T)=n}} \sum_{\boldsymbol{p} \in \mathcal{N}(T)} a_s\Big(\tilde{v}_l^{\boldsymbol{p}}, \sum_{K \in \widehat{\mathcal{T}}_m^{(l)}} \sum_{\boldsymbol{q} \in \mathcal{N}(K)} \tilde{w}_{l(K)}^{\boldsymbol{q}} \Big),$$

$$I_2 = \sum_{s=2}^{L} \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \sum_{\substack{K \in \mathcal{T}_s \backslash \mathcal{T}_{s-1} \\ \mathcal{G}(K)=m}} \sum_{\boldsymbol{q} \in \mathcal{N}(K)} a_s\Big(\sum_{T \in \widehat{\mathcal{M}}_n^{(s)}} \sum_{\boldsymbol{p} \in \mathcal{N}(T)} \tilde{v}_{l(T)}^{\boldsymbol{p}}, \tilde{w}_s^{\boldsymbol{q}} \Big).$$

An application of Lemma 6.1 and the localness of $\tilde{v}_l^{\boldsymbol{p}}, \tilde{w}_{l(K)}^{\boldsymbol{q}}$ shows that

$$I_1 \leq C \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \theta^{\frac{m-n}{2}} \sum_{l=1}^{L} \sum_{\substack{T \in \mathcal{T}_l \backslash \mathcal{T}_{l-1} \\ \mathcal{G}(T)=n}} \sum_{\boldsymbol{p} \in \mathcal{N}(T)} \big\| \tilde{v}_l^{\boldsymbol{p}} \big\|_{H^1(\Omega)} \Big(\sum_{K \in \widehat{\mathcal{T}}_m^{(0)}} \sum_{\boldsymbol{q} \in \mathcal{N}(K)} \big\| \tilde{w}_{l(K)}^{\boldsymbol{q}} \big\|_{H^1(\Omega_l^{\boldsymbol{p}})}^2 \Big)^{\frac{1}{2}}$$

$$\leq C \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \theta^{\frac{m-n}{2}} \Big(\sum_{T \in \widehat{\mathcal{T}}_n^{(0)}} \sum_{\boldsymbol{p} \in \mathcal{N}(T)} \big\| \tilde{v}_{l(T)}^{\boldsymbol{p}} \big\|_{H^1(\Omega)}^2 \Big)^{\frac{1}{2}} \Big(\sum_{K \in \widehat{\mathcal{T}}_m^{(0)}} \sum_{\boldsymbol{q} \in \mathcal{N}(K)} \big\| \tilde{w}_{l(K)}^{\boldsymbol{q}} \big\|_{H^1(\Omega)}^2 \Big)^{\frac{1}{2}}.$$

It is known that the infinite-dimensional matrix $\left(\theta^{|m-n|/2}\right)_{m,n=0}^{\infty}$ has the finite spectrum radius which only depends on θ. From (6.8) we know that

$$I_1 \leq C \Big(\sum_{m=0}^{\infty} \sum_{K \in \widehat{\mathcal{T}}_m^{(0)}} \sum_{\boldsymbol{p} \in \mathcal{N}(K)} \left\| v_{l(K)}^{\boldsymbol{p}} \right\|_{H^1(\Omega)}^2 \Big)^{\frac{1}{2}} \Big(\sum_{m=0}^{\infty} \sum_{K \in \widehat{\mathcal{T}}_m^{(0)}} \sum_{\boldsymbol{p} \in \mathcal{N}(K)} \left\| w_{l(K)}^{\boldsymbol{p}} \right\|_{H^1(\Omega)}^2 \Big)^{\frac{1}{2}}$$

$$\leq C \Big(\sum_{l=1}^{L} \sum_{i=1}^{N_l} \left\| v_l^i \right\|_{H^1(\Omega)}^2 \Big)^{1/2} \Big(\sum_{l=1}^{L} \sum_{i=1}^{N_l} \left\| w_l^i \right\|_{H^1(\Omega)}^2 \Big)^{1/2}. \tag{6.11}$$

Using Lemma 6.1 and similar arguments we have

$$I_2 \leq C \Big(\sum_{l=1}^{L} \sum_{i=1}^{N_l} \left\| v_l^i \right\|_{H^1(\Omega)}^2 \Big)^{1/2} \Big(\sum_{l=1}^{L} \sum_{i=1}^{N_l} \left\| w_l^i \right\|_{H^1(\Omega)}^2 \Big)^{1/2}. \tag{6.12}$$

The proof is completed by inserting (6.11)–(6.12) into (6.7) and combining (6.6). $\qquad\square$

From Theorems 4.1 and 6.1, we have indeed proved Theorem 3.1, that is, the uniform convergence of the local multigrid method for problem (2.4).

