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The Global Behavior of Finite Difference-Spatial Spectral Collocation Methods for a Partial Integro-differential Equation with a Weakly Singular Kernel

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Abstract. The z-transform is introduced to analyze a full discretization method for a partial integro-differential equation (PIDE) with a weakly singular kernel. In this method, spectral collocation is used for the spatial discretization, and, for the time stepping, the finite difference method combined with the convolution quadrature rule is considered. The global stability and convergence properties of complete discretization are derived and numerical experiments are reported.

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1. Introduction

We consider initial-boundary value problems of the form

$$u_t(\mathbf{x},t)=\int_0^t \beta(t-s)\Delta u(\mathbf{x},s)ds+f(\mathbf{x},t), \quad (\mathbf{x},t)\in\Omega\times(0,T], \tag{1.1}$$

where $\beta(t)=t^{-1/2}/\Gamma(1/2)$, which has a weak singularity at $t=0$ and $\Omega\equiv(-1,1)^2$, subject to the boundary condition

$$u(\mathbf{x},t)=0 \quad \text{on} \quad \partial\Omega, \quad t>0, \tag{1.2}$$

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and the initial condition

$$u(\mathbf{x},0) = u_0 \quad \text{in} \;\; \Omega. \tag{1.3}$$

Here $u_t(\mathbf{x},t) = \partial u(\mathbf{x},t)/\partial t$, Δ is the two-dimensional Laplacian operator, $\partial\Omega$ is the boundary of the unit square Ω and β is a real-valued and positive-definite kernel, i.e., $\beta \in L^{1,loc}(0,+\infty)$ and satisfies

$$\int_0^T \int_0^t \beta(t-s)\varphi(s)ds\varphi(t)dt \geq 0, \quad \forall T > 0, \quad \varphi \in C([0,T]). \tag{1.4}$$

Equations of the form (1.1) arise in problems concerned with heat conduction in materials with memory, population dynamics, viscoelasticity and theory of nuclear reactors (see Mustapha [19–21] and reference therein). The numerical solution of problems of the type (1.1) was studied extensively in the literature. See, for instance, Mclean and Thomée [17], Mclean et al. [18] and Pani et al. [23, 24] for the positive-type kernels, Chen [3], Sanz-Serna [25], López-Marcos [14], Lubich et al. [16], Mclean and Mustapha [19], Lin and Xu [11, 12], Tang [28] for weakly singular kernels, Da [31, 32] for completely monotonic kernels and Da [33] for log-convex kernels.

As we know, spectral methods have become increasingly popular and been widely used in spatial discretization of PDEs owing to its high order of accuracy (cf. [1, 2, 4–7, 26, 29]). Some work has been done along this line and we particularly point out that Kim and Choi [9] proposed and analyzed a spectral collocation method for the PIDEs with a weakly singular kernel, the spatial discretization is based on the pseudo-spectral method and the temporal discretization by finite difference methods was considered. Lin and Xu [11] proposed a finite difference scheme in time and Legendre spectral method in space for fractional diffusion-wave equation. Meanwhile, Li and Xu [10] proposed a spectral method in both temporal and spatial discretizations for this equation. In those papers [9–11], the error bounds of discretization in time are valid only on finite time intervals and pointwise. From a practical point of view, it is more interesting and challenging to develop and analyze high-order methods for PIDEs in a long time period.

This paper, motivated by [30], is devoted to approximate the problems (1.1)-(1.3) using spectral collocation in each spacial direction for the spatial discretization. Then the resulting systems of integro-differential equations in the time variable are discretized using backward Euler method, combined with the convolution quadrature rule, by employing a different approach involving the z-transform with respect to time sequence, we derive the global stability properties and associated error estimates for large T. It should be noted that the z-transform with respect to time sequence was employed by Sanz-Serna [25]. Our result is related to but different from [25].

The outline of this paper is as follows. In the next section, we first introduce the Sobolev spaces on a square and then define several projection operators from Sobolev spaces onto the space of polynomials with degree less than an integer N. In Section 3 we introduce the z-transform of a sequence $\{f_n\}_0^\infty$ and collect some of its properties. In Section 4, we establish stability and convergence of the full discrete scheme for (1.1). Numerical results in Section 5 validate the theoretical prediction in Section 4.

