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Error Estimates and Superconvergence of RT0 Mixed Methods for a Class of Semilinear Elliptic Optimal Control Problems

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Abstract. In this paper, we will investigate the error estimates and the superconvergence property of mixed finite element methods for a semilinear elliptic control problem with an integral constraint on control. The state and co-state are approximated by the lowest order Raviart-Thomas mixed finite element and the control variable is approximated by piecewise constant functions. We derive some superconvergence properties for the control variable and the state variables. Moreover, we derive L^∞- and H^{-1}-error estimates both for the control variable and the state variables. Finally, a numerical example is given to demonstrate the theoretical results.

AMS subject classifications: 49J20, 65N30

Key words: Semilinear elliptic equations, optimal control problems, superconvergence, error estimates, mixed finite element methods.

1. Introduction

As far as we know, there have been extensive studies in superconvergence of finite element approximations for optimal control problems, see, for example, [6, 17–19, 21, 24] for standard finite element methods and [4, 5, 8, 9, 20] for mixed finite element methods. In [21], Meyer and Rösch constructed a postprocessing projection operator and derived a quadratic superconvergence of the control by finite element methods. In [18], Liu and Yan considered recovery type superconvergence and a posteriori error estimates for control problem governed by Stokes equations. Next, Yan [24] analyzed the superconvergence property of finite element method for an optimal control problem governed by integral equations. A priori error estimates and superconvergence for an optimal control problem

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of bilinear type are obtained in [23]. Compared with standard finite element methods, the mixed finite element methods have many advantages. When the objective functional contains gradient of the state variable, we will firstly choose the mixed finite element methods. In [5], we used the postprocessing projection operator, which was defined by Meyer and Rösch (see [21]) to prove a quadratic superconvergence of the control by mixed finite element methods. We derived error estimates and superconvergence of mixed methods for convex optimal control problems in [9]. But in that paper, the convergence order is $h^{3/2}$ since the analysis was restricted by the low regularity of the control. Recently, in [8] we derived superconvergence and L^∞-error estimates of RT1 mixed finite element methods for semilinear elliptic control problems with an integral control constraint, however, we didn't considered the superconvergence property of the vector functions.

The goal of this paper is to derive the superconvergence property, the L^∞-error estimates and the H^{-1}-error estimates of the lowest order mixed finite element approximation for a semilinear elliptic control problem with an integral control constraint. Firstly, we derive the superconvergence property between average L^2 projection and the approximation of the control variable, the convergence order is h^2 instead of $h^{3/2}$ in [9], which is caused by the different admissible set. Then, we will derive some superconvergence properties for the state variables. We also derive the L^∞-error estimates for both the control variable and the state variables. Next, we give some applications of the superconvergence results. We derive a superconvergence result for the control variable by using a recovery operator instead of a projection of the discrete adjoint state z_h in the reference [8]. Furthermore, we shall obtain H^{-1}-error estimates for both the control variable and the state variables. Finally, we present a numerical experiment to demonstrate the practical side of the theoretical results about superconvergence and L^∞-error estimates.

We consider the following semilinear optimal control problems for the state variables $\boldsymbol{p}$, y, and the control u with an integral control constraint:

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\right\} \tag{1.1}$$

subject to the state equation

$$-\mathrm{div}(A(x)\,\mathbf{grad}\,y)+\phi(y)=Bu, \quad x\in\Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{p}+\phi(y)=Bu, \quad \boldsymbol{p}=-A(x)\,\mathbf{grad}\,y, \quad x\in\Omega, \tag{1.3}$$

and the boundary condition

$$y=0, \quad x\in\partial\Omega, \tag{1.4}$$

where Ω is a rectangular domain in $\mathbb{R}^2$. U_{ad} denotes the admissible set of the control variable, defined by

$$U_{ad}=\left\{u\in L^\infty(\Omega):\ \int_\Omega u\,dx\ge 0\right\}. \tag{1.5}$$

B is a linear continuous operator from U_{ad} to $L^2(\Omega)$. We assume that the function $\phi(\cdot) \in W^{2,\infty}(-R,R) \cap H^3(-R,R)$ for any $R > 0$, $\phi'(y) \in L^2(\Omega)$ for any $y \in H^1(\Omega)$, and $\phi' \geq 0$. Moreover, we assume that $y_d \in H^1(\Omega)$ and $\boldsymbol{p}_d \in (H^2(\Omega))^2$. ν is a fixed positive number. The coefficient $A(x) = (a_{ij}(x))$ is a symmetric matrix function with $a_{ij}(x) \in W^{1,\infty}(\Omega)$, which satisfies the ellipticity condition

$$c_*|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j, \quad \forall(\xi,x) \in \mathbb{R}^2 \times \bar{\Omega}, \quad c_* > 0.$$

The plan of this paper is as follows. In Section 2, we construct the mixed finite element approximation scheme for the optimal control problem (1.1)-(1.4) and give its necessary optimality conditions. The main results of this paper are stated in Section 3. In Section 3, we derive the superconvergence properties and the L^∞-error estimates for optimal control problem. In Section 4, we will give some applications of the results obtained in Section 3. In Section 5, we obtain H^{-1}-error estimates for both the control variable and the state variables. In Section 6, we present a numerical example to demonstrate our theoretical results. In the last section, we briefly summarize the results obtained and some possible future extensions.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\leq m} \|D^\alpha v\|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. In addition C denotes a general positive constant independent of h, where h is the spatial mesh-size for the control and state discretization.

2. Mixed methods for optimal control problems

In this section, we shall construct mixed finite element approximation scheme of the control problem (1.1)-(1.4). Now, we introduce the co-state elliptic equation

$$-\mathrm{div}(A(x)(\mathbf{grad}z + \boldsymbol{p} - \boldsymbol{p}_d)) + \phi'(y)z = y - y_d, \quad x \in \Omega, \tag{2.1}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{q} + \phi'(y)z = y - y_d, \quad \boldsymbol{q} = -A(x)(\mathbf{grad}z + \boldsymbol{p} - \boldsymbol{p}_d), \quad x \in \Omega, \tag{2.2}$$

and the boundary condition

$$z = 0, \quad x \in \partial\Omega. \tag{2.3}$$

By modifying the proofs of Lemma 2.1 in [7] and Lemma 3.2 in [12], using the regularity argument of elliptic problems in [2,3], we can get that $y, z \in H^3(\Omega)$.

In this paper, we shall employ duality respect to $H^1(\Omega)$ in place of $H_0^1(\Omega)$; i.e., if $\varphi \in L^2(\Omega)$, then

$$\|\varphi\|_{-1} = \|\varphi\|_{-1,2} = \sup_{0\neq\psi\in H^1(\Omega)} \frac{(\varphi,\psi)}{\|\psi\|_1}.$$

Nothing of interest would change if the usual dual space $H^{-1}(\Omega) = (H_0^1(\Omega))^*$ is used.

Let

$$\boldsymbol{V} = H(\mathrm{div};\Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^2,\ \ \mathrm{div}\boldsymbol{v} \in L^2(\Omega)\}, \quad W = L^2(\Omega). \tag{2.4}$$

We recast (1.1)-(1.4) as the following weak form: find $(\boldsymbol{p}, y, u) \in \boldsymbol{V} \times W \times U_{ad}$ such that

$$\begin{aligned}
&\min_{u\in U_{ad}} \left\{\frac{1}{2}\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{\nu}{2}\|u\|^2\right\}, && && (2.5a)\\
&(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, && \forall \boldsymbol{v} \in \boldsymbol{V}, && (2.5b)\\
&(\mathrm{div}\boldsymbol{p}, w) + (\phi(y), w) = (Bu, w), && \forall w \in W. && (2.5c)
\end{aligned}$$

If the triplet $(\boldsymbol{p}, y, u)$ is one of the solution of (2.5a)-(2.5c) then there exists a co-state $(\boldsymbol{q}, z) \in \boldsymbol{V} \times W$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$\begin{aligned}
&(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, && \forall \boldsymbol{v} \in \boldsymbol{V}, && (2.6a)\\
&(\mathrm{div}\boldsymbol{p}, w) + (\phi(y), w) = (Bu, w), && \forall w \in W, && (2.6b)\\
&(A^{-1}\boldsymbol{q}, \boldsymbol{v}) - (z, \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{v}), && \forall \boldsymbol{v} \in \boldsymbol{V}, && (2.6c)\\
&(\mathrm{div}\boldsymbol{q}, w) + (\phi'(y)z, w) = (y - y_d, w), && \forall w \in W, && (2.6d)\\
&(\nu u + B^*z, \tilde{u} - u) \geq 0, && \forall \tilde{u} \in U_{ad}, && (2.6e)
\end{aligned}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$ and B^* is the adjoint operator of B.

