Numer. Math. Theor. Meth. Appl. Vol. **6**, No. 2, pp. 408-423
doi: 10.4208/nmtma.2013.1127nm May 2013

Superconvergence and Asymptotic Expansions for Bilinear Finite Volume Element Approximations

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Received 2 October 2011; Accepted (in revised version) 13 March 2012

Available online 12 April 2013

Abstract. Aiming at the isoparametric bilinear finite volume element scheme, we initially derive an asymptotic expansion and a high accuracy combination formula of the derivatives in the sense of pointwise by employing the energy-embedded method on uniform grids. Furthermore, we prove that the approximate derivatives are convergent of order two. Finally, numerical examples verify the theoretical results.

AMS subject classifications: 65M10, 65M08, 41A60

Key words: Isoparametric bilinear finite volume element scheme, asymptotic expansion, high accuracy combination formula, superconvergence.

1. Introduction

Finite volume (FV) method has been one of the most commonly used numerical methods for solving partial differential equations due to its many attractive properties, such as preserving local conservation of certain physical quantities (mass, energy) and so on. The finite volume element (FVE) method is one important member of FV method. In 1982, Li and Zhu presented a generalized difference scheme [1], and proved the error estimate in H^1 norm on quadrilateral grids. The trial and test spaces are, respectively, chosen as bilinear finite element space and piecewise constant space. It is so-called the isoparametric bilinear finite volume element scheme. In 1993, Schmidt and Kiel constructed two types of box (diagonal box, center box) schemes [2], and obtained the saturated convergent order in H^1 norm and the superconvergent result on parallelogram grids based on the analysis of the eigenvalue problem for any partition element. Later, Porsching and Chou proposed a

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"Covolume Method" [3,4] which is actually a FVE method and widely applied in computational fluid dynamic problems. Simultaneously, some symmetric FVE schemes [5,6], high order FVE schemes [7–9] and new FVE schemes for three dimensional problems [10,11] were presented by some researches.

For the isoparametric bilinear finite volume element scheme, its optimal L^2 error estimate [12] is got more behind that on H^1 estimate. Recently, Li and Lv proved the optimal L^2 error results [13–15] for this scheme. Although the superconvergence for the finite element methods is abundantly studied [16–18], there are only some researches on superconvergence about the isoparametric bilinear finite volume element [1,15,19,20]. Furthermore, they are almost in the sense of average instead of pointwise. It urges us to study the superconvergence in the sense of pointwise.

In present paper, the innovative idea of our work is that we derive an asymptotic expansion for the isoparametric bilinear finite volume element solution. The derivation includes the achievement of the integral formula for the bilinear functional $A(u-u_I,v)$, the introduction of a proper auxiliary variational problem, and the employment of the discrete Green function and the energy-embedded method. Furthermore, we derive a high accuracy combination formula of the derivatives in the sense of pointwise on uniform grids for the first time, and prove that the approximate derivatives are convergent of order two. Numerical examples confirm the theoretical results.

The remainder of this paper is organized as follows. In Section 2, we introduce the isoparametric bilinear finite volume element scheme and some convergent results. In Section 3, we derive the asymptotic expansion for our finite volume element solution. In Section 4, we present a high accuracy combination formula of the approximate derivatives in the sense of pointwise on uniform grids and the corresponding superconvergence. Finally, we display numerical experiments to support our conclusions.

2. The isoparametric bilinear finite volume element scheme

We consider the following model problem

$$\begin{cases} -\nabla\cdot(\kappa\nabla u)=f, & \mathbf{x}\in\Omega,\\ u=0, & \mathbf{x}\in\partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega\subset\mathbb{R}^2$ is a convex polygonal domain with boundary $\partial\Omega$, $f(\mathbf{x})\in L^2(\Omega)$ and $\kappa(\mathbf{x})\in C^1(\Omega)$ satisfies

$$\kappa(\mathbf{x})\geq\kappa_0,$$

and κ_0 is a positive constant.

Let $\Omega_h=\{E_i,\ 1\leq i\leq M\}$ be the quadrilateral partition of Ω (see Fig. 1(a)), and $\mathscr{D}=\{P_i=(x_i^1,x_i^2),\ 1\leq i\leq N\}$ be the set of partition nodes in Ω_h, where M and N are, respectively, the numbers of elements and nodes. Denote $\Omega_h^*=\{b_{P_i},1\leq i\leq N\}$ as the dual partition of Ω_h, where b_{P_i} is the dual element (also called control volume) about node P_i (see Fig. 1(b)). In this paper, we always assume that Ω_h and Ω_h^* are all quasi-uniform, i.e.,

$$C_1h^2\leq S_E\leq C_2h^2,\quad E\in\Omega_h\quad\text{and}\quad C_1h^2\leq S_{P_i}\leq C_2h^2,\quad b_{P_i}\in\Omega_h^*,$$

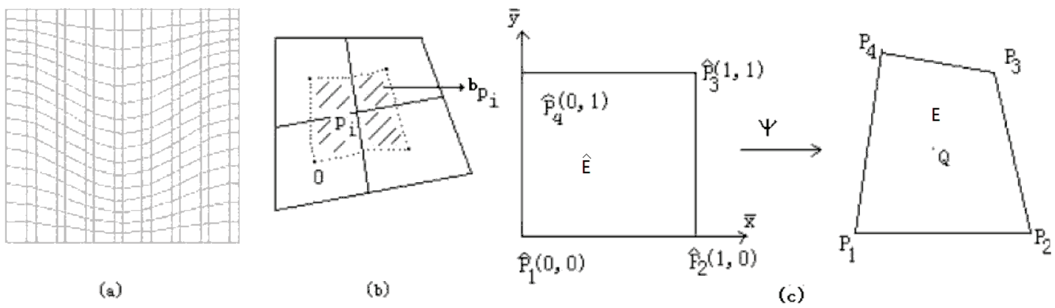

Figure 1: (a) Quadrilateral partition Ω_h. (b) Dual element b_{P_i}. (c) Transformation ψ.

where S_E and S_{P_i} are, respectively, the areas of element E and dual element b_{P_i}.