For any $1 \leq l \leq L$, we denote $\mathcal{E}_l = \{E_l^i : 1 \leq i \leq M_l\}$ and define

$$\mathbf{U}_l^i := \mathrm{Span} \left\{ \nabla b_l^{p_l^i} \right\}, \quad 1 \leq i \leq N_l \quad \text{and} \quad \mathbf{U}_l^{i+N_l} := \mathrm{Span} \left\{ \mathbf{b}_l^{E_l^i} \right\}, \quad 1 \leq i \leq M_l,$$

where $\mathbf{b}_l^E \in \mathbf{U}(\mathcal{T}_l)$ is the edge basis function belonging to E. For the initial mesh, we define $M_0 = 0$ and $\mathbf{U}_0^1 = \mathbf{U}(\mathcal{T}_0)$.

Theorem 6.2. *Let* (H1)–(H2) *be satisfied. Then for any* $\mathbf{v}_l^i, \mathbf{w}_l^i \in \mathbf{U}_l^i$ *with* $1 \leq i \leq N_l + M_l$ *and* $0 \leq l \leq L$, *the strengthened Cauchy-Schwartz inequality holds*

$$\sum_{l=0}^{L} \sum_{i=1}^{N_l+M_l} \sum_{(s,j)>(l,i)} a_v(\mathbf{v}_l^i, \mathbf{w}_s^j)$$

$$\leq C \Big(\sum_{l=0}^{L} \sum_{i=1}^{N_l+M_l} \left\| \mathbf{v}_l^i \right\|_{H(\mathbf{curl},\Omega)}^2 \Big)^{1/2} \Big(\sum_{l=0}^{L} \sum_{i=1}^{N_l+M_l} \left\| \mathbf{w}_l^i \right\|_{H(\mathbf{curl},\Omega)}^2 \Big)^{1/2}, \tag{6.13}$$

where the constant C *only depends on* θ *and* ρ_{max}.

Proof. This proof is quite similar to those of Lemma 6.1 and Theorem 6.1. It depends on scale separations for both nodal basis functions and edge basis functions. We do not elaborate on the details. $\qquad\square$

It is obvious that Theorem 5.2 and Theorem 6.2 leads directly to Theorem 3.1, namely, the uniform convergence of the local multigrid method for problem (2.5).

7. Numerical results

In this section, we solve the elliptic problem (1.1)–(1.2) and the Maxwell's problem (1.3)–(1.4) on a domain constructed by removing two small cubes from a L-shaped domain as shown in Fig. 2 (left). The initial mesh is shown in Fig. 2 (right). The adaptive algorithm is designed with the residual-based a posteriori error estimates and Dörfler's marking strategy (cf. [9, 13]). The codes are written in Matlab and run on a MAC Pro computer under Linux operating system.
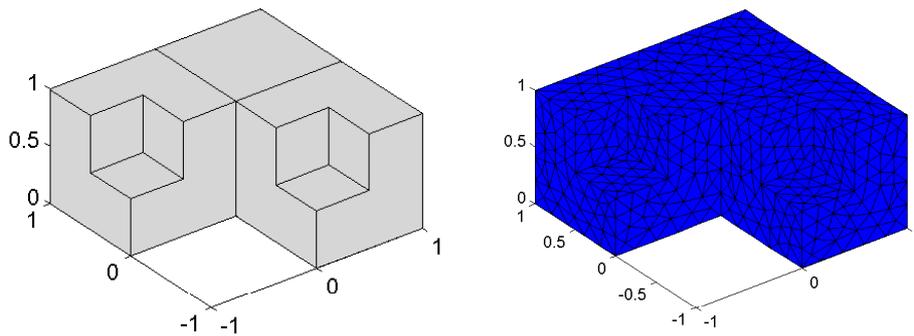

Figure 2: Geometry and initial mesh with 6,105 tetrahedrons for Example 7.1 and Example 7.2.