2. Preliminaries

We introduce some definitions and recall some basic results which will be used throughout the paper. We first introduce the Sobolev spaces on the square. For any integer $m \geq 0$ and any $\mathbf{x} = (x_1, x_2) \in \Omega$, the Sobolev space is defined by

$$H^m(\Omega) = \left\{ v \in L^2(\Omega) : \ \frac{\partial^{p+q} v}{\partial x_1^p \partial x_2^q} \in L^2(\Omega), \ (p,q) \in N^2, \ p+q \leq m \right\},$$

which is equipped with the norm

$$\|v\|_m = \left(\sum_{0 \leq p+q \leq m} \left\| \frac{\partial^{p+q} v}{\partial x_1^p \partial x_2^q} \right\|^2 \right)^{1/2}$$

and set

$$H_0^1 = H^1 \cap \{v | v = 0 \text{ on } \partial\Omega\}.$$

Further, for $v : [0,T] \to H^m(\Omega)$, define the norm $\|\cdot\|_{L^2(H^m)}$ by

$$\|v\|_{L^2(H^m)} = \left(\int_0^T \|v\|_m^2 dt \right)^{1/2}.$$

In particular, since $H^0(\Omega) = L^2(\Omega)$, we have

$$\|v\|_{L^2(H^0)} = \|v\|_{L^2(L^2)}.$$

For an integer $N > 0$, we set $\mathbb{P}_N = \tilde{\mathbb{P}}_N \times \tilde{\mathbb{P}}_N$, where $\tilde{\mathbb{P}}_N$ is the space of the polynomials of degree N in single variables. Further, we set $\mathbb{P}_N^0(\Omega) = \{p \in \mathbb{P}_N | p(\mathbf{x}) = 0 \text{ on } \partial\Omega\}$.

For our work, we require some spectral projection operators. First define the orthogonal projection operator $P_N : L^2 \mapsto \mathbb{P}_N(\Omega)$ by

$$(v - P_N v, \phi) = 0, \quad \forall \phi \in \mathbb{P}_N(\Omega). \tag{2.1}$$

The projection error is estimated as follows (see [1,2,4,5])

$$\|v - P_N v\| \leq C N^{-\sigma} \|v\|_\sigma, \quad \forall v \in H^\sigma(\Omega), \quad \sigma > 0. \tag{2.2}$$

We define the Ritz projection operator $\Pi_N : H_0^1 \to \mathbb{P}_N^0(\Omega)$ by

$$\big(-\Delta(v - \Pi_N v), \Phi \big) = 0, \quad \forall \Phi \in \mathbb{P}_N^0(\Omega). \tag{2.3}$$

Lemma 2.1. *The error estimate of the Ritz projection is: for all $v \in H^\sigma(\Omega) \cap H_0^1(\Omega)$ with $0 \leq \mu \leq \sigma$, $\sigma \geq 1$*

$$\|v - \Pi_N v\|_\mu \leq c N^{e(\mu)-\sigma} \|v\|_\sigma, \tag{2.4}$$

where $e(\mu) = \mu$ if $\mu \leq 1$ and $e(\mu) = 2\mu - 1$ if $\mu > 1$.

Proof. see [1,2,4,5]. □

We note that positive-definiteness of the kernel leads to global stability of the continuous problem (1.1). The notion of a positive kernel plays a central role, the following lemma, proved in [22], will often be used in this paper.

Lemma 2.2. *$\beta(t) \in L^{1,loc}(0,\infty)$ is of positive type if and only if*

$$Re\widehat{\beta}(s) \geq 0, \quad for \;\; s \in \{s \in \mathscr{C}, \; Res > 0\}, \tag{2.5}$$

where $\widehat{\beta}$ denotes the Laplace transform of β.

3. z-transform

In order to analyze the convergence of the discretization in time, we introduce the z-transform of a real sequence (or L^2-valued sequence) $\{f_n\}_{n=0}^{\infty}$, namely

$$\mathscr{Z}\left(\{f_n\}_{n=0}^{\infty}\right)(z) = \sum_{n=0}^{\infty} f_n z^{-n}, \tag{3.1}$$

and review some of its properties. If $\{f_n\}_{n=0}^{\infty}, \{g_n\}_{n=0}^{\infty}, \cdots$, are sequences, we shall denote by capital letters $F, G, \cdots$, their respective transforms.

Proposition 3.1. (see [13]) *(Convolution Sum Theorem)*

$$\mathscr{Z}\Big(\sum_{k=0}^{n} f_k g_{n-k}\Big)(z) = F(z)G(z). \tag{3.2}$$

Proposition 3.2. (see [13]) *Let*

$$g_n = \begin{cases} 0, & \text{for} \;\; n < 1, \\ f_{n-1}, & \text{for} \;\; n \geq 1, \end{cases} \qquad G(z) = z^{-1}F(z). \tag{3.3}$$

For any $\varepsilon > 0$, suppose $f_\varepsilon(t)$ is an impulse function, defined by

$$f_\varepsilon(t) = \sum_{n=0}^{\infty} f_n \delta(t - n\varepsilon),$$

where $\delta(t - n\varepsilon)$ is an unit pulse function at $t = n\varepsilon$.

The following proposition is necessary for our stability analysis, which is an extension of Proposition 3.4 in [30].