We introduce an invertible bilinear transformation ψ_E which maps $\hat{E}$ onto convex element E (see Fig. 1(c))

$$\begin{cases} x_1 = x_1^1 + A_1\hat{x}_1 + A_2\hat{x}_2 + A_3\hat{x}_1\hat{x}_2, \\ x_2 = x_2^1 + B_1\hat{x}_1 + B_2\hat{x}_2 + B_3\hat{x}_1\hat{x}_2, \end{cases} \tag{2.2}$$

where (x_1^l, x_2^l), $l = 1,2,3,4$ are four nodal coordinates of element E, and

$$\begin{aligned} &A_1 = x_1^2 - x_1^1, \qquad A_2 = x_1^4 - x_1^1, \qquad A_3 = x_1^1 - x_1^2 + x_1^3 - x_1^4, \\ &B_1 = x_2^2 - x_2^1, \qquad B_2 = x_2^4 - x_2^1, \qquad B_3 = x_2^1 - x_2^2 + x_2^3 - x_2^4. \end{aligned}$$

Especially, for a rectangle partition, the transformation (2.2) becomes

$$\begin{cases} x_1 = x_1^1 + \hat{x}_1 h_1, \\ x_2 = x_2^1 + \hat{x}_2 h_2, \end{cases} \tag{2.3}$$

where $h_1 = A_1$, $h_2 = B_2$.

Let V_B and V_h, respectively, be the trial and test function spaces

$$V_B = \{v \in L^2(\Omega) : v|_{b_{P_i}} = \text{constant}, \ b_{P_i} \in \Omega_h^*\},$$

and

$$V_h = \{u_h \in C(\bar{\Omega}) : u_h|_E = P_{\hat{E}} \circ \psi_E^{-1}, \ P_{\hat{E}} \in \hat{\mathscr{P}}_{1,1}, \ E \in \Omega_h, u_h|_{\partial\Omega} = 0\},$$

where $\hat{\mathscr{P}}_{1,1}$ is the set of bilinear functions on $\hat{E}$, and $P_{\hat{E}} \circ \psi_E^{-1}$ is the composite function of $P_{\hat{E}}$ and ψ_E^{-1}.

We introduce two interpolation operators in the following. The first one is $\Pi^* : V_h \mapsto V_B$,

$$\Pi^* v(\mathbf{x}) = \begin{cases} v(P_i), & \mathbf{x} \in b_{P_i}, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall v(\mathbf{x}) \in V_h, \tag{2.4}$$

and the second one is $\Pi : H_0^1(\Omega) \cap H^2(\Omega) \to V_h$,

$$\Pi u = \sum_{i=1}^{N} u_i \phi_i, \quad u \in H_0^1(\Omega) \cap H^2(\Omega), \tag{2.5}$$

where $u_i = u(P_i)$, and ϕ_i is the Lagrange interpolation basis function about node P_i.

In the following, we also denote Πu as u_I for the sake of convenience.

Lemma 2.1 (see [15]). *If $u \in H_0^1(\Omega) \cap H^2(\Omega)$, then*

$$|u - \Pi u|_0 \lesssim C h^{2-m} |u|_2, \quad m = 0, 1, \quad |u - \Pi^* u|_0 \lesssim C h |u|_1.$$

The isoparametric bilinear finite volume element solution $u_h \in V_h$ of problem (2.1) satisfies

$$A(u_h, v) = (f, \Pi^* v), \quad \forall v \in V_h, \tag{2.6}$$

where

$$A(u_h, v) = -\sum_{b_{P_i} \in \Omega_h^*} \int_{\partial b_{P_i}} \kappa \frac{\partial u}{\partial \mathbf{n}} \Pi^* v ds = \sum_{E \in \Omega_h} A(u, v)\Big|_E,$$

$$A(u, v)\Big|_E = -\sum_{i=1}^{4} \int_{\partial b_{P_i} \cap E} \kappa \frac{\partial u}{\partial \mathbf{n}} \Pi^* v ds, \tag{2.7a}$$

$$(f, \Pi^* v) = \sum_{b_{P_i} \in \Omega_h^*} \int_{b_{P_i}} f \Pi^* v dx. \tag{2.7b}$$

We define the discrete H^1 semi-norm on space V_h as follows

$$|w|_{1,h} = \Big(\sum_E |w|_{1,h,E}^2 \Big)^{\frac{1}{2}}, \quad w \in V_h, \tag{2.8}$$

where $|w|_{1,h,E}^2 = (w_2 - w_1)^2 + (w_3 - w_2)^2 + (w_4 - w_3)^2 + (w_1 - w_4)^2$, $w_i = w(P_i)$.

Lemma 2.2 (see [15]). *The semi-norm $|w|_{1,h}$ is equivalent with $|w|_1$, i.e., there exist two positive constants β_1, β_2 independent of the mesh size h, such that*

$$\beta_1 |w|_{1,h} \le |w|_1 \le \beta_2 |w|_{1,h}, \quad \forall w \in V_h.$$

The following results (see [1, 14, 15]) hold.