In [10], the expression of the control variable is given. Here, we adopt the same method to derive the following operator

$$u = (\max\{0, \overline{B^*z}\} - B^*z)/\nu, \tag{2.7}$$

where $\overline{B^*z} = \int_\Omega B^*z / \int_\Omega 1$ denotes the integral average on Ω of the function B^*z.

Let $\mathscr{T}_h$ denote a uniform rectangulation of the rectangular domain Ω, h_T denotes the diameter of T and $h = \max h_T$. Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denote the lowest order Raviart-Thomas mixed finite element space [22], namely,

$$\begin{aligned}
&\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{V} : \forall T \in \mathscr{T}_h,\ \ \boldsymbol{v}_h|_T \in Q_{1,0}(T) \times Q_{0,1}(T)\}, && (2.8a)\\
&W_h := \{w_h \in W : \forall T \in \mathscr{T}_h,\ \ w_h|_T \in Q_{0,0}(T)\}, && (2.8b)
\end{aligned}$$

where $Q_{m,n}(T)$ indicates the space of polynomials of degree no more than m and n in x and y on T, respectively.

And the approximated space of control is given by

$$U_h := \{\tilde{u}_h \in U_{ad} : \forall T \in \mathcal{T}_h, \ \ \tilde{u}_h|_T = \text{constant}\}. \tag{2.9}$$

Before the mixed finite element scheme is given, we introduce two operators. Firstly, we define the standard $L^2(\Omega)$-projection [13] $P_h : W \to W_h$, which satisfies: for any $\phi \in W$

$$\begin{aligned}
&(P_h\phi - \phi, w_h) = 0, && \forall w_h \in W_h, && (2.10a)\\
&\|\phi - P_h\phi\|_{-r} \le Ch^{1+r}\|\phi\|_1, \quad r = 0,1, && \forall \phi \in H^1(\Omega), && (2.10b)\\
&\|\phi - P_h\phi\|_{0,\rho} \le Ch\|\phi\|_{1,\rho}, \quad 2 \le \rho \le \infty, && \forall \phi \in W^{1,\rho}(\Omega). && (2.10c)
\end{aligned}$$

Next, recall the Fortin projection (see [1] and [13]) $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q} \in \boldsymbol{V}$

$$\begin{aligned}
&(\mathrm{div}(\Pi_h\boldsymbol{q} - \boldsymbol{q}), w_h) = 0, && \forall w_h \in W_h, && (2.11a)\\
&\|\boldsymbol{q} - \Pi_h\boldsymbol{q}\|_{0,\rho} \le Ch\|\boldsymbol{q}\|_{1,\rho}, \quad 2 \le \rho \le \infty, && \forall \boldsymbol{q} \in (W^{1,\rho}(\Omega))^2, && (2.11b)\\
&\|\mathrm{div}(\boldsymbol{q} - \Pi_h\boldsymbol{q})\|_{-r} \le Ch^{1+r}\|\mathrm{div}\boldsymbol{q}\|_1, \quad r = 0,1, && \forall \mathrm{div}\boldsymbol{q} \in H^1(\Omega). && (2.11c)
\end{aligned}$$

We have the commuting diagram property

$$\mathrm{div} \circ \Pi_h = P_h \circ \mathrm{div} : \boldsymbol{V} \to W_h \quad \text{and} \quad \mathrm{div}(I - \Pi_h)\boldsymbol{V} \perp W_h,$$

where and after, I denotes the identity operator.

We assume that

$$\begin{aligned}
&\|\boldsymbol{v}_h\|_{0,\infty} \le Ch^{-1}\|\boldsymbol{v}_h\|, && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, && (2.12a)\\
&\|w_h\|_{0,\infty} \le Ch^{-1}\|w_h\|, && \forall w_h \in W_h. && (2.12b)
\end{aligned}$$

Then the mixed finite element discretization of (2.5a)-(2.5c) is as follows: find the triplet $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times W_h \times U_h$ such that

$$\begin{aligned}
&\min_{u_h \in U_h} \left\{\frac{1}{2}\|\boldsymbol{p}_h - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y_h - y_d\|^2 + \frac{\nu}{2}\|u_h\|^2\right\}, && && (2.13a)\\
&(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \mathrm{div}\boldsymbol{v}_h) = 0, && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, && (2.13b)\\
&(\mathrm{div}\boldsymbol{p}_h, w_h) + (\phi(y_h), w_h) = (Bu_h, w_h), && \forall w_h \in W_h. && (2.13c)
\end{aligned}$$

Similarly, if the triplet $(\boldsymbol{p}_h, y_h, u_h)$ is one of the solution of (2.13a)-(2.13c) then there exists a co-state $(\boldsymbol{q}_h, z_h) \in \boldsymbol{V}_h \times W_h$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$\begin{aligned}
&(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \mathrm{div}\boldsymbol{v}_h) = 0, && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, && (2.14a)\\
&(\mathrm{div}\boldsymbol{p}_h, w_h) + (\phi(y_h), w_h) = (Bu_h, w_h), && \forall w_h \in W_h, && (2.14b)\\
&(A^{-1}\boldsymbol{q}_h, \boldsymbol{v}_h) - (z_h, \mathrm{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h - \boldsymbol{p}_d, \boldsymbol{v}_h), && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, && (2.14c)\\
&(\mathrm{div}\boldsymbol{q}_h, w_h) + (\phi'(y_h)z_h, w_h) = (y_h - y_d, w_h), && \forall w_h \in W_h, && (2.14d)\\
&(\nu u_h + B^*z_h, \tilde{u}_h - u_h) \ge 0, && \forall \tilde{u}_h \in U_h. && (2.14e)
\end{aligned}$$

In the rest of the paper, we shall use some intermediate variables. For any control function $\tilde{u} \in U_{ad}$, we first define the state solution $(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u})) \in (\boldsymbol{V} \times W)^2$ associated with $\tilde{u}$ that satisfies

$$
\begin{aligned}
&(A^{-1}\boldsymbol{p}(\tilde{u}), \boldsymbol{v}) - (y(\tilde{u}), \mathrm{div}\boldsymbol{v}) = 0, && \forall \boldsymbol{v} \in \boldsymbol{V}, && \text{(2.15a)}\\
&(\mathrm{div}\boldsymbol{p}(\tilde{u}), w) + (\phi(y(\tilde{u})), w) = (B\tilde{u}, w), && \forall w \in W, && \text{(2.15b)}\\
&(A^{-1}\boldsymbol{q}(\tilde{u}), \boldsymbol{v}) - (z(\tilde{u}), \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p}(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}), && \forall \boldsymbol{v} \in \boldsymbol{V}, && \text{(2.15c)}\\
&(\mathrm{div}\boldsymbol{q}(\tilde{u}), w) + (\phi'(y(\tilde{u}))z(\tilde{u}), w) = (y(\tilde{u}) - y_d, w), && \forall w \in W. && \text{(2.15d)}
\end{aligned}
$$