Proposition 3.3. *Let $\{f_n\}$, $\{h_n\}$ are two sequences of $L^2(\Omega)$. assume we have two sampled functions $F_\varepsilon(s)$ and $H_\varepsilon(s)$ as the Laplace transform of $f_\varepsilon(t)$ and $h_\varepsilon(t)$ respectively. We have*

$$\begin{aligned}\sum_{n=0}^{\infty}(f_n,h_n)_N e^{-n\varepsilon s} =&\mathscr{L}\{(f(t),h(t))_N\delta_\varepsilon(t)\}\\ =&\frac{\varepsilon}{2\pi i}\int_{c-i\frac{\pi}{\varepsilon}}^{c+i\frac{\pi}{\varepsilon}}\Big(F_\varepsilon(q),\overline{H_\varepsilon(s-q)}\Big)_N dq,\end{aligned} \tag{3.4}$$

where $\overline{z}$ denotes complex conjugate of z, $\mathscr{L}(u)$ denotes the Laplace transform of a function u defined on $[0,\infty)$, the discrete inner product will be defined the next section.

Proof. Since the proof follows the same procedure as [30], we omit the details (for details on these conditions and the results see Proposition 3.4 in [30]). □

Both $F_\varepsilon(s)$ and $H_\varepsilon(s)$ are function of $e^{\varepsilon s}$ only, we can replace $e^{\varepsilon s}$ by z and $e^{\varepsilon q}$ by p, so that $F_\varepsilon(q)$ becomes a function of p. that is, $F(p)$ and $H_\varepsilon(s-q)$ becomes a function of z/p, that is, $H(z/p)$. As required by this mapping

$$dq=\Big(\frac{1}{\varepsilon\ e^{\varepsilon q}}\Big)dp=\Big(\frac{p^{-1}}{\varepsilon}\Big)dp.$$

With this substitution equation (3.4) becomes

$$\sum_{n=0}^{\infty}(f_n,h_n)_N z^{-n}=\mathscr{L}\{(f(t),h(t))_N\delta_\varepsilon(t)\}=\frac{1}{2\pi i}\int_\Gamma p^{-1}\Big(F(p),\overline{H\Big(\frac{z}{p}\Big)}\Big)_N dp, \tag{3.5}$$

where Γ is a contour contain unit circle while not the poles of $H(z/p)$ in the p-plane.

4. Full discretization schemes

In this section, we discuss the full discretization based on the spectral collocation methods in space and the backward difference quotient combined with the convolution quadrature rule in time. Let $\xi_i' s$ be the nodes of the Gauss-Lobatto integration formula of degree N and ω_i the corresponding Legendre weight at ξ_i. Then, we see that

$$\int_{-1}^{1}p(\xi)d\xi=\sum_{j=0}^{N}p(\xi_j)\omega_j,\quad \forall p\in\tilde{\mathbb{P}}_{2N-1}. \tag{4.1}$$

We define the interpolation operator $I_N: C^0(\overline{\Omega})\mapsto\mathbb{P}_N(\Omega)$by

$$I_N v(x_{ij})=v(x_{ij}),\quad 0\le i,j\le N,$$

where $x_{ij}=(\xi_i,\xi_j)$ for $0\le i,j\le N$. For any real μ, σ such that $0\le\mu\le 1<\sigma$, the interpolation error is estimated as follows (see [8]):

$$\|v-I_N v\|_\mu\le cN^{\mu-\sigma}\|v\|_\sigma,\quad \forall v\in H^\sigma(\Omega). \tag{4.2}$$

Thus our spatial semi-discrete spectral approximation of (1.1) is the following collocation problem: we look for a mapping $U \in C^1(\mathbb{P}_N^0(\Omega))$ such that, for any $t \in (0,T)$,

$$U_t(x_{ij}) - \int_0^t \beta(t-s)\Delta U(x_{ij},s)ds = f(x_{ij}), \qquad 1 \le i,j \le N-1, \tag{4.3a}$$

$$U(x_{0j}) = U(x_{i0}) = U(x_{Nj}) = U(x_{iN}) = 0, \tag{4.3b}$$

$$U(x_{ij},0) = u_0(x_{ij}), \qquad 0 \le i,j \le N. \tag{4.3c}$$

Remark 4.1. The global behavior of spatial semi-discretization spectral collocation methods for such equation is discussed in [27].

We now define a discrete inner product:

$$(\phi,\psi)_N = \sum_{i,j=0}^{N} \phi(x_{ij})\psi(x_{ij})\omega_i\omega_j, \quad \forall \phi,\psi \in C^0(\overline{\Omega}).$$

By (4.1) it follows that

$$(\phi,\psi)_N = (\phi,\psi), \quad \forall \phi,\psi : \phi \cdot \psi \in \mathbb{P}_{2N-1}(\Omega).$$

The discrete norm

$$\|\phi\|_N = (\phi,\phi)_N^{1/2}, \quad \forall \phi \in C^0(\overline{\Omega}),$$

is equivalent to the L^2-norm, namely (see [1,2,4,5])

$$\|\phi\| \le \|\phi\|_N \le 2\|\phi\|, \quad \forall \phi \in \mathbb{P}_N(\Omega). \tag{4.4}$$