Lemma 2.3. *The bilinear functional $A(\cdot,\cdot)$ defined by* (2.7a) *satisfies*

$$A(u_h, u_h) \gtrsim |u_h|_1^2, \quad A(u_h, v_h) \lesssim |u_h|_1 |v_h|_1, \quad \forall u_h, v_h \in V_h. \tag{2.9}$$

Lemma 2.4. *Let $u \in H_0^1(\Omega) \cap H^3(\Omega)$ and $u_h \in V_h$ be the solutions of problems* (2.1) *and* (2.6)*, respectively. Then we have*

$$\|u - u_h\|_1 \lesssim h |u|_2, \quad \|u - u_h\|_0 \lesssim h^2 \|u\|_3. \tag{2.10}$$

For convenience, we shall derive an asymptotic expansion for the finite volume element solution u_h and prove the corresponding superconvergence on uniform rectangle grids whose step sizes along both x_1 and x_2 directions are all equal to h.

3. Asymptotic expansion

To obtain the asymptotic expansion, we firstly display some lemmas.

Lemma 3.1 (see [21]). *Assume that $\hat{u}_I$ is the bilinear interpolation function of $\hat{u}$ on $\hat{E}$, and $\hat{u}_I=\sum_{l=1}^4 \hat{u}_l\hat{\phi}_l$, where $\hat{\phi}_l$ is the Lagrange basis function about node $\hat{P}_l$ (see Fig. 2(b)). We have*

(1) If $\hat{u}\in\mathscr{P}_2(\hat{E})$, then

$$\hat{u}-\hat{u}_I=\frac{1}{2}\Big(\frac{\partial^2\hat{u}}{\partial\hat{x}_1^2}\hat{x}_1(\hat{x}_1-1)+\frac{\partial^2\hat{u}}{\partial\hat{x}_2^2}\hat{x}_2(\hat{x}_2-1)\Big).$$

(2) If $\hat{u}\in\mathscr{P}_3(\hat{E})$, then

$$\begin{aligned}\hat{u}-\hat{u}_I=&\hat{x}_1(\hat{x}_1-1)\Big(\frac{1}{6}\frac{\partial^3\hat{u}}{\partial\hat{x}_1^3}\hat{x}_1+\frac{1}{2}\frac{\partial^3\hat{u}}{\partial\hat{x}_1^2\partial\hat{x}_2}\hat{x}_2+C_1\Big)\\&+\hat{x}_2(\hat{x}_2-1)\Big(\frac{1}{6}\frac{\partial^3\hat{u}}{\partial\hat{x}_2^3}\hat{x}_2+\frac{1}{2}\frac{\partial^3\hat{u}}{\partial\hat{x}_1\partial\hat{x}_2^2}\hat{x}_1+C_2\Big),\end{aligned}$$

where C_1, C_2 are constants:

$$C_1=\frac{1}{2}\frac{\partial^2\hat{u}}{\partial\hat{x}_1^2}\Big|_{\hat{x}_1=\frac{1}{3},\hat{x}_2=0},\quad C_2=\frac{1}{2}\frac{\partial^2\hat{u}}{\partial\hat{x}_2^2}\Big|_{\hat{x}_1=0,\hat{x}_2=\frac{1}{3}}.$$

We will consider the estimate for $A(u-u_I,v)$, $(v\in V_h)$ in the following.

Lemma 3.2. *Assume that $\kappa\in C^1(\Omega)$, $u\in H^3(\Omega)\cap H_0^1(\Omega)$, and u_I is the interpolation function defined by* (2.5). *Then, for any $\mathbf{x}=(x_1,x_2)\in E$, we have*

$$\Big|\int_{\overrightarrow{OM_l}}(\kappa(\mathbf{x})-\kappa(\mathbf{x}^0))\frac{\partial(u-u_I)}{\partial x_1}dx_2\Big|\lesssim h^3|u|_{3,E},\quad l=1,3, \tag{3.1a}$$

$$\Big|\int_{\overrightarrow{OM_l}}(\kappa(\mathbf{x})-\kappa(\mathbf{x}^0))\frac{\partial(u-u_I)}{\partial x_2}dx_1\Big|\lesssim h^3|u|_{3,E},\quad l=2,4, \tag{3.1b}$$

where $\mathbf{x}^0=(x_1^0,x_2^0)$ is the center of E (see Fig. 2(a)).

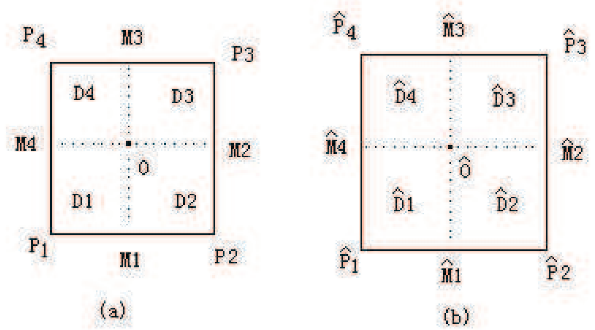

Figure 2: (a) E and D_l, $1\le l\le 4$. (b) $\hat{E}$ and $\hat{D}_l$, $1\le l\le 4$.