Then, we define the discrete state solution $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u})) \in (\boldsymbol{V}_h \times W_h)^2$ associated with $\tilde{u}$ that satisfies

$$
\begin{aligned}
&(A^{-1}\boldsymbol{p}_h(\tilde{u}), \boldsymbol{v}_h) - (y_h(\tilde{u}), \mathrm{div}\boldsymbol{v}_h) = 0, && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, && \text{(2.16a)}\\
&(\mathrm{div}\boldsymbol{p}_h(\tilde{u}), w_h) + (\phi(y_h(\tilde{u})), w_h) = (B\tilde{u}, w_h), && \forall w_h \in W_h, && \text{(2.16b)}\\
&(A^{-1}\boldsymbol{q}_h(\tilde{u}), \boldsymbol{v}_h) - (z_h(\tilde{u}), \mathrm{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}_h), && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, && \text{(2.16c)}\\
&(\mathrm{div}\boldsymbol{q}_h(\tilde{u}), w_h) + (\phi'(y_h(\tilde{u}))z_h(\tilde{u}), w_h) = (y_h(\tilde{u}) - y_d, w_h), && \forall w_h \in W_h. && \text{(2.16d)}
\end{aligned}
$$

Thus, as we defined, the exact solution and its approximation can be written in the following way:

$$
\begin{aligned}
&(\boldsymbol{p}, y, \boldsymbol{q}, z) = (\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u)),\\
&(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) = (\boldsymbol{p}_h(u_h), y_h(u_h), \boldsymbol{q}_h(u_h), z_h(u_h)).
\end{aligned}
$$

3. Superconvergence and L^∞-error estimates

In this section, we will derive superconvergence and L^∞-error estimates for the optimal control problem. In the rest of this paper, we assume that $B = a(x) \in W^{1,\infty}(\Omega)$.

By modifying the proof of Lemma 4.2 in [8], we derive the following superconvergence results for the intermediate solutions which are very important for our following work.

Lemma 3.1. *Let $(\boldsymbol{p}, y, \boldsymbol{q}, z) \in (\boldsymbol{V} \times W)^2$ and $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u)) \in (\boldsymbol{V}_h \times W_h)^2$ be the solutions of* (2.15a)-(2.15d) *and* (2.16a)-(2.16d) *with $\tilde{u} = u$ respectively. Assume that*

$$\boldsymbol{p},\ \boldsymbol{q} \in (H^2(\Omega))^2, \quad y,\ z \in W^{1,\infty}(\Omega),$$

then we have

$$
\begin{aligned}
&\|P_h y - y_h(u)\| \le Ch^2, && \text{(3.1a)}\\
&\|P_h z - z_h(u)\| \le Ch^2. && \text{(3.1b)}
\end{aligned}
$$

Let $(\boldsymbol{p}_h(P_h u), y_h(P_h u), \boldsymbol{q}_h(P_h u), z_h(P_h u))$ and $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u))$ be the solutions of (2.16a)-(2.16d) with $\tilde{u} = P_h u$ and $\tilde{u} = u$, respectively. We can get the following error equations

$$
\begin{aligned}
&(A^{-1}(\boldsymbol{p}_h(P_hu)-\boldsymbol{p}_h(u)),\boldsymbol{v}_h)-(y_h(P_hu)-y_h(u),\mathrm{div}\boldsymbol{v}_h)=0, && (3.2a)\\
&(\mathrm{div}(\boldsymbol{p}_h(P_hu)-\boldsymbol{p}_h(u)),w_h)+(\phi(y_h(P_hu))-\phi(y_h(u)),w_h)\\
&\quad=((a-P_ha)(P_hu-u),w_h), && (3.2b)\\
&(A^{-1}(\boldsymbol{q}_h(P_hu)-\boldsymbol{q}_h(u)),\boldsymbol{v}_h)-(z_h(P_hu)-z_h(u),\mathrm{div}\boldsymbol{v}_h)\\
&\quad=-(\boldsymbol{p}_h(P_hu)-\boldsymbol{p}_h(u),\boldsymbol{v}_h), && (3.2c)\\
&(\mathrm{div}(\boldsymbol{q}_h(P_hu)-\boldsymbol{q}_h(u)),w_h)+(\phi'(y_h(P_hu))z_h(P_hu)-\phi'(y_h(u))z_h(u),w_h)\\
&\quad=(y_h(P_hu)-y_h(u),w_h), && (3.2d)
\end{aligned}
$$

for any $\boldsymbol{v}_h\in\boldsymbol{V}_h$ and $w_h\in W_h$.

Lemma 3.2. *Let $(\boldsymbol{p}_h(P_hu),y_h(P_hu),\boldsymbol{q}_h(P_hu),z_h(P_hu))$ and $(\boldsymbol{p}_h(u),y_h(u),\boldsymbol{q}_h(u),z_h(u))$ be the solutions of* (2.16a)-(2.16d) *with $\tilde{u}=P_hu$ and $\tilde{u}=u$, respectively. Assume that $u\in H^1(\Omega)$ and all the conditions in Lemma* 3.1 *are valid, then we have*

$$
\begin{aligned}
&\|y_h(u)-y_h(P_hu)\|+\|\boldsymbol{p}_h(u)-\boldsymbol{p}_h(P_hu)\|\le Ch^2, && (3.3a)\\
&\|z_h(u)-z_h(P_hu)\|+\|\boldsymbol{q}_h(u)-\boldsymbol{q}_h(P_hu)\|\le Ch^2. && (3.3b)
\end{aligned}
$$

Proof. Note that

$$
((a-P_ha)(P_hu-u),w_h)\le Ch^2\|a\|_{1,\infty}\|u\|_1\|w_h\|,\quad \forall w_h\in W_h. \tag{3.4}
$$

Then, it follows from (3.2a)-(3.2b), (3.4), the assumptions on A and ϕ, and the standard stability estimate that

$$
\|y_h(u)-y_h(P_hu)\|+\|\boldsymbol{p}_h(u)-\boldsymbol{p}_h(P_hu)\|\le Ch^2\|a\|_{1,\infty}\|u\|_1. \tag{3.5}
$$

Now, we rewrite (3.2d) as

$$
\begin{aligned}
&(\mathrm{div}(\boldsymbol{q}_h(P_hu)-\boldsymbol{q}_h(u)),w_h)+(\phi'(y_h(P_hu))(z_h(P_hu)-z_h(u)),w_h)\\
=&((\phi'(y_h(u))-\phi'(y_h(P_hu)))z_h(u),w_h)+(y_h(P_hu)-y_h(u),w_h). && (3.6)
\end{aligned}
$$

For the first term on the right-hand side of (3.6), we have

$$
\begin{aligned}
&((\phi'(y_h(u))-\phi'(y_h(P_hu)))z_h(u),w_h)\\
\le& C\|\phi\|_{2,\infty}\|y_h(u)-y_h(P_hu)\|\cdot\|z_h(u)\|_{0,\infty}\|w_h\|\\
\le& C\|\phi\|_{2,\infty}(\|z\|_{0,\infty}+\|z-P_hz\|_{0,\infty}+\|P_hz-z_h(u)\|_{0,\infty})\|y_h(u)-y_h(P_hu)\|\cdot\|w_h\|\\
\le& C\|\phi\|_{2,\infty}(\|z\|_{0,\infty}+h^{-1}\|P_hz-z_h(u)\|)\|y_h(u)-y_h(P_hu)\|\cdot\|w_h\|. && (3.7)
\end{aligned}
$$

It follows from (3.1b), (3.2c), (3.5)-(3.7), the assumptions on A and ϕ, and the standard stability estimate that

$$
\begin{aligned}
&\|z_h(u)-z_h(P_hu)\|+\|\boldsymbol{q}_h(u)-\boldsymbol{q}_h(P_hu)\|\\
\le& C(\|y_h(u)-y_h(P_hu)\|+\|\boldsymbol{p}_h(u)-\boldsymbol{p}_h(P_hu)\|)+Ch^{-1}\|P_hz-z_h(u)\|\cdot\|y_h(u)-y_h(P_hu)\|\\
\le& Ch^2. && (3.8)
\end{aligned}
$$

Thus, we complete the proof. $\square$

By modifying the proof of Theorem 4.1 in [20], we derive the following L^∞-error estimate for the control variable.