Finally, for any $v \in C^0(\overline{\Omega})$, we define $E(v)$ by

$$(E(v),\phi) = (v,\phi)_N - (v,\phi), \quad \forall \phi \in C^0(\overline{\Omega}).$$

Lemma 4.1. *For any $v \in C^0(\overline{\Omega})$, it can be shown that*

$$|(E(v),\phi)| \le c\{\|v - P_{N-1}v\| + \|v - I_N v\|\}\|\phi\|, \quad \forall \phi \in \mathbb{P}_N,$$

from (4.2), for $v \in H^\sigma(\Omega)$, we have

$$|(E(v),\phi)| \le cN^{-\sigma}\|v\|_\sigma\|\phi\|, \quad \forall \phi \in \mathbb{P}_N. \tag{4.5}$$

Proof. see [2]. □

The semi-discrete approximation (4.3) gives a system of ordinary differential equations in the time variable. Let $k > 0$ be a time step and let $U^n \in \mathbb{P}_N^0(\Omega)$ be the approximation of the exact solution of (1.1) at time $t_n = nk$. The time discretization considered will be based on the backward difference quotient

$$\overline{\partial}_t U^n = (U^n - U^{n-1})/k.$$

Writing $S^{\beta}_{t_n}(\phi)=\int_0^{t_n}\beta(t_n-s)\phi(s)ds$, we consider the convolution quadrature rule, suggested by Lubich [15], for approximating $S^{\beta}_{t_n}(\phi)$, i.e.,

$$q^{\beta}_n(\phi)=k^{1/2}\sum_{j=1}^{n}w_{n-j}\phi^j, \tag{4.6}$$

where the quadrature weights w_j are the coefficients of the power series,

$$\widehat{\beta}(1-z)=(1-z)^{-\frac{1}{2}}=\sum_{j=0}^{\infty}w_jz^j. \tag{4.7}$$

We have the following quadrature error.

Lemma 4.2. *Let $S^{\beta}_{t_n}(\phi)$ and $q^{\beta}_n(\phi)$ are defined accordingly, then*

$$\begin{aligned}\|S^{\beta}_{t_n}(\phi)-q^{\beta}_n(\phi)\|\le & ck^{\frac{1}{2}}/\sqrt{n}\|\phi(0)\|+ck^{\frac{1}{2}}\int_{t_{n-1}}^{t_n}\|\phi_t(s)\|ds\\ &+ck\int_0^{t_{n-1}}(t_n-s)^{-\frac{1}{2}}\|\phi_t(s)\|ds.\end{aligned} \tag{4.8}$$

Proof. see formula (6.9) of [30] with $\alpha=1/2$. □

We next describe the stability and convergence properties of complete discretization. For this purpose, our fully discretized scheme is defined by

$$\overline{\partial}_tU^n_{ij}+k^{1/2}\sum_{m=1}^{n}w_{n-m}AU^m_{ij}=f^n_{ij},\qquad \text{for } 1\le i,j\le N-1,\quad n\ge1, \tag{4.9a}$$

$$U^m_{0j}=U^m_{i0}=U^m_{Nj}=U^m_{iN}=0,\qquad \text{for } 0\le i,j\le N,\quad m\ge0, \tag{4.9b}$$

$$U^0_{ij}=u_0(x_{ij}),\qquad \text{for } 0\le i,j\le N, \tag{4.9c}$$

where $U^n_{ij}=U(x_{ij},t_n)$ and $A=-\Delta$. Furthermore, we can rewrite the above as in the variational form

$$(\overline{\partial}_tU^n,\chi)_N+k^{1/2}\sum_{j=1}^{n}w_{n-j}(AU^j,\chi)_N=(f^n,\chi)_N,\qquad \forall\chi\in\mathbb{P}^0_N(\Omega),\quad n\ge1, \tag{4.10a}$$

$$U^0=I_Nu_0,\qquad \text{for } 0\le i,j\le N. \tag{4.10b}$$

Theorem 4.1. *For the backward Euler method* (4.9), *with U^n and q^{β}_n are defined by* (4.10) *and* (4.6), *respectively, then for any $c>0$:*

$$\sum_{n=1}^{\infty}e^{-nc}\|U^n\|^2_N\le\frac{4}{(1-e^{-\frac{c}{2}})^2}\Big\{e^{-c}\|U^0\|^2_N+k^2\Big(\sum_{n=1}^{\infty}\|f^n\|_Ne^{-n\frac{c}{2}}\Big)^2\Big\}. \tag{4.11}$$

Proof. By using Propositions 3.1, 3.2 and formulation (4.7), the application of the z-transform to (4.10) leads to

$$(1-z^{-1})(\widetilde{U}(z),\chi)_N + k^{3/2}\widehat{\beta}(1-z^{-1})(A\widetilde{U}(z),\chi)_N = k(\widetilde{F}(z),\chi)_N + z^{-1}(U^0,\chi)_N, \quad (4.12)$$

where

$$\widetilde{U}(z) = U^1 z^{-1} + U^2 z^{-2} + \cdots + U^n z^{-n} + \cdots,$$
$$\widetilde{F}(z) = f^1 z^{-1} + f^2 z^{-2} + \cdots + f^n z^{-n} + \cdots.$$