Proof. We first prove (3.1a). Let $F(u)$ be a linear functional as follows

$$F(u)=\int_{\overrightarrow{OM_l}}(\kappa(\mathbf{x})-\kappa(\mathbf{x}^0))\frac{\partial(u-u_I)}{\partial x_1}dx_2,\quad l=1,3.$$

By (2.3),

$$F(u)=F(\hat{u})=\int_{\overrightarrow{\hat{O}\hat{M}_l}}(\kappa(\hat{\mathbf{x}})-\kappa(\hat{\mathbf{x}}^0))\frac{\partial(\hat{u}-\hat{u}_I)}{\partial \hat{x}_1}d\hat{x}_2, \tag{3.2}$$

where $\hat{\mathbf{x}}=(\hat{x}_1,\hat{x}_2)$, $\hat{\mathbf{x}}^0=(\hat{x}_1^0,\hat{x}_2^0)$ (see Fig. 2(b)).

One can easily see that the following inequality holds.

$$|\kappa(\hat{\mathbf{x}})-\kappa(\hat{\mathbf{x}}^0)|\le Ch,\quad \hat{\mathbf{x}}\in\hat{E}.$$

The above inequality and the trace theorem imply that

$$|F(\hat{u})|\lesssim Ch\int_{\overrightarrow{\hat{O}\hat{M}_l}}\left|\frac{\partial(\hat{u}-\hat{u}_I)}{\partial \hat{x}_1}\right|d\hat{x}_2\lesssim Ch\|\hat{u}\|_{2,\hat{E}}\lesssim Ch\|\hat{u}\|_{3,\hat{E}}. \tag{3.3}$$

Hence, $F(\hat{u})$ is a boundary linear functional in $H^3(\hat{E})$.

From Lemma 3.1, for any $\hat{u}\in\mathscr{P}_2(\hat{E})$, we have

$$\frac{\partial(\hat{u}-\hat{u}_I)}{\partial \hat{x}_1}\bigg|_{\hat{x}_1=\frac{1}{2}}=0.$$

Then, by (3.2), we can obtain

$$F(\hat{u})=0. \tag{3.4}$$

Combing (3.3) with (3.4), and using the Bramble-Hilbert lemma, we have

$$|F(\hat{u})|\lesssim Ch|\hat{u}|_{3,\hat{E}}.$$

By the above inequality, the scaling technique and (3.2),

$$|F(u)|\lesssim Ch|\hat{u}|_{3,\hat{E}}\lesssim Ch^3|u|_{3,E}.$$

Then, we can obtain (3.1a). Similarly, we can prove that (3.1b) holds. This completes the proof of Lemma 3.2. □

We define two linear functionals

$$\begin{aligned}B(u,v)|_E=&(v_2-v_1)\int_{\overrightarrow{M_1O}}\frac{\partial(u-u_I)}{\partial x_1}dx_2+(v_3-v_4)\int_{\overrightarrow{OM_3}}\frac{\partial(u-u_I)}{\partial x_1}dx_2\\&+(v_4-v_1)\int_{\overrightarrow{M_4O}}\frac{\partial(u-u_I)}{\partial x_2}dx_1+(v_3-v_2)\int_{\overrightarrow{OM_2}}\frac{\partial(u-u_I)}{\partial x_2}dx_1\end{aligned} \tag{3.5}$$

and

$$H(u,v)|_E = -\frac{h^2}{24}\int_E \Big[\Big(\frac{\partial^3 u}{\partial x_1^3} + 2\frac{\partial^3 u}{\partial x_1 \partial x_2^2}\Big)\frac{\partial v}{\partial x_1} + \Big(\frac{\partial^3 u}{\partial x_2^3} + 2\frac{\partial^3 u}{\partial x_1^2 \partial x_2}\Big)\frac{\partial v}{\partial x_2}\Big] d\mathbf{x}. \tag{3.6}$$

By (2.3),

$$B(u,v)|_E = \hat{B}(\hat{u},\hat{v})|_{\hat{E}}, \quad H(u,v)|_E = \hat{H}(\hat{u},\hat{v})|_{\hat{E}}, \tag{3.7}$$

where

$$\begin{aligned}\hat{B}(\hat{u},\hat{v})|_{\hat{E}} =& (\hat{v}_2 - \hat{v}_1)\int_0^{\frac{1}{2}} \frac{\partial(\hat{u}-\hat{u}_I)}{\partial \hat{x}_1}\Big|_{\hat{x}_1=\frac{1}{2}} d\hat{x}_2 + (\hat{v}_3 - \hat{v}_4)\int_{\frac{1}{2}}^{1} \frac{\partial(\hat{u}-\hat{u}_I)}{\partial \hat{x}_1}\Big|_{\hat{x}_1=\frac{1}{2}} d\hat{x}_2 \\ &+ (\hat{v}_4 - \hat{v}_1)\int_0^{\frac{1}{2}} \frac{\partial(\hat{u}-\hat{u}_I)}{\partial \hat{x}_2}\Big|_{\hat{x}_2=\frac{1}{2}} d\hat{x}_1 + (\hat{v}_3 - \hat{v}_2)\int_{\frac{1}{2}}^{1} \frac{\partial(\hat{u}-\hat{u}_I)}{\partial \hat{x}_2}\Big|_{\hat{x}_2=\frac{1}{2}} d\hat{x}_1\end{aligned} \tag{3.8}$$

and

$$\hat{H}(\hat{u},\hat{v})|_{\hat{E}} = -\frac{1}{24}\int_{\hat{E}} \Big[\Big(\frac{\partial^3 \hat{u}}{\partial \hat{x}_1^3} + 2\frac{\partial^3 \hat{u}}{\partial \hat{x}_1 \partial \hat{x}_2^2}\Big)\frac{\partial \hat{v}}{\partial \hat{x}_1} + \Big(\frac{\partial^3 \hat{u}}{\partial \hat{x}_2^3} + 2\frac{\partial^3 \hat{u}}{\partial \hat{x}_1^2 \partial \hat{x}_2}\Big)\frac{\partial \hat{v}}{\partial \hat{x}_2}\Big] d\hat{\mathbf{x}}. \tag{3.9}$$