Theorem 3.1. *Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z, u) \in (\boldsymbol{V}\times W)^2 \times U_{ad}$ *and* $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h) \in (\boldsymbol{V}_h \times W_h)^2 \times U_h$ *be the solutions of* (2.6a)-(2.6e) *and* (2.14a)-(2.14e) *respectively. Assume that* $\boldsymbol{p}, \boldsymbol{q} \in (H^2(\Omega))^2$ *and* $u \in W^{1,\infty}(\Omega)$*, then we have*

$$\|u - u_h\|_{0,\infty} \leq Ch. \tag{3.9}$$

We assume that we have a sequence of functional $J_h(\cdot) : L^2(\Omega) \to \mathbb{R}$:

$$J_h(u) = \frac{1}{2}\|\boldsymbol{p}_h(u) - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y_h(u) - y_d\|^2 + \frac{\nu}{2}\|u\|^2. \tag{3.10}$$

It is can be shown that

$$(J_h'(u), v) = (\nu u + B^* z_h(u), v), \tag{3.11a}$$

$$(J_h'(u_h), v) = (\nu u_h + B^* z_h, v), \tag{3.11b}$$

$$(J_h'(P_h u), v) = (\nu P_h u + B^* z_h(P_h u), v). \tag{3.11c}$$

In many applications, $J_h(u)$ is local convex near the optimal solution u. The local convexity of $J_h(\cdot)$ is closely related to the second order sufficient conditions of the control problem, which are assumed in many studies on numerical methods of the problem. Then, there exists a constant $c > 0$, independent of sufficiently small h, such that

$$(J_h'(P_h u) - J_h'(u_h), P_h u - u_h) \geq c\|P_h u - u_h\|^2, \tag{3.12}$$

where u and u_h are solutions of (2.6a)-(2.6e) and (2.14a)-(2.14e) respectively, $P_h u$ is the orthogonal projection of u which is defined in (2.10a).

Theorem 3.2. *Let u be the solution of* (2.6a)-(2.6e) *and u_h be the solution of* (2.14a)-(2.14e)*, respectively. Assume that all the conditions in Lemmas* 3.1 *and* 3.2 *are valid. Then, we have*

$$\|P_h u - u_h\| \leq Ch^2. \tag{3.13}$$

Proof. We choose $\tilde{u} = u_h$ in (2.6e) and $\tilde{u}_h = P_h u$ in (2.14e) to get the following two inequalities:

$$(\nu u + B^* z, u_h - u) \geq 0 \tag{3.14}$$

and

$$(\nu u_h + B^* z_h, P_h u - u_h) \geq 0. \tag{3.15}$$

Note that $u_h - u = u_h - P_h u + P_h u - u$. Adding the two inequalities (3.14) and (3.15), we have

$$(\nu u_h + B^* z_h - \nu u - B^* z, P_h u - u_h) + (\nu u + B^* z, P_h u - u) \geq 0. \tag{3.16}$$

Thus, by (3.12) and (3.16), we find that

$$\begin{aligned}
c\|P_h u - u_h\|^2 \leq& (J_h'(P_h u) - J_h'(u_h), P_h u - u_h) \\
=& \nu(P_h u - u_h, P_h u - u_h) + (B^* z_h(P_h u) - B^* z_h, P_h u - u_h) \\
=& \nu(P_h u - u, P_h u - u_h) + \nu(u - u_h, P_h u - u_h) \\
&+ (B^* z_h(P_h u) - B^* z_h, P_h u - u_h) \\
\leq& (B^* z_h - B^* z, P_h u - u_h) + (\nu u + B^* z, P_h u - u) \\
&+ (B^* z_h(P_h u) - B^* z_h, P_h u - u_h) \\
=& (B^* z_h(P_h u) - B^* z_h(u), P_h u - u_h) + (B^* z_h(u) - B^* P_h z, P_h u - u_h) \\
&+ (B^* P_h z - B^* z, P_h u - u_h) + (\nu u + B^* z, P_h u - u). \qquad (3.17)
\end{aligned}$$

By Lemma 3.1 and Lemma 3.2, we find that

$$(B^* z_h(u) - B^* P_h z, P_h u - u_h) \leq C h^4 + \frac{c}{4}\|P_h u - u_h\|^2 \tag{3.18}$$

and

$$(B^* z_h(P_h u) - B^* z_h(u), P_h u - u_h) \leq C h^4 + \frac{c}{4}\|P_h u - u_h\|^2. \tag{3.19}$$

For the third term on the right-hand side of (3.17), we have

$$\begin{aligned}
(B^* P_h z - B^* z, P_h u - u_h) =& ((a - P_h a)(P_h z - z), P_h u - u_h) \\
\leq& C h^4 \|a\|_{1,\infty}^2 \|z\|_1^2 + \frac{c}{4}\|P_h u - u_h\|^2. \qquad (3.20)
\end{aligned}$$

From (2.7), we know that

$$\nu u + B^* z = \max\{0, \overline{B^* z}\} = \text{const}. \tag{3.21}$$

Thus, we have

$$(\nu u + B^* z, P_h u - u) = (\nu u + B^* z) \int_\Omega (P_h u - u) = 0. \tag{3.22}$$

Combining (3.17)-(3.20) with (3.22), we derive (3.13). □

From Eqs. (2.6a)-(2.6d) and (2.14a)-(2.14d), using (2.10a) and (2.11a), we can easily obtain the following error equations

$$(A^{-1}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h), \boldsymbol{v}_h) - (P_h y - y_h, \mathrm{div}\boldsymbol{v}_h) = -(A^{-1}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \boldsymbol{v}_h), \tag{3.23a}$$

$$(\mathrm{div}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h), w_h) = -(\phi(y) - \phi(y_h), w_h) + (a(u - u_h), w_h), \tag{3.23b}$$

$$\begin{aligned}
&(A^{-1}(\Pi_h \boldsymbol{q} - \boldsymbol{q}_h), \boldsymbol{v}_h) - (P_h z - z_h, \mathrm{div}\boldsymbol{v}_h) \\
&\quad = -(A^{-1}(\boldsymbol{q} - \Pi_h \boldsymbol{q}), \boldsymbol{v}_h) - (\boldsymbol{p} - \Pi_h \boldsymbol{p}, \boldsymbol{v}_h) - (\Pi_h \boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{v}_h), \qquad (3.23c)
\end{aligned}$$

$$(\mathrm{div}(\Pi_h \boldsymbol{q} - \boldsymbol{q}_h), w_h) = (\phi'(y_h) z_h - \phi'(y) z, w_h) + (P_h y - y_h, w_h), \tag{3.23d}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$.

Similar to Lemma 3.1, we can obtain

Theorem 3.3. *Let* (y,z) *and* (y_h,z_h) *be the solutions of* (2.6a)-(2.6e) *and* (2.14a)-(2.14e) *respectively. Assume that all the conditions in Lemmas* 3.1 *and* 3.2 *are valid, then we have*

$$\|P_h y - y_h\| + \|P_h z - z_h\| \le Ch^2. \tag{3.24}$$

Now, we can derive the following superconvergence properties for vector-valued functions.