The stopping rule of the multigrid algorithm is described as follows. Denote the finite element linear algebraic systems to (2.4) and (2.5) on the meshes $\mathcal{T}_l$ by

$$\tilde{A}_l \tilde{u}_l = \tilde{f}_l, \qquad l = 0, 1, \cdots, L,$$

and denote the corresponding multigrid scheme at the lth level by

$$\tilde{u}_l^{(k+1)} = \tilde{u}_l^{(k)} + \tilde{B}_l\big(\tilde{f}_l - \tilde{A}_l\tilde{u}_l^{(k)}\big), \qquad k = 0, 1, 2, \cdots.$$

At the lth level, we set $u_l^{(0)} = u_{l-1}$, the multigrid solution of the previous level, and terminate the multigrid iteration when the following relationship is satisfied

$$\left\|\tilde{f}_l - \tilde{A}_l\tilde{u}_l^{(k+1)}\right\|_\infty \times \left\|\tilde{f}_l - \tilde{A}_l\tilde{u}_l^{(0)}\right\|_\infty^{-1} \leq 10^{-6}.$$

Example 7.1. The elliptic problem (1.1)–(1.2) on the domain in Fig. 2 (left). The right-hand side is set by $f \equiv 1$ in Ω.

Fig. 3 shows the mesh and several slices of the discrete solution for Example 7.1 after 18 adaptive finite element iterations. It is clear that the solution has singularities at the two vertices located at $(-0.5, 0.5, 0.5)$ and $(0.5, -0.5, 0.5)$ and along the edges starting
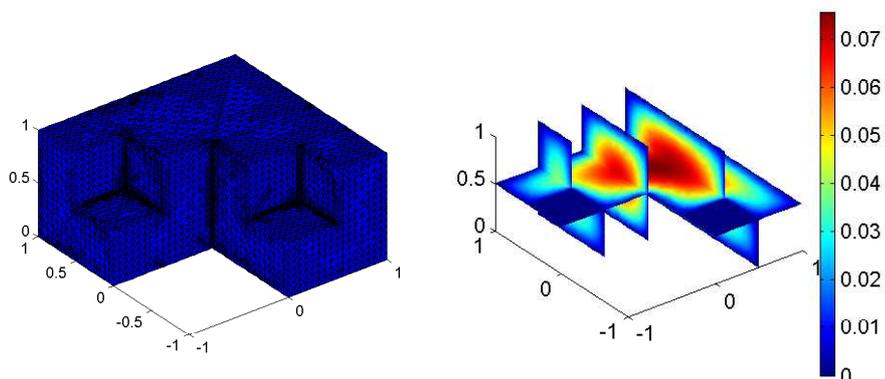
Figure 3: The mesh plot and the slice plot of the discrete solution for Example 7.1 after 18 local refinements. The mesh contains 274,219 tetrahedrons.
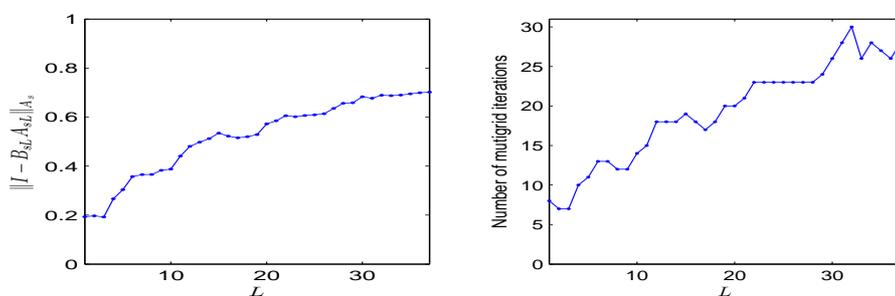