Lemma 3.3. *If $\hat{u} \in \mathscr{P}_3(\hat{E})$ and $\hat{v} \in \hat{\mathscr{P}}_{1,1}(\hat{E})$, then $\hat{B}(\hat{u},\hat{v})|_{\hat{E}} = \hat{H}(\hat{u},\hat{v})|_{\hat{E}}$.*

Proof. If $\hat{u} \in \mathscr{P}_3(\hat{E})$, then, from Lemma 3.1, we have

$$\frac{\partial(\hat{u}-\hat{u}_I)}{\partial \hat{x}_1}\Big|_{\hat{x}_1=\frac{1}{2}} = -\frac{1}{24}\frac{\partial^3 \hat{u}}{\partial \hat{x}_1^3} + \frac{1}{2}\frac{\partial^3 \hat{u}}{\partial \hat{x}_1 \partial \hat{x}_2^2}\hat{x}_2(\hat{x}_2 - 1), \tag{3.10a}$$

$$\frac{\partial(\hat{u}-\hat{u}_I)}{\partial \hat{x}_2}\Big|_{\hat{x}_2=\frac{1}{2}} = -\frac{1}{24}\frac{\partial^3 \hat{u}}{\partial \hat{x}_2^3} + \frac{1}{2}\frac{\partial^3 \hat{u}}{\partial \hat{x}_1^2 \partial \hat{x}_2}\hat{x}_1(\hat{x}_1 - 1). \tag{3.10b}$$

Substituting (3.10a) and (3.10b) into (3.8), and noting that the third order derivatives of $\hat{u}$ are all constants, we have

$$\begin{aligned}\hat{B}(\hat{u},\hat{v})|_{\hat{E}} =& -\frac{1}{48}\Big(\frac{\partial^3 \hat{u}}{\partial \hat{x}_1^3} + 2\frac{\partial^3 \hat{u}}{\partial \hat{x}_1 \partial \hat{x}_2^2}\Big)\big[(\hat{v}_2 - \hat{v}_1) + (\hat{v}_3 - \hat{v}_4)\big] \\ &- \frac{1}{48}\Big(\frac{\partial^3 \hat{u}}{\partial \hat{x}_2^3} + 2\frac{\partial^3 \hat{u}}{\partial \hat{x}_1^2 \partial \hat{x}_2}\Big)\big[(\hat{v}_4 - \hat{v}_1) + (\hat{v}_3 - \hat{v}_2)\big].\end{aligned} \tag{3.11}$$

Since $\hat{v} \in \mathscr{P}_{1,1}(\hat{E})$, we have $\hat{v} = \sum_{l=1}^4 \hat{v}_l \hat{\phi}_l$. Then we can obtain

$$\int_{\hat{E}} \frac{\partial \hat{v}}{\partial \hat{x}_1} d\hat{x}_1 d\hat{x}_2 = \frac{1}{2}\big[(\hat{v}_1 - \hat{v}_2) + (\hat{v}_3 - \hat{v}_4)\big], \tag{3.12a}$$

$$\int_{\hat{E}} \frac{\partial \hat{v}}{\partial \hat{x}_2} d\hat{x}_1 d\hat{x}_2 = \frac{1}{2}\big[(\hat{v}_4 - \hat{v}_1) + (\hat{v}_3 - \hat{v}_2)\big]. \tag{3.12b}$$

Hence, combining (3.10a), (3.10b), (3.12), (3.8) with (3.11), we have

$$\begin{aligned}\hat{H}(\hat{u},\hat{v})|_{\hat{E}} =& -\frac{1}{24}\int_{\hat{E}}\Big[\Big(\frac{\partial^3\hat{u}}{\partial\hat{x}_1^3}+2\frac{\partial^3\hat{u}}{\partial\hat{x}_1\partial\hat{x}_2^2}\Big)\frac{\partial\hat{v}}{\partial\hat{x}_1}+\Big(\frac{\partial^3\hat{u}}{\partial\hat{x}_2^3}+2\frac{\partial^3\hat{u}}{\partial\hat{x}_1^2\partial\hat{x}_2}\Big)\frac{\partial\hat{v}}{\partial\hat{x}_2}\Big]d\hat{\mathbf{x}}\\ =&\hat{B}(\hat{u},\hat{v})|_{\hat{E}}.\end{aligned}$$

This completes the proof of Lemma 3.3. □

Lemma 3.4. *If $\hat{v}\in\mathscr{P}_{1,1}(\hat{E})$ and $\hat{u}\in H^4(\hat{E})$, then*

$$|\hat{B}(\hat{u},\hat{v})|_{\hat{E}}|\lesssim|\hat{v}|_{1,\hat{h},\hat{E}}\|\hat{u}\|_{4,\hat{E}}, \tag{3.13a}$$

$$|\hat{H}(\hat{u},\hat{v})|_{\hat{E}}|\lesssim|\hat{v}|_{1,\hat{h},\hat{E}}\|\hat{u}\|_{4,\hat{E}}, \tag{3.13b}$$

where $|\hat{v}|^2_{1,\hat{h},\hat{E}}=(\hat{v}_2-\hat{v}_1)^2+(\hat{v}_3-\hat{v}_2)^2+(\hat{v}_4-\hat{v}_3)^2+(\hat{v}_1-\hat{v}_4)^2$*,* $\hat{v}_i=\hat{v}(\hat{P}_i)$*,* $i=1,2,3,4$*.*