Theorem 3.4. *Let* $(\boldsymbol{p},\boldsymbol{q})$ *and* $(\boldsymbol{p}_h,\boldsymbol{q}_h)$ *be the solutions of* (2.6a)-(2.6e) *and* (2.14a)-(2.14e) *respectively. Assume that* $\boldsymbol{p},\ \boldsymbol{q} \in (H^2(\Omega))^2$*,* $y,\ z \in W^{1,\infty}(\Omega)$ *and* $u \in H^1(\Omega)$*, then we have*

$$\|\Pi_h \boldsymbol{p} - \boldsymbol{p}_h\| + \|\Pi_h \boldsymbol{q} - \boldsymbol{q}_h\| \le Ch^{\frac{3}{2}}. \tag{3.25}$$

Proof. Choosing $\boldsymbol{v}_h = \Pi_h \boldsymbol{p} - \boldsymbol{p}_h$ in (3.23a) and $w_h = P_h y - y_h$ in (3.23b), respectively. Then adding the two equations to get

$$\begin{aligned}(A^{-1}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h), \Pi_h \boldsymbol{p} - \boldsymbol{p}_h) = &- (A^{-1}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \Pi_h \boldsymbol{p} - \boldsymbol{p}_h) - (\phi(y) - \phi(y_h), P_h y - y_h) \\ &+ (a(u-u_h), P_h y - y_h).\end{aligned} \tag{3.26}$$

By applying the proof of Theorems 4.1, 5.1 and Example 6.2 in [14], we can prove

$$(A^{-1}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \Pi_h \boldsymbol{p} - \boldsymbol{p}_h) \le Ch^{\frac{3}{2}}(\|\boldsymbol{p}\|_2 \|\Pi_h \boldsymbol{p} - \boldsymbol{p}_h\| + \|\boldsymbol{p}\|_1 \|\mathrm{div}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h)\|). \tag{3.27}$$

Using (2.10a) and (2.10c), for some function $\tilde{y}$, we get

$$\begin{aligned}&(\phi(y) - \phi(y_h), P_h y - y_h) = (\phi'(\tilde{y})(y - y_h), P_h y - y_h) \\ =&((\phi'(\tilde{y}) - P_h(\phi'(\tilde{y})))(y - P_h y), P_h y - y_h) + (\phi'(\tilde{y})(P_h y - y_h), P_h y - y_h) \\ \le& Ch^2 \|\phi\|_{2,\infty} \|y\|_1 \|P_h y - y_h\| + C\|\phi\|_{1,\infty} \|P_h y - y_h\|^2.\end{aligned} \tag{3.28}$$

Moreover,

$$\begin{aligned}(a(u-u_h), P_h y - y_h) =&(a(u - P_h u), P_h y - y_h) + (a(P_h u - u_h), P_h y - y_h) \\ =&((a - P_h a)(u - P_h u), P_h y - y_h) + (a(P_h u - u_h), P_h y - y_h) \\ \le& C(h^2 \|a\|_{1,\infty} \|u\|_1 + \|a\|_{0,\infty} \|P_h u - u_h\|) \|P_h y - y_h\|.\end{aligned} \tag{3.29}$$

Using (3.26)-(3.29), Theorems 3.2 and 3.3 and the standard stability estimate, we find that

$$\|\Pi_h \boldsymbol{p} - \boldsymbol{p}_h\| \le Ch^{\frac{3}{2}}. \tag{3.30}$$

Choosing $\boldsymbol{v}_h = \Pi_h \boldsymbol{q} - \boldsymbol{q}_h$ in (3.23c) and $w_h = P_h z - z_h$ in (3.23d), respectively. Then adding the two equations to get

$$\begin{aligned}(A^{-1}(\Pi_h \boldsymbol{q} - \boldsymbol{q}_h), \Pi_h \boldsymbol{q} - \boldsymbol{q}_h) = &- (A^{-1}(\boldsymbol{q} - \Pi_h \boldsymbol{q}), \Pi_h \boldsymbol{q} - \boldsymbol{q}_h) - (\boldsymbol{p} - \Pi_h \boldsymbol{p}, \Pi_h \boldsymbol{q} - \boldsymbol{q}_h) \\ &- (\Pi_h \boldsymbol{p} - \boldsymbol{p}_h, \Pi_h \boldsymbol{q} - \boldsymbol{q}_h) + (P_h y - y_h, P_h z - z_h) \\ &+ (\phi'(y_h) z_h - \phi'(y) z, P_h z - z_h).\end{aligned} \tag{3.31}$$

Similar to (3.27), we have

$$(A^{-1}(\boldsymbol{q}-\Pi_h\boldsymbol{q}),\Pi_h\boldsymbol{q}-\boldsymbol{q}_h)\le Ch^{\frac{3}{2}}(\|\boldsymbol{q}\|_2\|\Pi_h\boldsymbol{q}-\boldsymbol{q}_h\|+\|\boldsymbol{q}\|_1\|\mathrm{div}(\Pi_h\boldsymbol{q}-\boldsymbol{q}_h)\|) \tag{3.32}$$

and

$$(\boldsymbol{p}-\Pi_h\boldsymbol{p},\Pi_h\boldsymbol{q}-\boldsymbol{q}_h)\le Ch^{\frac{3}{2}}(\|\boldsymbol{p}\|_2\|\Pi_h\boldsymbol{q}-\boldsymbol{q}_h\|+\|\boldsymbol{p}\|_1\|\mathrm{div}(\Pi_h\boldsymbol{q}-\boldsymbol{q}_h)\|). \tag{3.33}$$

From (3.30) and Theorem 3.3, we can see that

$$(\Pi_h\boldsymbol{p}-\boldsymbol{p}_h,\Pi_h\boldsymbol{q}-\boldsymbol{q}_h)\le Ch^{\frac{3}{2}}\|\Pi_h\boldsymbol{q}-\boldsymbol{q}_h\| \tag{3.34}$$

and

$$(P_hy-y_h,P_hz-z_h)\le Ch^2\|P_hz-z_h\|. \tag{3.35}$$

Finally, for the last term on the right-hand side of (3.31), using (2.10a), (2.10c), (3.24) and the inverse inequality, we have

$$\begin{aligned}
&(\phi'(y_h)z_h-\phi'(y)z,P_hz-z_h)\\
=&(\phi'(y_h)z_h-\phi'(y)z_h,P_hz-z_h)+(\phi'(y)z_h-\phi'(y)z,P_hz-z_h)\\
=&((\phi'(y_h)-\phi'(y))(z_h-z),P_hz-z_h)+((\phi'(y_h)-\phi'(y))z,P_hz-z_h)\\
&+(\phi'(y)(z_h-P_hz),P_hz-z_h)+(\phi'(y)(P_hz-z),P_hz-z_h)\\
=&((\phi'(y_h)-\phi'(y))(z_h-P_hz),P_hz-z_h)+((\phi'(y_h)-\phi'(y))(P_hz-z),P_hz-z_h)\\
&+((\phi'(y_h)-\phi'(P_hy))z,P_hz-z_h)+((\phi'(y)-P_h(\phi'(y)))(P_hz-z),P_hz-z_h)\\
&+((\phi'(P_hy)-\phi'(y))z,P_hz-z_h)+(\phi'(y)(z_h-P_hz),P_hz-z_h)\\
\le&C\|\phi\|_{2,\infty}(\|y-P_hy\|+\|P_hy-y_h\|)\|z_h-P_hz\|_{0,\infty}\|P_hz-z_h\|\\
&+Ch\|\phi\|_{2,\infty}(\|y-P_hy\|+\|P_hy-y_h\|)\|z\|_{1,\infty}\|P_hz-z_h\|\\
&+C\|\phi\|_{2,\infty}\|z\|_{0,\infty}\|P_hy-y_h\|\cdot\|P_hz-z_h\|+Ch^2\|\phi\|_2\|z\|_{1,\infty}\|P_hz-z_h\|\\
&+C\|\phi\|_{1,\infty}\|P_hz-z_h\|^2+((\phi'(P_hy)-\phi'(y))z,P_hz-z_h)\\
\le&Ch^2\|P_hz-z_h\|+((\phi'(P_hy)-\phi'(y))z,P_hz-z_h).
\end{aligned} \tag{3.36}$$