Figure 4: The reduction factor $\left\|I - B_{sL}A_{sL}\right\|_{a_s}$ (left) and the number of multigrid V-cycle iterations (right) for Example 7.1, $1 \leq L \leq 37$.
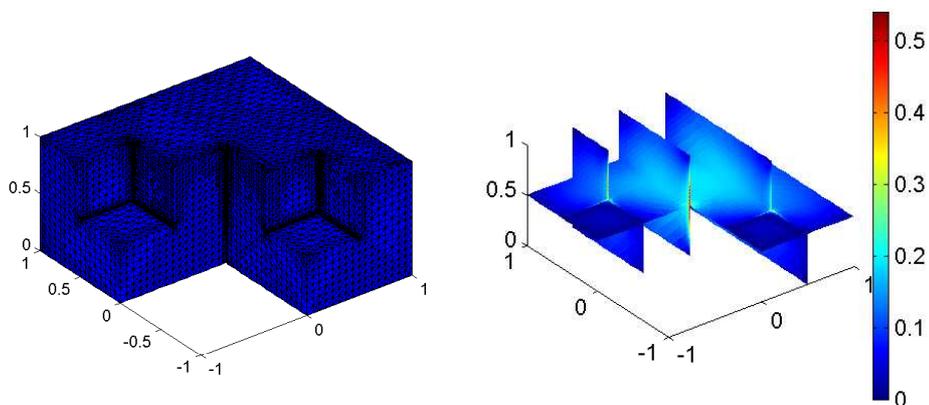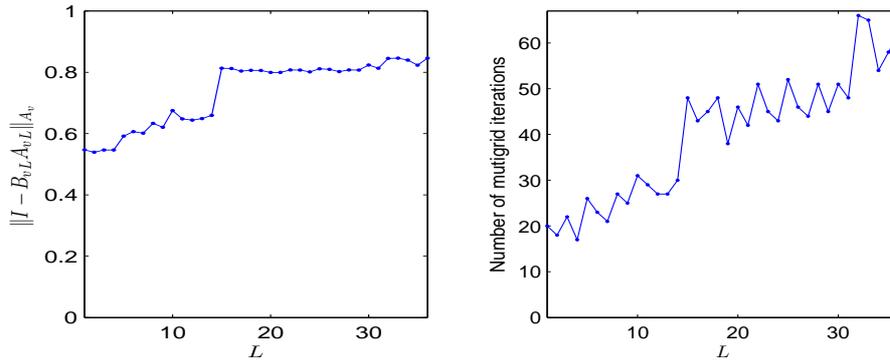

Figure 5: The mesh plot and the slice plot of the the amplitude of the discrete solution for Example 7.2 after 24 local refinements. The mesh contains 258,849 tetrahedrons.

Figure 6: The reduction factor $\left\| I - B_{vL}A_{vL} \right\|_{a_v}$ (left) and the number of multigrid V-cycle iterations (right) for Example 7.2, $1 \leq L \leq 37$.

at the two vertices and the edge through the origin. The mesh is much finer at those places. Fig. 4 shows the reduction factor $\left\| I - B_{sL}A_{sL} \right\|_{a_s}$ (left) and the number of multigrid V-cycle iterations (right) for Example 7.1. Since $I - B_{sL}A_{sL}$ is symmetric with respect to the bilinear form $a_s(\cdot, \cdot)$, $\left\| I - B_{sL}A_{sL} \right\|_{a_s}$ is equal to the maximum eigenvalue of $I - B_{sL}A_{sL}$ which is computed by the power method in this paper. We observe that the convergence rate is independent of the number of levels L as predicted by our theoretical analysis.

Example 7.2. The Maxwell's problem (1.3)–(1.4) on the domain in Fig. 2 (left). The righthand side is set by $\mathbf{f} \equiv (1, 1, 1)$ in Ω.

Fig. 3 shows the mesh plot and a slice plot of the amplitude of the discrete solution for Example 7.2 after 24 adaptive finite element iterations. The mesh is much finer where the solution has singularities. Fig. 6 shows the reduction factor $\left\| I - B_{vL}A_{vL} \right\|_{a_v}$ (left) and the number of multigrid V-cycle iterations (right) for Example 7.2. We observe that the convergence rate is independent of the number of levels L as predicted by our theoretical analysis.