Proof. Using the trace theorem, the embedded theorem and

$$\Big(\int_{\hat{E}}\Big[\Big(\frac{\partial^2\hat{u}}{\partial\hat{x}_1^2}\Big)^2+\Big(\frac{\partial^2\hat{u}}{\partial\hat{x}_1\partial\hat{x}_2}\Big)^2\Big]d\hat{\mathbf{x}}\Big)^{\frac{1}{2}}\lesssim|\hat{u}|_{2,\hat{E}},$$

for $\alpha=0$ or $1/2$, we can obtain

$$\begin{aligned}&\Big|\int_{\alpha}^{\alpha+\frac{1}{2}}\frac{\partial(\hat{u}-\hat{u}_I)}{\partial\hat{x}_1}\Big|_{\hat{x}_1=\frac{1}{2}}d\hat{x}_2\Big|\lesssim\Big\|\frac{\partial(\hat{u}-\hat{u}_I)}{\partial\hat{x}_1}\Big\|_{1,\hat{E}}\\ \lesssim&\Big(|\hat{u}|^2_{2,\hat{E}}+\int_{\hat{E}}\Big|\frac{\partial^2\hat{u}_I}{\partial\hat{x}_1\partial\hat{x}_2}\Big|^2d\hat{\mathbf{x}}\Big)^{\frac{1}{2}}\lesssim|\hat{u}|_{2,\hat{E}}.\end{aligned} \tag{3.14}$$

Similarly, we have

$$\Big|\int_{\alpha}^{\alpha+\frac{1}{2}}\frac{\partial(\hat{u}-\hat{u}_I)}{\partial\hat{x}_2}\Big|_{\hat{x}_2=\frac{1}{2}}d\hat{x}_1\Big|\lesssim|\hat{u}|_{2,\hat{E}},\quad\alpha=0,\frac{1}{2}. \tag{3.15}$$

By (3.8), (3.14) and (3.15),

$$\Big|\hat{B}(\hat{u},\hat{v})|_{\hat{E}}\Big|\lesssim|\hat{v}|_{1,\hat{h},\hat{E}}\|\hat{u}\|_{4,\hat{E}}. \tag{3.16}$$

In the same way, using (3.9) and noticing that $|\hat{v}|_{1,\hat{E}}\cong|\hat{v}|_{1,h,E}$, we have

$$\Big|\hat{H}(\hat{u},\hat{v})|_{\hat{E}}\Big|\lesssim\Big(\int_{\hat{E}}\Big(\Big(\frac{\partial\hat{v}}{\partial\hat{x}_1}\Big)^2+\Big(\frac{\partial\hat{v}}{\partial\hat{x}_2}\Big)^2\Big)d\hat{\mathbf{x}}\Big)^{\frac{1}{2}}|\hat{u}|_{4,\hat{E}}\lesssim|\hat{v}|_{1,\hat{h},\hat{E}}\|\hat{u}\|_{4,\hat{E}}.$$

This completes the proof of Lemma 3.4. □

Lemma 3.5. *If $v \in V_h$ and $u \in H^4(\Omega) \cap H_0^1(\Omega)$, then*

$$B(u,v)|_E = H(u,v)|_E + \mathscr{O}(h^3)|v|_{1,h,E}|u|_{4,E}.$$

Proof. By (3.7),

$$\Big|B(u,v)|_E - H(u,v)|_E\Big| = \Big|\hat{B}(\hat{u},\hat{v})|_{\hat{E}} - \hat{H}(\hat{u},\hat{v})|_{\hat{E}}\Big|.$$

Combining Lemma 3.3 with Lemma 3.4, and using the Bramble-Hilbert lemma, the scaling technique and $|v|_{1,h,E} \cong |\hat{v}|_{1,\hat{h},\hat{E}}$, we can obtain

$$\Big|\hat{B}(\hat{u},\hat{v})|_{\hat{E}} - \hat{H}(\hat{u},\hat{v})|_{\hat{E}}\Big| \lesssim |\hat{v}|_{1,\hat{h},\hat{E}}|\hat{u}|_{4,\hat{E}} \lesssim h^3|v|_{1,h,E}|u|_{4,E}.$$

This completes the proof of Lemma 3.5. ☐

Now, we consider the estimate on $A(u-u_I,v)|_E$.

$$\begin{aligned} A(u-u_I,v)|_E =&(v_2-v_1)\int_{\overrightarrow{M_1O}} \kappa\frac{\partial(u-u_I)}{\partial x_1}dx_2 + (v_3-v_4)\int_{\overrightarrow{OM_3}} \kappa\frac{\partial(u-u_I)}{\partial x_1}dx_2 \\ &+(v_4-v_1)\int_{\overrightarrow{M_4O}} \kappa\frac{\partial(u-u_I)}{\partial x_2}dx_1 + (v_3-v_2)\int_{\overrightarrow{OM_2}} \kappa\frac{\partial(u-u_I)}{\partial x_2}dx_1 \\ =&\kappa(\mathbf{x}^0)B(u,v)|_E + R(u,v), \end{aligned} \tag{3.17}$$

where denote $\delta\kappa$ as the difference $\kappa(\mathbf{x}) - \kappa(\mathbf{x}^0)$, and

$$\begin{aligned} R(u,v) =&(v_2-v_1)\int_{\overrightarrow{M_1O}} \delta\kappa\frac{\partial(u-u_I)}{\partial x_1}dx_2 + (v_3-v_4)\int_{\overrightarrow{OM_3}} \delta\kappa\frac{\partial(u-u_I)}{\partial x_1}dx_2 \\ &+(v_4-v_1)\int_{\overrightarrow{M_4O}} \delta\kappa\frac{\partial(u-u_I)}{\partial x_2}dx_1 + (v_3-v_2)\int_{\overrightarrow{OM_2}} \delta\kappa\frac{\partial(u-u_I)}{\partial x_2}dx_1. \end{aligned}$$