For some $\hat{y}$, we have

$$\phi'(P_hy)-\phi'(y)=\phi''(\hat{y})(P_hy-y). \tag{3.37}$$

Thus, we can get

$$\begin{aligned}
((\phi'(P_hy)-\phi'(y))z,P_hz-z_h)=&(z\phi''(\hat{y})(P_hy-y),P_hz-z_h)\\
=&((z\phi''(\hat{y})-P_h(z\phi''(\hat{y})))(P_hy-y),P_hz-z_h)\\
\le&Ch^2\|z\phi''(\hat{y})\|_1\|y\|_{1,\infty}\|P_hz-z_h\|\\
\le&Ch^2\|\phi\|_3\|z\|_{1,\infty}\|y\|_{1,\infty}\|P_hz-z_h\|.
\end{aligned} \tag{3.38}$$

Substituting (3.38) into (3.36), we get

$$(\phi'(y_h)z_h - \phi'(y)z, P_h z - z_h) \le Ch^2 \|P_h z - z_h\|. \tag{3.39}$$

Using (3.31)-(3.35), (3.39), Theorem 3.3 and the standard stability estimate, we find that

$$\|\Pi_h \boldsymbol{q} - \boldsymbol{q}_h\| \le Ch^{\frac{3}{2}}. \tag{3.40}$$

Thus, we complete the proof. □

Now, combining (2.10c), (2.11b), Theorem 3.3, Theorem 3.4 with the inverse inequality, we give the following L^∞-error estimates for the state variables.

Theorem 3.5. *Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z, u) \in (\boldsymbol{V} \times W)^2 \times U_{ad}$ *and* $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h) \in (\boldsymbol{V}_h \times W_h)^2 \times U_h$ *be the solutions of* (2.6a)-(2.6e) *and* (2.14a)-(2.14e) *respectively. Assume that* $\boldsymbol{p}, \boldsymbol{q} \in (H^2(\Omega))^2 \cap (W^{1,\infty}(\Omega))^2$ *and* $y, z \in W^{1,\infty}(\Omega)$*, then we have*

$$\|y - y_h\|_{0,\infty} + \|z - z_h\|_{0,\infty} \le Ch, \tag{3.41a}$$

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{0,\infty} + \|\boldsymbol{q} - \boldsymbol{q}_h\|_{0,\infty} \le Ch^{\frac{1}{2}}. \tag{3.41b}$$

4. Application

In this paper, some applications of the results derived in Section 3 will be presented.

First, we will apply a higher order interpolation postprocessing method presented by Lin and Yan [16] to obtain global superconvergence for the mixed finite element approximations. We first construct a larger rectangular elements partition $\mathcal{T}_{2h}$, which is the coarse meshes of $\mathcal{T}_h$. That is, each element τ of $\mathcal{T}_{2h}$ of composed of four neighboring rectangular elements of $\mathcal{T}_h$. Based on this coarse meshes, we denote $\boldsymbol{V}_{2h} \times W_{2h}$ to express the order $k = 1$ Raviart-Thomas mixed finite element spaces:

$$\boldsymbol{V}_{2h} := \{\boldsymbol{v} \in \boldsymbol{V} : \forall \tau \in \mathcal{T}_{2h}, \ \ \boldsymbol{v}|_\tau \in Q_{2,1}(\tau) \times Q_{1,2}(\tau)\}, \tag{4.1a}$$

$$W_{2h} := \{w \in W : \forall \tau \in \mathcal{T}_{2h}, \ \ w|_\tau \in Q_{1,1}(\tau)\}, \tag{4.1b}$$

and the related Raviart-Thomas projection (see [13] and [22]):

$$\Pi_{2h} \times P_{2h} : \boldsymbol{V} \times W \to \boldsymbol{V}_{2h} \times W_{2h},$$

which satisfies the following properties [16]:

$$P_{2h} P_h = P_{2h} \quad \text{and} \quad \|P_{2h} w_h\| \le C\|w_h\|, \qquad \text{for all } \ w_h \in W_h. \tag{4.2a}$$

$$\Pi_{2h} \Pi_h = \Pi_{2h} \quad \text{and} \quad \|\Pi_{2h} \boldsymbol{v}_h\| \le C\|\boldsymbol{v}_h\|, \qquad \text{for all } \ \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.2b}$$

By using the interpolation operators Π_{2h} and P_{2h} and their properties, we immediately obtain the following global superconvergence theorem.

Theorem 4.1. *Let* $(y,\boldsymbol{p},z,\boldsymbol{q})$ *and* $(y_h,\boldsymbol{p}_h,z_h,\boldsymbol{q}_h)$ *be the solutions of* (2.6a)-(2.6e) *and* (2.14a)-(2.14e)*, respectively. Assume that all the conditions in Theorems* 3.2-3.5 *are valid. Then we have*

$$\|y-P_{2h}y_h\|+\|z-P_{2h}z_h\|\le Ch^2, \tag{4.3a}$$

$$\|\boldsymbol{p}-\Pi_{2h}\boldsymbol{p}_h\|+\|\boldsymbol{q}-\Pi_{2h}\boldsymbol{q}_h\|\le Ch^{\frac{3}{2}}. \tag{4.3b}$$

Proof. From the property (4.2a) of the operator P_{2h}, we get

$$y-P_{2h}y_h=y-P_{2h}y+P_{2h}(P_hy-y_h).$$

Thus, by the approximation property, the property (4.2a) of the operator P_{2h}, and Theorem 3.3 that

$$\|y-P_{2h}y_h\|\le\|y-P_{2h}y\|+C\|P_hy-y_h\|\le Ch^2.$$

Similarly, we can estimate other terms of Theorem 4.1. □

Secondly, let us construct the recovery operator G_h. Let G_hv be a continuous piecewise linear function (without zero boundary constraint). The value of G_hv on the nodes are defined by least-squares argument on an element patches surrounding the nodes, the details can be refer to the definition of R_h in [17].

Theorem 4.2. *Let u and u_h be the solutions of* (2.6a)-(2.6e) *and* (2.14a)-(2.14e)*, respectively. Assume that all the conditions in Theorem* 3.2 *are valid and* $u\in H^2(\Omega)$*. Then we have*

$$\|u-G_hu_h\|\le Ch^2. \tag{4.4}$$

Proof. Let P_hu be defined in (2.10a). Then

$$\|u-G_hu_h\|\le\|u-G_hu\|+\|G_hu-G_hP_hu\|+\|G_hP_hu-G_hu_h\|. \tag{4.5}$$

It can be proved by the standard technique (see, e.g., [11]) that

$$\|u-G_hu\|\le Ch^2\|u\|_2. \tag{4.6}$$

Using the definition of G_h, we find that

$$G_hu=G_hP_hu \tag{4.7}$$

and

$$\|G_hP_hu-G_hu_h\|\le C\|P_hu-u_h\|. \tag{4.8}$$

Combining (4.5)-(4.8) with Theorem 3.2, we complete the proof. □

5. H^{-1}-error estimates

In this section, we will obtain H^{-1}-error estimates for the optimal control problem. First, we can derive the H^{-1}-error estimates for the scalar functions.

Theorem 5.1. *Let* (y,z,u) *and* (y_h,z_h,u_h) *be the solutions of* (2.6a)-(2.6e) *and* (2.14a)-(2.14e) *respectively. Assume that all the conditions in Theorem* 3.5 *are valid. Then we have*

$$\|u-u_h\|_{-1} \le Ch^2, \tag{5.1a}$$

$$\|y-y_h\|_{-1}+\|z-z_h\|_{-1} \le Ch^2. \tag{5.1b}$$

Proof. Using (2.10b) and Theorem 3.2, it is easy to see that

$$\begin{aligned}\|u-u_h\|_{-1} \le&\|u-P_hu\|_{-1}+\|P_hu-u_h\|_{-1}\\ \le&Ch^2\|u\|_1+C\|P_hu-u_h\|\\ \le&Ch^2.\end{aligned} \tag{5.2}$$

Similarly, we can derive (5.1b). Thus, we complete the proof. □

Next, we consider the H^{-1}-error estimates for the divergence of the vector-valued functions.