Appendix: Abstract framework of multigrid V-cycle method

The purpose of this appendix is to provide the standard framework of multigrid V-cycle method with smoothers defined by successive subspace correction methods (cf. e.g., [32, 33, 37]).

Suppose that we have a sequence of nested finite element spaces $H_0 \subset H_1 \subset \cdots \subset H_L$ and we consider the following variational problem: Find $\xi \in H_L$ such that

$$a(\xi, v) = f(v), \qquad \forall v \in H_L, \tag{A.1}$$

where a is a positive definite bilinear form on H_L and $f \in H_L'$, the dual space of H_L. On the l-th level, we define a linear operator $A_l \colon H_l \to H_l$ by

$$(A_l \eta, v) := a(\eta, v), \qquad \forall \eta, v \in H_l, \quad 0 \le l \le L,$$

where $(\cdot, \cdot)$ is the L^2-inner product. Then (A.1) is equivalent to the operator equation on H_L: Find $\xi \in H_L$ such that

$$A_L \xi = F, \tag{A.2}$$

where $F \in H_L$ is defined by the Riesz representation of f, namely, $(F, v) = f(v)$ for all $v \in H_L$. The standard V-cycle multigrid algorithm which solves (A.2) is defined by the following iterative scheme:

$$\xi_{k+1} = \xi_k + B_L(F - A_L \xi_k), \qquad k = 0, 1, \cdots. \tag{A.3}$$

Here $B_l \colon H_l \to H_l$, $0 \le l \le L$ are defined recursively by the multilevel algorithm of smoothing and corrections:

Algorithm A.1. *Let $B_0 = A_0^{-1}$. For $l > 0$ and $g \in H_l$, we define $B_l g = v$ where v is defined by three steps of operations:*

1. *Pre-smoothing:* $\quad v \leftarrow R_l g$.

2. *Correction:* $\quad\quad v \leftarrow v + B_{l-1} Q_{l-1}(g - A_l v)$.

3. *Post-smoothing:* $\quad v \leftarrow v + R_l^t(g - A_l v)$.

In Algorithm A.1, $Q_l \colon H_L \to H_l$ stands for the $L^2(\Omega)$-orthogonal projection. Let $H_l^1, \cdots, H_l^{n_l}$ be one family of subspaces of H_l satisfying

$$\sum_{i=1}^{n_l} H_l^i \subseteq H_l, \ 0 \le l \le L \qquad \text{and} \qquad \sum_{l=0}^{L} \sum_{i=1}^{n_l} H_l^i = H_L. \tag{A.4}$$

For $l = 0$, we set $n_0 = 1$ and $H_0^{n_0} := H_0$. Then the smoother R_l is defined by one step of successive subspace correction on the l-th level:

Algorithm A.2. *Given $g \in H_l$ and the initial guess $v_0 = 0$, we define $R_l g = v_{n_l}$ as follows:*

> *for* $i = 1, 2, \cdots, n_l$
>
>> *1. Compute the residual:* $g \leftarrow g - A_l v_{i-1}$
>>
>> *2. Solve the error equation:*   *Find $e_i \in H_l^i$ such that*
>>
>> $$a(e_i, w) = (g, w), \qquad \forall\, w \in H_l^i$$
>>
>> *3. Correction:* $v_i \leftarrow v_{i-1} + e_i$
>
> *endfor*

Substituting (A.2) into (A.3), we obtain the error propagating equation:

$$\xi - \xi_{k+1} = (I - B_L A_L)(\xi - \xi_k) = \cdots = (I - B_L A_L)^{k+1}(\xi - \xi_0).$$

In the following we shall give a framework for proving the contraction property of the error propagation operator $I - B_L A_L$, namely,

$$\|I - B_L A_L\|_A < 1,$$

which indicates the convergence of the iterative scheme (A.3). Let $P_l^i: H_L \to H_l^i$ be orthogonal projections:

$$\forall\, \eta \in H_L, \quad a(P_l^i \eta, w) = a(\eta, w), \qquad \forall\, w \in H_l^i, \quad 1 \le i \le n_l,\ 1 \le l \le L.$$

Then R_l can be represented as follows:

$$R_l = (I - E_l) A_l^{-1}, \qquad E_l := (I - P_l^{n_l})(I - P_l^{n_l - 1}) \cdots (I - P_l^1),$$