Taking the inner product of (4.12) with $\widetilde{U}(z)$, we find that

$$\begin{aligned}&(1-z^{-1})\|\widetilde{U}(z)\|_N^2 + k^{3/2}\widehat{\beta}(1-z^{-1})(A\widetilde{U}(z),\widetilde{U}(z))_N \\ =&k(\widetilde{F}(z),\widetilde{U}(z))_N + z^{-1}(U^0,\widetilde{U}(z))_N. \end{aligned} \quad (4.13)$$

Let $z = e^{s_0+i\eta}$ with any $s_0 > 0$. and note that

$$Re(1-z^{-1}) = 1 - e^{-s_0}\cos\eta \geq 1 - e^{-s_0} > 0.$$

Applying Lemma 2.2, we then have

$$Re\widehat{\beta}(1-z^{-1}) \geq 0. \quad (4.14)$$

We take the real part of (4.13) and obtain, from the positivity of the Laplacian operator A, an estimate of the form

$$\|\widetilde{U}(z)\|_N \leq \frac{1}{1-e^{-s_0}}\left\{e^{-s_0}\|U^0\|_N + k\sum_{n=1}^{\infty}\|f^n\|_N e^{-ns_0}\right\}. \quad (4.15)$$

Now applying (3.5), we have that

$$\sum_{n=1}^{\infty} e^{-nc}\|U^n\|_N^2 = \frac{1}{2\pi i}\oint_{C_{e^{\frac{c}{2}}}} p^{-1}\left(\widetilde{U}(p),\overline{\widetilde{U}\left(\frac{e^c}{p}\right)}\right)_N dp, \quad (4.16)$$

where, $C_{e^{c/2}}$ is taken along the radius $e^{c/2}$ circle, Therefore

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-nc}\|U^n\|_N^2 =&\frac{1}{2\pi}\int_0^{2\pi}\left(\widetilde{U}(e^{\frac{c}{2}+i\eta}),\overline{\widetilde{U}(e^{\frac{c}{2}-i\eta})}\right)_N d\eta \\ =&\frac{1}{2\pi}\int_0^{2\pi}\|\widetilde{U}(e^{\frac{c}{2}+i\eta})\|_N^2 d\eta. \end{aligned} \quad (4.17)$$

From the inequality (4.15) we can get

$$\sum_{n=1}^{\infty} e^{-nc}\|U^n\|_N^2 \leq \frac{4}{(1-e^{-\frac{c}{2}})^2}\left\{e^{-c}\|U^0\|_N^2 + k^2\Big(\sum_{n=1}^{\infty}\|f^n\|_N e^{-n\frac{c}{2}}\Big)^2\right\},$$

here, we have used a simple inequality, that is

$$(|a|+|b|)^2 \le [2\max(|a|,|b|)]^2 \le 4(|a|^2+|b|^2), \quad \text{for any } a,b\in\mathscr{R},$$

then result follows. □

We are in position to prove the error estimate for the fully discretized scheme (4.9).

Theorem 4.2. *Let u and $\{U^n\}$ be the solutions of* (1.1) *and* (4.10) *respectively.* $f\in H^\sigma(\Omega)$, *with q_n^β given by* (4.6). *Provided β satisfies the conditions of Lemma 2.2, then for any $\delta>0$, we have that*

$$\begin{aligned}
&\sum_{n=1}^{\infty} e^{-n\delta}\|U^n-u(t_n)\|^2\\
\le&\frac{4c}{(1-e^{-\frac{\delta}{2}})^2}\Big\{N^{-2\sigma}\Big[e^{-\delta}\|u_0\|_\sigma^2+\sum_{n=1}^{\infty}e^{-n\delta}\|u\|_\sigma^2+\Big(\sum_{n=1}^{\infty}e^{-n\frac{\delta}{2}}\|u_t\|_{\sigma,L_1(I_n)}\Big)^2\Big]\\
&+k^2\Big[\Big(\sum_{n=1}^{\infty}e^{-n\frac{\delta}{2}}\|E(f^n)\|\Big)^2+\Big(\sum_{n=1}^{\infty}e^{-n\frac{\delta}{2}}\|u_{tt}\|_{L_1(I_n)}\Big)^2+k\Big(\sum_{n=1}^{\infty}e^{-n\frac{\delta}{2}}n^{-\frac12}\|Au_0\|\Big)^2\\
&+k\Big(\sum_{n=1}^{\infty}e^{-n\frac{\delta}{2}}\|Au_t\|_{L_1(I_n)}\Big)^2+k^2\Big(\sum_{n=1}^{\infty}e^{-n\frac{\delta}{2}}\|\beta(t_n-\cdot)Au_t(\cdot)\|_{L_1(0,t_n)}\Big)^2\Big]\Big\},
\end{aligned}\tag{4.18}$$

where $c>0$ is a generic constant which may not be the same at different occurrences.