Theorem 5.2. *Let* $(\boldsymbol{p},\boldsymbol{q})$ *and* $(\boldsymbol{p}_h,\boldsymbol{q}_h)$ *be the solutions of* (2.6a)-(2.6e) *and* (2.13b)-(2.14c), *respectively. Assume that all the conditions in Theorem* 5.1 *are valid. Then we have*

$$\|div(\boldsymbol{p}-\boldsymbol{p}_h)\|_{-1}+\|div(\boldsymbol{q}-\boldsymbol{q}_h)\|_{-1} \le Ch^2. \tag{5.3}$$

Proof. It follows from (3.23a)-(3.23d), (2.10c), (2.11b)-(2.11c) and the standard stability estimate that

$$\|y-y_h\|+\|\boldsymbol{p}-\boldsymbol{p}_h\|_{\rm div} \le Ch, \tag{5.4a}$$

$$\|z-z_h\|+\|\boldsymbol{q}-\boldsymbol{q}_h\|_{\rm div} \le Ch. \tag{5.4b}$$

Let $\varphi \in H^1(\Omega)$. Then, by (3.23b), (2.10a) and (2.10c),

$$\begin{aligned}({\rm div}(\boldsymbol{p}-\boldsymbol{p}_h),\varphi)=&({\rm div}(\boldsymbol{p}-\boldsymbol{p}_h),P_h\varphi)+({\rm div}(\boldsymbol{p}-\boldsymbol{p}_h),\varphi-P_h\varphi)\\ =&(a(u-u_h),P_h\varphi)-(\phi(y)-\phi(y_h),P_h\varphi)+({\rm div}(\boldsymbol{p}-\boldsymbol{p}_h),\varphi-P_h\varphi)\\ =&((a-P_ha)(u-P_hu),P_h\varphi)+(a(P_hu-u_h),P_h\varphi)\\ &-((\phi'(\tilde{y})-P_h(\phi'(\tilde{y})))(y-P_hy),P_h\varphi)\\ &-(\phi(P_hy)-\phi(y_h),P_h\varphi)+({\rm div}(\boldsymbol{p}-\boldsymbol{p}_h),\varphi-P_h\varphi)\\ \le&Ch^2\|a\|_{1,\infty}\|u\|_1\|P_h\varphi\|+C\|a\|_{0,\infty}\|P_hu-u_h\|\cdot\|P_h\varphi\|\\ &+Ch^2\|\phi\|_{2,\infty}\|y\|_1\|P_h\varphi\|+C\|\phi\|_{1,\infty}\|P_hy-y_h\|\cdot\|P_h\varphi\|\\ &+Ch\|{\rm div}(\boldsymbol{p}-\boldsymbol{p}_h)\|\cdot\|\varphi\|_1.\end{aligned} \tag{5.5}$$

Using (5.4a)-(5.5) and Theorems 3.2 and 3.3, we find that

$$\|\mathrm{div}(\boldsymbol{p}-\boldsymbol{p}_h)\|_{-1} \le Ch^2. \tag{5.6}$$

Similarly, by (3.23d) and (2.10c),

$$\begin{aligned}(\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h),\varphi) =&(\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h),P_h\varphi)+(\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h),\varphi-P_h\varphi)\\ =&(P_hy-y_h,P_h\varphi)+(\phi'(y_h)z_h-\phi'(y)z,P_h\varphi)+(\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h),\varphi-P_h\varphi)\\ =&(P_hy-y_h,P_h\varphi)+(\phi'(y_h)(z_h-z),P_h\varphi-\varphi)\\ &+((\phi'(y_h)-\phi'(y))z,P_h\varphi-\varphi)+(\phi'(y_h)(z_h-z),\varphi)\\ &+(\phi'(y_h)-\phi'(P_hy),z\varphi)+(\phi'(P_hy)-\phi'(y),z\varphi-P_h(z\varphi))\\ &+(\phi'(P_hy)-\phi'(y),P_h(z\varphi))+(\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h),\varphi-P_h\varphi)\\ \le& C\|P_hy-y_h\|\cdot\|P_h\varphi\|+Ch\|\phi\|_{1,\infty}\|z-z_h\|\cdot\|\varphi\|_1\\ &+Ch\|\phi\|_{2,\infty}\|z\|_{0,\infty}\|y-y_h\|\cdot\|\varphi\|_1+C\|z-z_h\|_{-1}\|\phi\|_{2,\infty}\|\varphi\|_1\\ &+C\|\phi\|_{2,\infty}\|y_h-P_hy\|\cdot\|z\|_{0,\infty}\|\varphi\|+Ch^2\|\phi\|_{2,\infty}\|y\|_1\|z\|_{1,\infty}\|\varphi\|_1\\ &+Ch^2\|\phi\|_3\|y\|_{1,\infty}\|z\|_{0,\infty}\|\varphi\|+Ch\|\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h)\|\cdot\|\varphi\|_1.\end{aligned} \tag{5.7}$$

It follows from (5.1b), (5.4a)-(5.4b), (5.7) and Theorem 3.3 that

$$\|\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h)\|_{-1} \le Ch^2. \tag{5.8}$$

Combining (5.6) and (5.8), we complete the proof. □

Finally, we consider the H^{-1}-error estimates for the vector-valued functions.

Theorem 5.3. *Assume that all the conditions in Theorem* 3.5 *are valid. Let* $(\boldsymbol{p},\boldsymbol{q})$ *and* $(\boldsymbol{p}_h,\boldsymbol{q}_h)$ *be the solutions of* (2.6a)-(2.6e) *and* (2.14a)-(2.14e)*, respectively. Then we have*

$$\|\boldsymbol{p}-\boldsymbol{p}_h\|_{-1}+\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1} \le Ch^2. \tag{5.9}$$

Proof. For $\boldsymbol{\psi}\in(H^1(\Omega))^2$, let $\varphi\in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of the Dirichlet problem

$$-\mathrm{div}(A\nabla\varphi)=\mathrm{div}\boldsymbol{\psi}, \qquad x\in\Omega, \tag{5.10a}$$
$$\varphi=0, \qquad x\in\partial\Omega. \tag{5.10b}$$

Then,

$$\|\varphi\|_2 \le C\|\mathrm{div}\boldsymbol{\psi}\| \le C\|\boldsymbol{\psi}\|_1. \tag{5.11}$$

Furthermore, $\boldsymbol{\psi}=-A\nabla\varphi+\boldsymbol{\theta}$, where $\mathrm{div}\boldsymbol{\theta}=0$ and

$$\|\boldsymbol{\theta}\|_1 \le C\|\boldsymbol{\psi}\|_1. \tag{5.12}$$

Now,

$$\begin{aligned}(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h),\boldsymbol{\psi}) =&-(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h),A\nabla\varphi)+(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h),\boldsymbol{\theta})\\ =&(\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h),\varphi)+(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h),\boldsymbol{\theta}).\end{aligned} \tag{5.13}$$

Using (5.8) and (5.11), we have

$$(\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h),\varphi)\leq C\|\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h)\|_{-1}\|\varphi\|_1\leq Ch^2\|\boldsymbol{\psi}\|_1. \tag{5.14}$$

Then, since $\mathrm{div}\boldsymbol{\theta}=0$ and by (3.23c), (2.11b) and (5.4a)-(5.4b)

$$\begin{aligned}(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h),\boldsymbol{\theta})=&(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h),\Pi_h\boldsymbol{\theta})+(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h),\boldsymbol{\theta}-\Pi_h\boldsymbol{\theta})\\ =&(P_hz-z_h,\mathrm{div}\Pi_h\boldsymbol{\theta})-(\boldsymbol{p}-\boldsymbol{p}_h,\Pi_h\boldsymbol{\theta})+(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h),\boldsymbol{\theta}-\Pi_h\boldsymbol{\theta})\\ =&(P_hz-z_h,\mathrm{div}\boldsymbol{\theta})-(\boldsymbol{p}-\boldsymbol{p}_h,\Pi_h\boldsymbol{\theta}-\boldsymbol{\theta})\\ &-(\boldsymbol{p}-\boldsymbol{p}_h,\boldsymbol{\theta})+(A^{-1}(\boldsymbol{q}-\boldsymbol{q}_h),\boldsymbol{\theta}-\Pi_h\boldsymbol{\theta})\\ \leq&Ch(\|\boldsymbol{p}-\boldsymbol{p}_h\|+\|\boldsymbol{q}-\boldsymbol{q}_h\|)\|\boldsymbol{\theta}\|_1+C\|\boldsymbol{p}-\boldsymbol{p}_h\|_{-1}\|\boldsymbol{\theta}\|_1\\ \leq&C(h^2+\|\boldsymbol{p}-\boldsymbol{p}_h\|_{-1})\|\boldsymbol{\theta}\|_1.\end{aligned} \tag{5.15}$$