Following [32, Sec. 2] and [8, (3.4)], the error propagation operator can be represented as the product of projection operators:

$$I - B_L A_L = \Big[\prod_{l=0}^{L} \prod_{i=1}^{n_l} \big(I - P_l^i \big) \Big]^* \Big[\prod_{l=0}^{L} \prod_{i=1}^{n_l} \big(I - P_l^i \big) \Big],$$

where L^* denotes the adjoint operator of the linear operator $L: H_L \to H_L$ with respect to the inner product $a(\cdot, \cdot)$. Let $\|v\|_A := \sqrt{a(v, v)}$ be the energy norm on H_L. Then

$$\|I - B_L A_L\|_A = \sup_{0 \neq v \in H_L} \frac{a\big((I - B_L A_L)v, v\big)}{\|v\|_A} = \Big\| \prod_{l=0}^{L} \prod_{i=1}^{n_l} \big(I - P_l^i \big) \Big\|_A^2.$$

From (A.4) we know that any function $v \in H_L$ admits a multilevel decomposition:

$$v = \sum_{l=0}^{L} \sum_{i=1}^{n_l} v_l^i, \qquad v_l^i \in H_l^i. \tag{A.5}$$

From the identity of Xu and Zikatanov [37, Corollary 4.3], the error reduction rate by one Multigrid iteration is

$$\|I - B_L A_L\|_A = \frac{c_0}{1 + c_0}, \tag{A.6}$$

where

$$c_0 = \sup_{\|v\|_A = 1} \inf_{\sum_{l=0}^{L} \sum_{i=1}^{n_l} v_l^i = v} \sum_{l=0}^{L} \sum_{i=1}^{n_l} \left\| P_l^i \sum_{(s,j)>(l,i)} v_s^j \right\|_A^2, \tag{A.7}$$

and $(s,j) > (l,i)$ means that either $s > l$ or $s = l$ but $j > i$.

Theorem A.1. *Let H_L satisfy the multilevel decomposition in (A.4). Suppose that there exist two positive constants C_{stab} and C_{orth} such that*

1. *Stability of the space decomposition:*

$$\inf\left\{ \sum_{l=0}^{L} \sum_{i=1}^{n_l} \left\| v_l^i \right\|_A^2 : \sum_{l=0}^{L} \sum_{i=1}^{n_l} v_l^i = v \right\} \leq C_{\text{stab}} \|v\|_A^2, \quad \forall v \in H_L.$$

2. *Strengthened Cauchy-Schwartz inequality: for any $v_l^i, w_l^i \in H_l^i$ with $1 \leq i \leq n_l$ and $0 \leq l \leq L$,*

$$\sum_{l=0}^{L} \sum_{i=1}^{n_l} \sum_{(s,j)>(l,i)} \mathsf{a}(v_l^i, w_s^j) \leq C_{\text{orth}} \left(\sum_{l=0}^{L} \sum_{i=1}^{n_l} \left\| v_l^i \right\|_A^2 \right)^{1/2} \left(\sum_{l=0}^{L} \sum_{i=1}^{n_l} \left\| w_l^i \right\|_A^2 \right)^{1/2}.$$

Then $\|I - B_L A_L\|_A \leq 1 - \left(1 + C_{\text{orth}}^2 C_{\text{stab}} \right)^{-1}$.

Proof. From (A.6) we need only estimate the constant c_0 in (A.7). For any $v \in H_L$, since H_L is finite-dimensional, the infimum in the stability assumption can be obtained, i.e., there exists a splitting of v such that

$$v = \sum_{l=0}^{L} \sum_{i=1}^{n_l} v_l^i, \quad v_l^i \in H_l^i \quad \text{and} \quad \sum_{l=0}^{L} \sum_{i=1}^{n_l} \left\| v_l^i \right\|_A^2 \leq C_{\text{stab}} \|v\|_A^2.$$