Proof. For any $t\ge0$, we set $\widetilde{u}=\Pi_N u$, then $\widetilde{u}$ satisfies the variational equation: for all $\chi\in\mathbb{P}_N^0(\Omega)$,

$$(\widetilde{u}_t,\chi)_N=\int_0^t\beta(t-s)(\Delta\widetilde{u}(s),\chi)_N ds-(u_t,\chi)+(\widetilde{u}_t,\chi)_N+(f,\chi),\tag{4.19}$$

and set $e^n=U^n-\widetilde{u}^n+\widetilde{u}^n-u^n=\theta^n+\rho^n$, Subtracting (4.19) from (4.10) at time $t=t_n$,

$$(\overline{\partial}_t U^n,\chi)_N-(u_t^n,\chi)+q_n^\beta(AU,\chi)_N=(E(f^n),\chi)+S_{t_n}^\beta(A\widetilde{u},\chi)_N.$$

It follows that

$$\begin{aligned}
(\overline{\partial}_t\theta^n,\chi)_N+q_n^\beta(A\theta,\chi)_N=&(E(f^n),\chi)+\{(u_t^n,\chi)-(\overline{\partial}_t\widetilde{u}^n,\chi)_N\}\\
&+\{S_{t_n}^\beta(A\widetilde{u},\chi)_N-q_n^\beta(A\widetilde{u},\chi)_N\}.
\end{aligned}\tag{4.20}$$

Applying (2.3), we can rewrite the above as follows

$$\begin{aligned}
&(\overline{\partial}_t\theta^n,\chi)_N+q_n^\beta(A\theta,\chi)_N\\
=&(E(f^n),\chi)+\{(u_t^n,\chi)-(\overline{\partial}_t\widetilde{u}^n,\chi)_N\}+\{S_{t_n}^\beta(Au,\chi)-q_n^\beta(Au,\chi)\}\\
=&(E(f^n),\chi)+\{(u_t^n-\overline{\partial}_t u^n,\chi)-(\overline{\partial}_t\rho^n,\chi)-(E(\overline{\partial}_t\rho^n),\chi)\\
&-(E(\overline{\partial}_t u^n),\chi)\}+\{S_{t_n}^\beta(Au,\chi)-q_n^\beta(Au,\chi)\}.
\end{aligned}\tag{4.21}$$

We denote τ_1^n, τ_2^n and τ_3^n as follows:

$$\begin{aligned}
\tau_1^n &= (E(f^n),\chi),\\
\tau_2^n &= (u_t^n - \overline{\partial}_t u^n,\chi) - (\overline{\partial}_t \rho^n,\chi) - (E(\overline{\partial}_t \rho^n),\chi) - (E(\overline{\partial}_t u^n),\chi),\\
\tau_3^n &= S_{t_n}^{\beta}(Au,\chi) - q_n^{\beta}(Au,\chi).
\end{aligned}$$

We now turn to the estimates for τ_1^n, τ_2^n and τ_3^n. from (4.4), we obtain

$$|\tau_1^n| \le \|E(f^n)\|\|\chi\|_N. \tag{4.22}$$

Using the Taylor formula with the integral form of the remainder will give

$$|u_t^n - \overline{\partial}_t u^n| \le \int_{t_{n-1}}^{t_n} |u_{tt}|ds, \quad n \ge 1.$$

Hence, from (2.4) and (4.5), we have immediately

$$|(E(\overline{\partial}_t \rho^n),\chi)| \le c\|\overline{\partial}_t \rho^n\|\|\chi\|_N \le \frac{c}{k}\int_{t_{n-1}}^{t_n} \Big\|\frac{\partial \rho}{\partial t}(s)\Big\| ds\|\chi\|_N \le \frac{cN^{-\sigma}}{k}\int_{t_{n-1}}^{t_n} \|u_t\|_\sigma ds\|\chi\|_N,$$

$$|(E(\overline{\partial}_t u^n),\chi)| \le cN^{-\sigma}\|\overline{\partial}_t u^n\|_\sigma\|\chi\|_N \le \frac{cN^{-\sigma}}{k}\int_{t_{n-1}}^{t_n} \|u_t\|_\sigma ds\|\chi\|_N.$$

We have at once the following estimate for τ_2^n

$$|\tau_2^n| \le \bigg(\int_{t_{n-1}}^{t_n} \|u_{tt}\|ds + \frac{CN^{-\sigma}}{k}\int_{t_{n-1}}^{t_n} \|u_t\|_\sigma ds\bigg)\|\chi\|_N. \tag{4.23}$$