By Lemma 3.2,

$$|R(u,v)| \lesssim h^3|v|_{1,h,E}|u|_{3,E} \lesssim h^3|v|_{1,h,E}\|u\|_{4,E}. \tag{3.18}$$

When $\kappa \in C^1(\bar{\Omega})$, from Lemma 3.5, we have

$$\begin{aligned} \kappa(\mathbf{x}^0)B(u,v)|_E =&\kappa(\mathbf{x}^0)H(u,v)|_E + \mathscr{O}(h^3)|v|_{1,h,E}\|u\|_{4,E} \\ =&H_\kappa(u,v)|_E + \mathscr{O}(h^3)|v|_{1,h,E}\|u\|_{4,E}, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} H_\kappa(u,v)|_E = -\frac{h^2}{24}\int_E \kappa(\mathbf{x})\Big[\Big(\frac{\partial^3 u}{\partial x_1^3} + 2\frac{\partial^3 u}{\partial x_1\partial x_2^2}\Big)\frac{\partial v}{\partial x_1} \\ + \Big(\frac{\partial^3 u}{\partial x_2^3} + 2\frac{\partial^3 u}{\partial x_1^2\partial x_2}\Big)\frac{\partial v}{\partial x_2}\Big]d\mathbf{x}. \end{aligned} \tag{3.20}$$

By (3.17), (3.18) and (3.19),

$$A(u-u_I,v)|_E = H_\kappa(u,v)|_E + \mathscr{O}(h^3)|v|_{1,h,E}\|u\|_{4,E}. \tag{3.21}$$

Lemma 3.6. *Assume that u is the solution of* (2.1) *and* $u \in H^4(\Omega)\cap H_0^1(\Omega)$, *then*

$$A(u-u_I,v)=\frac{h^2}{24}\int_\Omega q\Pi^* v d\mathbf{x}+\mathcal{O}(h^3)\|u\|_4|v|_1,\quad \forall v\in V_h, \tag{3.22}$$

where $A(u,v)$ *is defined by* (2.7a), *and*

$$q(\mathbf{x})=\frac{\partial}{\partial x_1}\Big(\kappa\Big(\frac{\partial^3 u}{\partial x_1^3}+2\frac{\partial^3 u}{\partial x_1\partial x_2^2}\Big)\Big)+\frac{\partial}{\partial x_2}\Big(\kappa\Big(\frac{\partial^3 u}{\partial x_2^3}+2\frac{\partial^3 u}{\partial x_1^2\partial x_2}\Big)\Big). \tag{3.23}$$

Proof. Taking the sum of two sides of (3.21) about element E, we have

$$A(u-u_I,v)=-\sum_E H_\kappa(u,v)|_E+\mathcal{O}(h^3)\sum_E \|u\|_{4,E}|v|_{1,E}. \tag{3.24}$$

For the second term in (3.24), using the Cauchy inequality, we can obtain

$$\sum_E \|u\|_{4,E}|v|_{1,E}\le\Big(\sum_E\|u\|_{4,E}^2\Big)^{\frac{1}{2}}\Big(\sum_E|v|_{1,E}^2\Big)^{\frac{1}{2}}=\|u\|_4|v|_1. \tag{3.25}$$

For the first term in (3.24), using (3.20), the Green formula and noticing the fact that

$$v\in V_h,\quad v|_{\partial\Omega}=0,$$

we have

$$\begin{aligned}
&-\sum_E H_\kappa(u,v)|_E\\
=&-\frac{h^2}{24}\sum_E\int_E\kappa\Big[\Big(\frac{\partial^3 u}{\partial x_1^3}+2\frac{\partial^3 u}{\partial x_1\partial x_2^2}\Big)\frac{\partial v}{\partial x_1}+\Big(\frac{\partial^3 u}{\partial x_2^3}+2\frac{\partial^3 u}{\partial x_1^2\partial x_2}\Big)\frac{\partial v}{\partial x_2}\Big]d\mathbf{x}\\
=&-\frac{h^2}{24}\int_\Omega\kappa\Big[\Big(\frac{\partial^3 u}{\partial x_1^3}+2\frac{\partial^3 u}{\partial x_1\partial x_2^2}\Big)\frac{\partial v}{\partial x_1}+\Big(\frac{\partial^3 u}{\partial x_2^3}+2\frac{\partial^3 u}{\partial x_1^2\partial x_2}\Big)\frac{\partial v}{\partial x_2}\Big]d\mathbf{x}\\
=&\frac{h^2}{24}\int_\Omega\Big\{\frac{\partial}{\partial x_1}\Big(\kappa\Big(\frac{\partial^3 u}{\partial x_1^3}+2\frac{\partial^3 u}{\partial x_1\partial x_2^2}\Big)\Big)+\frac{\partial}{\partial x_2}\Big(\kappa\Big(\frac{\partial^3 u}{\partial x_2^3}+2\frac{\partial^3 u}{\partial x_1^2\partial x_2}\Big)\Big)\Big\}v d\mathbf{x}\\
=&\frac{h^2}{24}\int_\Omega q v d\mathbf{x}.
\end{aligned} \tag{3.26}$$