Define $w_l^i := P_l^i \sum_{(s,j)>(l,i)} v_s^j$ for any (l,i). Then using the Strengthened Schwartz inequality, we find that

$$\sum_{l=0}^{L} \sum_{i=1}^{n_l} \left\| w_l^i \right\|_A^2 = \sum_{l=0}^{L} \sum_{i=1}^{n_l} \sum_{(s,j)>(l,i)} \mathsf{a}(w_l^i, v_s^j)$$

$$\leq C_{\text{orth}} \left(\sum_{l=0}^{L} \sum_{i=1}^{n_l} \left\| w_l^i \right\|_A^2 \right)^{1/2} \left(\sum_{s=0}^{L} \sum_{j=1}^{n_s} \left\| v_s^j \right\|_A^2 \right)^{1/2}.$$

The stability of the decomposition of v yields that

$$\sum_{l=0}^{L}\sum_{i=1}^{n_l}\left\|w_l^i\right\|_A^2 \le C_{\text{orth}}^2 \sum_{s=0}^{L}\sum_{j=1}^{n_s}\left\|v_s^j\right\|_A^2 \le C_{\text{orth}}^2 C_{\text{stab}}\left\|v\right\|_A^2.$$

From (A.7) and the arbitrariness of v we obtain $c_0 \le C_{\text{orth}}^2 C_{\text{stab}}$. $\square$

References

[1] D. ARNOLD, R. FALK, AND R. WINTHER, *Multigrid in H(div) and H(curl)*, Numer. Math., 85 (2000), pp. 175–195.

[2] D. ARNOLD, R. FALK, AND R. WINTHER, *Finite element exterior calculus, homological techniques, and applications*, Acta Numerica, 15 (2006), pp. 1–155.

[3] D. BAI AND A. BRANDT, *Local mesh refinement multilevel techniques*, SIAM J. Sci. Stat. Comput., 8 (1987), pp. 109–134.

[4] R. BECK, P. DEUFLHARD, R. HIPTMAIR, R. HOPPE, AND B. WOHLMUTH, *Adaptive multilevel methods for edge element discretizations of Maxwell's equations*, Surveys on Mathematics for Industry, 8 (1998), pp. 271–312.

[5] R. BECK, R. HIPTMAIR, R. HOPPE, AND B. WOHLMUTH, *Residual based a-posteriori error estimators for eddy current computation*, M^{2}AN, 34 (2000), pp. 159–182.

[6] P. BINEV, W. DAHMEN, AND R. DEVORE, *Adaptive finite element methods with convergence rates*, Numerische Mathematik, 97 (2004), pp. 219–268.

[7] F. BORNEMANN AND H. YSERENTANT, *A basic norm equivalence for the theory of multilevel methods*, Numer. Math., 64 (1993), pp. 455–476.

[8] J. BRAMBLE, *Multigrid Methods*, vol. 294 of Pitman Research Notes in Mathematical Sciences, Longman, Essex, 1993.

[9] J. M. CASCON, C. KREUZER, R. H. NOCHETTO, AND K. G. SIEBERT, *Quasi-optimal convergence rate for an adaptive finite element method*, SIAM J. Numer. Anal., 46 (2008), pp. 2524–2550.

[10] Z. CHEN AND S. DAI, *On the efficiency of adaptive finite element methods for elliptic problems with discontinuous coefficients*, SIAM J. Sci. Comput., 24 (2002), p. 443ÍC462.

[11] Z. CHEN, L. WANG, AND W. ZHENG, *An adaptive multilevel method for time-harmonic maxwell equations with singularities*, SIAM J. Sci. Comp., 29 (2007), pp. 118–138.

[12] P. CLÉMENT, *Approximation by finite element functions using local regularization*, RAIRO Anal. Numér., 2 (1975), pp. 77–84.

[13] W. DÖRFLER, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal., 33 (1996), pp. 1106–1124.

[14] R. HIPTMAIR, *Multigrid method for Maxwell's equations*, Tech. Report 374, Institut für Mathematik, Universität Augsburg, 1997. USE HIP 99.

[15] R. HIPTMAIR, *Multigrid method for Maxwell's equations*, SIAM J. Numer. Anal., 36 (1999), pp. 204–225.

[16] R. HIPTMAIR, *Finite elements in computational electromagnetism*, Acta Numerica, 11 (2002), pp. 237–339.

[17] R. HIPTMAIR, *Analysis of multilevel methods for eddy current problems*, Math. Comp., 72 (2003), pp. 1281–1303.