Finally, to estimate τ_3^n, using Lemma 4.2, we have

$$\begin{aligned}
|\tau_3^n| =& |S_{t_n}^{\beta}(Au,\chi) - q_n^{\beta}(Au,\chi)| \le \|S_{t_n}^{\beta}(Au) - q_n^{\beta}(Au)\|\|\chi\|_N\\
\le& \Big(ck^{\frac12}n^{-\frac12}\|Au(0)\| + ck^{\frac12}\int_{t_{n-1}}^{t_n}\|Au_t(s)\|ds + ck\int_0^{t_{n-1}}(t_n-s)^{-\frac12}\|Au_t(s)\|ds\Big)\|\chi\|_N.
\end{aligned}\tag{4.24}$$

Applying z-transform to (4.21), then following the same line as in the derivation of (4.11) and using the above estimates yields

$$\begin{aligned}
\sum_{n=1}^{\infty} e^{-n\delta}\|\theta_n\|_N^2 \le& \frac{4c}{(1-e^{-\frac{\delta}{2}})^2}\bigg\{N^{-2\sigma}\bigg[e^{-\delta}\|u_0\|_\sigma^2 + \Big(\sum_{n=1}^{\infty} e^{-n\frac{\delta}{2}}\|u_t\|_{\sigma,L_1(I_n)}\Big)^2\bigg]\\
&+ k^2\bigg[\Big(\sum_{n=1}^{\infty} e^{-n\frac{\delta}{2}}\|E(f^n)\|\Big)^2 + \Big(\sum_{n=1}^{\infty} e^{-n\frac{\delta}{2}}\|u_{tt}\|_{L_1(I_n)}\Big)^2\\
&+ k\Big(\sum_{n=1}^{\infty} e^{-n\frac{\delta}{2}}n^{-\frac12}\|Au_0\|\Big)^2 + k\Big(\sum_{n=1}^{\infty} e^{-n\frac{\delta}{2}}\|Au_t\|_{L_1(I_n)}\Big)^2\\
&+ k^2\Big(\sum_{n=1}^{\infty} e^{-n\frac{\delta}{2}}\|\beta(t_n-\cdot)Au_t(\cdot)\|_{L_1(0,t_n)}\Big)^2\bigg]\bigg\}.
\end{aligned}\tag{4.25}$$

Where the first term of the right-hand side of the above inequality follows a simple calculation:

$$\begin{aligned}\|\theta(0)\|_N =& \|U^0-\tilde{u}^0\|_N = \|I_N u_0 - \tilde{u}(0)\|_N \\ \le& 2\|I_N u_0 - \tilde{u}(0)\| \le 2(\|u_0 - I_N u_0\| + \|u_0 - \tilde{u}(0)\|) \\ \le& cN^{-\sigma}\|u_0\|_\sigma.\end{aligned}$$

Finally we get

$$\begin{aligned}&\sum_{n=1}^{\infty} e^{-n\delta}\|U^n-u(t_n)\|^2 \le \sum_{n=1}^{\infty} e^{-n\delta}(4\|\theta^n\|_N^2 + 2\|\rho^n\|^2) \\ \le& \frac{4c}{(1-e^{-\frac{\delta}{2}})^2}\Big\{N^{-2\sigma}\Big[e^{-\delta}\|u_0\|_\sigma^2 + \sum_{n=1}^{\infty} e^{-n\delta}\|u\|_\sigma^2 + \Big(\sum_{n=1}^{\infty} e^{-n\frac{\delta}{2}}\|u_t\|_{\sigma,L_1(I_n)}\Big)^2\Big] \\ &+k^2\Big[\Big(\sum_{n=1}^{\infty} e^{-n\frac{\delta}{2}}\|E(f^n)\|\Big)^2 + \Big(\sum_{n=1}^{\infty} e^{-n\frac{\delta}{2}}\|u_{tt}\|_{L_1(I_n)}\Big)^2 + k\Big(\sum_{n=1}^{\infty} e^{-n\frac{\delta}{2}} n^{-\frac{1}{2}}\|Au_0\|\Big)^2 \\ &+k\Big(\sum_{n=1}^{\infty} e^{-n\frac{\delta}{2}}\|Au_t\|_{L_1(I_n)}\Big)^2 + k^2\Big(\sum_{n=1}^{\infty} e^{-n\frac{\delta}{2}}\|\beta(t_n-\cdot)Au_t(\cdot)\|_{L_1(0,t_n)}\Big)^2\Big]\Big\}.\end{aligned}$$

This completes the proof. □

5. Numerical experiments

In this section, numerical examples for both 1-dimensional (1D) and 2-dimensional (2D) partial integro-differential equations are given.

Example 5.1. (1D numerical example).