Using (5.15), we conclude that

$$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1}\leq C(h^2+\|\boldsymbol{p}-\boldsymbol{p}_h\|_{-1}). \tag{5.16}$$

Similarly, we can prove

$$\|\boldsymbol{p}-\boldsymbol{p}_h\|_{-1}\leq Ch^2. \tag{5.17}$$

Thus, we complete the proof. □

6. Numerical experiments

In this section, we present below an example to illustrate the theoretical results. The optimization problems were solved numerically by projected gradient methods, with codes developed based on AFEPack [15]. The discretization was already described in previous sections: the control function u was discretized by piecewise constant functions, whereas the state $(y,\boldsymbol{p})$ and the co-state $(z,\boldsymbol{q})$ were approximated by the lowest order Raviart-Thomas mixed finite element functions. In the following example, we choose the domain $\Omega=[0,1]\times[0,1]$, $\nu=1$ and $A=I$ and $B=I$.

Example 6.1. We consider the following two-dimensional elliptic optimal control problem

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{1}{2}\|u\|^2\right\} \tag{6.1}$$

subject to the state equation

$$\mathrm{div}\boldsymbol{p}+y^3=f+u,\quad \boldsymbol{p}=-\mathbf{grad}y, \tag{6.2}$$

where

$$y=\sin(\pi x_1)\sin(\pi x_2),\qquad z=\sin(2\pi x_1)\sin(2\pi x_2), \tag{6.3a}$$

$$u=\max(0,\bar{z})-z,\qquad f=2\pi^2y+y^3-u, \tag{6.3b}$$

$$y_d=y-8\pi^2z-3y^2z,\qquad \boldsymbol{p}_d=-\begin{pmatrix}\pi\cos(\pi x_1)\sin(\pi x_2)\\ \pi\sin(\pi x_1)\cos(\pi x_2)\end{pmatrix}. \tag{6.3c}$$

Table 1: The errors of $\|u-u_h\|$, $\|u-u_h\|_{0,\infty}$, $\|P_hu-u_h\|$ and $\|u-G_hu_h\|$.

Resolution	$\|u-u_h\|$	$\|u-u_h\|_{0,\infty}$	$\|P_hu-u_h\|$	$\|u-G_hu_h\|$
16×16	6.52138e-02	1.80716e-01	3.18016e-03	4.97599e-02
32×32	3.27741e-02	9.07962e-02	8.01988e-04	1.24764e-02
64×64	1.63034e-02	4.57632e-02	2.00159e-04	3.11442e-03
128×128	8.24428e-03	2.29653e-02	5.06837e-05	8.07687e-04

Table 2: The errors of $\|y-y_h\|_{0,\infty}$, $\|z-z_h\|_{0,\infty}$, $\|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\infty}$ and $\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\infty}$.

Resolution	$\|y-y_h\|_{0,\infty}$	$\|z-z_h\|_{0,\infty}$	$\|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\infty}$	$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\infty}$
16×16	9.07822e-02	1.80955e-01	2.69974e-01	1.05850e+00
32×32	4.49827e-02	9.08409e-02	1.92094e-01	7.57435e-01
64×64	2.25492e-02	4.57749e-02	1.35471e-01	5.31751e-01
128×128	1.14949e-02	2.30624e-02	9.68302e-02	3.79788e-01

In the numerical implementation, we choose the solution u which satisfies $\int_\Omega u dx = 0$. In Table 1, the errors $\|u-u_h\|$, $\|u-u_h\|_{0,\infty}$, $\|P_hu-u_h\|$ and $\|u-G_hu_h\|$ obtained on a sequence of uniformly refined meshes are shown. Table 2 displays the errors $\|y-y_h\|_{0,\infty}$, $\|z-z_h\|_{0,\infty}$, $\|\boldsymbol{p}-\boldsymbol{p}_h\|_{0,\infty}$ and $\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,\infty}$. In Fig. 1, the profile of the numerical solution of u on the 64×64 mesh grid is plotted. Moreover, in Figs. 2 and 3, we show the convergence orders by slopes. Theoretical results are clearly recognized from the data.

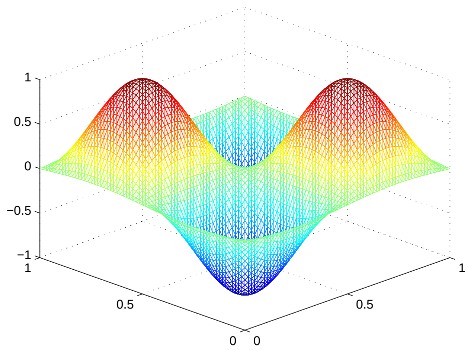

Figure 1: The profile of the numerical solution of u on 64×64 mesh.

7. Conclusions

In this paper, we discussed the lowest order Raviart-Thomas mixed finite element methods for the semilinear elliptic optimal control problem (1.1)-(1.4). We have derived some superconvergence results of the mixed finite element methods for the control problem.

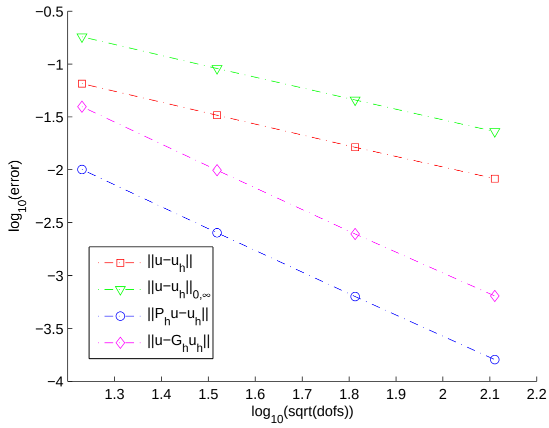

Figure 2: Convergence orders of $u-u_h$, P_hu-u_h and $u-G_hu_h$ in different norms.

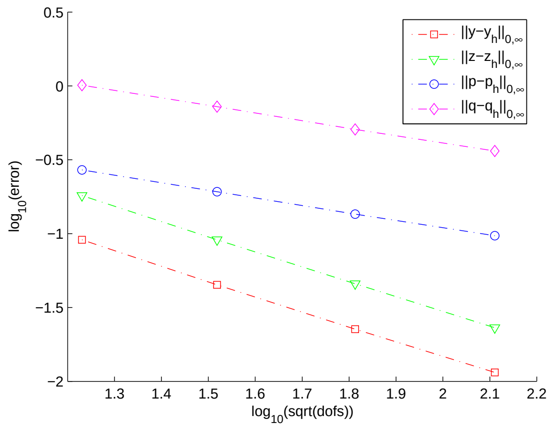

Figure 3: Convergence orders of $y-y_h$, $z-z_h$, $\boldsymbol{p}-\boldsymbol{p}_h$ and $\boldsymbol{q}-\boldsymbol{q}_h$ in L^∞-norm.

Moreover, we derive L^∞- and H^{-1}-error estimates both for the control variable and the state variables. We also give some applications of the superconvergence results. In our future work, we will investigate the superconvergence of mixed finite element methods for optimal control problems governed by nonlinear parabolic equations.

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