[18] R. HIPTMAIR AND J. XU, *Nodal auxiliary space preconditioning in H(curl) and H(div) spaces*, SIAM J. Numer. Anal., 45 (2007), pp. 2483–2509.

[19] R. HIPTMAIR AND W. ZHENG, *Local multigrid in H(curl)*, J. Comp. Math., 27 (2009), pp. 573–603.

[20] I. KOSSACZKÝ, *A recursive approach to local mesh refinement in two and three dimensions*, J. Comput. Appl. Math., 55 (1994), pp. 275–288.

[21] J. MAUBACH, *Local bisection refinement for n–simplicial grids generated by reflection*, SIAM J. Sci. Stat. Comp., 16 (1995), pp. 210–227.

[22] W. MITCHELL, *Optimal multilevel iterative methods for adaptive grids*, SIAM J. Sci. Stat. Comput, 13 (1992), pp. 146–167.

[23] P. OSWALD, *Multilevel finite element approximation*, Teubner Skripten zur Numerik, B.G. Teubner, Stuttgart, 1994.

[24] A. SCHMIDT AND K. SIEBERT, *Alberta-an adaptive hierarchical finite element toolbox*, tech. report, ALBERTA is available online from: http://www.alberta-fem.de.

[25] J. SCHÖBERL, *Commuting quasi-interpolation operators for mixed finite elements*, Preprint ISC-01-10-MATH, Texas A&M University, College Station, TX, 2001.

[26] J. SCHÖBERL, *A posteriori error estimates for Maxwell equations*, Math. Comp., 77 (2008), pp. 633–649.

[27] L. R. SCOTT AND Z. ZHANG, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493.

[28] O. STERZ, A. HAUSER, AND G. WITTUM, *Adaptive local multigrid methods for solving time-harmonic eddy current problems*, IEEE Trans. Magnetics, 42 (2006), pp. 309–318.

[29] R. STEVENSON, *Optimality of a standard adaptive finite element method*, Foundations of Computational Mathematics, 7 (2007), pp. 245–269.

[30] A. TOSELLI AND O. WIDLUND, *Domain Decomposition Methods–Algorithms and Theory*, Springer-Verlag, Berlin Heiderberg, 2005.

[31] R. VERFÜRTH, *A review of a posteriori error estimation and adaptive mesh-refinement rechniques*, Wiley, Chichester, Stuttgart, 1996.

[32] H.-J. WU AND Z.-M. CHEN, *Uniform convergence of multigrid V-cycle on adaptively refined finite element meshes for second order elliptic problems*, Science in China: Series A Mathematics, 49 (2006), pp. 1–28.

[33] J. XU, *Iterative methods by space decomposition and subspace correction*, SIAM Review, 34 (1992), pp. 581–613.

[34] J. XU, *An introduction to multilevel methods*, in Wavelets, Multilevel Methods and Elliptic PDEs, M. Ainsworth, K. Levesley, M. Marletta, and W. Light, eds., Numerical Mathematics and Scientific Computation, Clarendon Press, Oxford, 1997, pp. 213–301.

[35] J. XU, L. CHEN, AND R. NOCHETTO, *Optimal multilevel methods for* H(grad)*,* H(curl)*,* H(div) *systems on graded and unstructured grids*, in Multiscale, Nonlinear and Adaptive Approximation, R. DeVore and A. Kunoth, eds., Springers, 2009, pp. 599–659.

[36] J. XU AND Y.-R. ZHU, *Uniformly convergent multigrid methods for elliptic problems with strongly discontinuous coefficients*, Math. Models Methods Appl. Sci., 18 (2008), pp. 77–105.

[37] J. XU AND L. ZIKATANOV, *The method of alternating projections and the method of subspace corrections in Hilbert space*, J. Am. Math. Soc., 15 (2002), pp. 573–597.

[38] H. YSERENTANT, *On the multi–level splitting of finite element spaces*, Numer. Math., 58 (1986), pp. 379–412.

[39] Z. ZHENG, Z. CHEN, AND L. WANG, *An adaptive finite element method for the* **h**–ψ *formulation of time-dependent eddy current problems*, Numer. Math., 103 (2006), pp. 667–689.