In our test problem, we choose the weakly-singular kernel

$$\beta(t) = (\pi t)^{-\frac{1}{2}}, \quad \text{for } t > 0.$$

So, with $u_0(x) = 0$ and

$$f(x,t) = \frac{3}{2}(1-x^2)t^{\frac{1}{2}} + \frac{3\sqrt{\pi}}{4}t^2,$$

the exact solution is

$$u(t,x) = t^{\frac{3}{2}}(1-x^2).$$

Denoting

$$U^n = [U(x_0,t_n), U(x_1,t_n), \cdots, U(x_N,t_n)]^T \text{ and } F^n = [f(x_0,t_n), f(x_1,t_n), \cdots, f(x_N,t_n)]^T.$$

From (4.9), we can obtain an equation of the matrix form:

$$\mathbf{Z_N}\Big((I-k^{3/2}D_N^{(2)})U^n\Big) = \mathbf{Z_N}\Big(U^{n-1} + k^{3/2}\sum_{j=1}^{n-1}\omega_{n-j}D_N^{(2)}U^j + kF^n\Big),$$

Table 1: L^2 errors in u, at $t = 2$, 1D case.

N	L^2 error in u ($k = 1.0E-3$)	L^2 error in u ($k = 1.0E-4$)
4	1.529e-1	1.526e-1
6	3.408e-3	3.089e-3
8	7.676e-4	9.363e-5
10	8.502e-4	8.578e-5
12	9.284e-4	9.366e-5

Table 2: L^2 errors in u, at $t = 100$, 1D case.

N	L^2 error in u ($k = 1.0E-3$)	L^2 error in u ($k = 1.0E-4$)
6	1.127e+0	1.107e+0
8	5.495e-2	1.883e-2
10	5.692e-2	5.698e-3
12	6.215e-2	6.215e-3
14	6.697e-2	6.697e-3

where $D_N^{(2)}$ is the second order Legendre collocation differentiation matrix (see formula (2.3.29) of [2]), I denotes the unit matrix of order $N+1$ and $\mathbf{Z_N}$ is the matrix which represents setting the first and last points of a vector to zero. From which we obtain an explicit equations of U^n. Considering the initial conditions, the L^2-errors by using our spectral collocation and backward Euler with convolution quadrature approach are presented in Tables 1 and 2 for $t = 2$ and $t = 100$, respectively.

The first order convergence rate for the proposed method are observed from the above table and it is observed that for fixed values of k the error decreases until $N = 8$ and then remains almost unchanged. This implies that for $N \geq 8$, the error is dominated by that of the time discretization. Due to the small values of the time step, the effect of rounding errors can also be observed.

Example 5.2. (2D numerical example).

In our test problem, the kernel $\beta(t) = (\pi t)^{-1/2}$ and the forcing function $f(x, y, t)$ is chosen so that

$$u(x, y, t) = t^{3/2}(1 - x^2)(1 - y^2)$$

is the exact solution of the partial integro-differential equation problem.

In the implementation of the spectral collocation methods, let U^n and F^n be two matrices of order $(N-1) \times (N-1)$ such that

$$U^n = (U(\xi_i, \xi_j, t_n))_{i,j=1}^{N-1}, \quad F^n = (f(\xi_i, \xi_j, t_n))_{i,j=1}^{N-1}.$$

Then, (4.9) becomes the matrix equation

$$U^n = U^{n-1} + k^{\frac{3}{2}} \sum_{j=1}^{n} \omega_{n-j} D U^j + k F^n,$$

Table 3: L^2 errors in u, at $t=2$, 2D case.

N	L^2 error in u $(k=1.0E-3)$	L^2 error in u $(k=1.0E-4)$
4	1.959e-1	1.954e-1
6	5.825e-3	5.425e-3
8	1.363e-3	1.775e-4
10	1.679e-3	1.679e-4
12	2.003e-3	2.003e-4

Table 4: L^2 errors in u, at $t=100$, 2D case.

N	L^2 error in u $(k=1.0E-2)$	L^2 error in u $(k=1.0E-3)$
6	1.953e+0	1.929e-0
8	1.016e-1	4.109e-2
10	1.148e-1	1.150e-2
12	1.369e-1	1.369e-2
14	1.589e-1	1.589e-2

which can also be written as a standard linear system,

$$\big[I\otimes I-k^{\frac{3}{2}}(I\otimes D+D\otimes I)\big]\vec{u}^n=\vec{u}^{n-1}+k^{\frac{3}{2}}\sum_{j=1}^{n-1}\omega_{n-j}(I\otimes D+D\otimes I)\vec{u}^j+k\vec{f}^n,$$

where I is the identity matrix and $D=D_N^{(2)}(2:N,2:N)$, $\vec{f}^n$ and $\vec{u}^n$ are vectors of length $(N-1)^2$ formed by the columns of F^n and U^n and $\otimes$ denotes the tensor product of matrices.

The similar error estimates for 1D PIDE can be obtained in the same way as we have done for 2D PIDE case. In Tables 3 and 4, we list the L^2-errors for $t=2$ and $t=100$, respectively.

It is seen from Table 2 and Table 4 that our scheme is valid for large time computations both in 1D and 2D cases.

6. Concluding remarks

We have formulated and analyzed the spectral collocation method associated with backward Euler method and the first-order convolution quadrature rule on Eq. (1.1) under properly regularity conditions which yield the global behavior of stability and convergence of fully discretized numerical scheme in a straightforward way. Numerical results support the analysis.

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