Combining (3.24), (3.25) with (3.26), we have

$$A(u-u_I,v)=\frac{h^2}{24}\int_\Omega q\Pi^* v d\mathbf{x}+\frac{h^2}{24}\int_\Omega q(v-\Pi^* v)d\mathbf{x}+\mathcal{O}(h^3)\|u\|_4|v|_1.$$

From the Hölder inequality and Lemma 2.1, we can obtain

$$\frac{h^2}{24}\int_\Omega q(v-\Pi^* v)d\mathbf{x}\lesssim h^2\|q\|_0\|v-\Pi^* v\|_0\lesssim\mathcal{O}(h^3)|u|_4|v|_1.$$

Hence,

$$A(u-u_I,v)=\frac{h^2}{24}\int_\Omega q\Pi^* v d\mathbf{x}+\mathcal{O}(h^3)\|u\|_4|v|_1.$$

This completes the proof of Lemma 3.6. □

Now, we shall begin to derive the asymptotic expansion.

The discrete Green function $g_h^z \in V_h$ satisfies

$$A(v_h,g_h^z)=v_h(z),\quad v_h\in V_h. \tag{3.27}$$

Lemma 3.7 (see [16,17]). *The discrete Green function $g_h^z\in V_h$ in (3.27) satisfies that*

$$|g_h^z|_1\lesssim |\ln h|^{\frac{1}{2}},\quad g_h^z\in V_h. \tag{3.28}$$

Lemma 3.8. *If $w\in H^2(\Omega)$, then, for any $0<\epsilon<1$, we have*

$$\|w-w_I\|_{0,\infty}\lesssim h^{1-\epsilon}\|w\|_2,$$

where w_I is defined by (2.5).

If $u\in H^4(\Omega)\cap H_0^1(\Omega)$, then we can introduce an auxiliary problem

$$\begin{cases} -\nabla\cdot(\kappa\nabla w)=q, & x\in\Omega,\\ w|_{\partial\Omega}=0, \end{cases} \tag{3.29}$$

where q is defined by (3.23).

From the regularity of solution for problem (3.29), we have

$$\|w\|_2\lesssim \|q\|_0\lesssim |u|_4. \tag{3.30}$$

The weak solution of problem (3.29) $w\in H^2(\Omega)\cap H_0^1(\Omega)$ satisfies

$$A(w,v)=(q,\Pi^* v),\quad \forall v\in H_0^1(\Omega). \tag{3.31}$$

Assume that $w_h\in V_h$ is the isoparametric bilinear finite volume element solution of problem (3.29). Then we have

$$A(w_h,v^h)=(q,\Pi^* v^h),\quad \forall v^h\in V_h. \tag{3.32}$$

Setting $v=v_h\in V_h$ in (3.31), and subtracting (3.32) from (3.31), we have

$$A(w-w_h,v^h)=0,\quad \forall v^h\in V_h. \tag{3.33}$$

Lemma 3.9. *If $u\in H^4(\Omega)\cap H_0^1(\Omega)$, then for any $0<\epsilon<1$, we have*

$$\|w-w_h\|_{0,\infty}\lesssim \mathcal{O}(h^{1-\epsilon})|u|_4.$$

Proof. Using the embedded inequality, the triangle inequality, the interpolation theory and Theorem 2.4, we have

$$\begin{aligned}\|w_I - w_h\|_{0,\infty} \lesssim& |\ln h|^{\frac{1}{2}} \|w_I - w_h\|_1 \\ \lesssim& |\ln h|^{\frac{1}{2}} (\|w_I - w\|_1 + \|w - w_h\|_1) \lesssim |\ln h|^{\frac{1}{2}} h \|w\|_2. \end{aligned} \tag{3.34}$$

By Lemma 3.8,

$$\|w - w_I\|_{0,\infty} \lesssim h^{1-\epsilon} \|w\|_2. \tag{3.35}$$

Combining (3.34), (3.35) with (3.30), we have

$$\begin{aligned}\|w - w_h\|_{0,\infty} \leq& \|w - w_I\|_{0,\infty} + \|w_I - w_h\|_{0,\infty} \\ \lesssim& h^{1-\epsilon} \|w\|_2 + |\ln h|^{\frac{1}{2}} h \|w\|_2 \lesssim h^{1-\epsilon} |u|_4. \end{aligned}$$

This completes the proof of Lemma 3.9. □

From Lemma 3.6 and (3.32), the following result holds.

Lemma 3.10. *Assume that $u \in H^4(\Omega) \cap H_0^1(\Omega)$ is the solution of* (2.1)*, u_h, $w_h \in V_h$, respectively, are solutions of* (2.6) *and* (3.32)*, and $u_I \in V_h$ is the interpolation function of u. Then we have*

$$A\Big(u_h - u_I - \frac{h^2}{24} w_h, v\Big) = \mathcal{O}(h^3) \|u\|_4 |v|_1, \quad \forall v \in V_h. \tag{3.36}$$

Theorem 3.1. *Under the hypotheses of Lemma* 3.10*, we have*

$$(u_h - u_I)(\mathbf{z}) = \frac{h^2}{24} w(\mathbf{z}) + \mathcal{O}(h^{3-\epsilon}) \|u\|_4, \tag{3.37}$$

where $\mathbf{z}$ is an inner node and $0 < \epsilon < 1$ is an arbitrary constant.

Proof. From (3.27), Lemma 3.10 and Lemma 3.7, we have

$$\begin{aligned}\Big(u_h - u_I - \frac{h^2}{24} w_h\Big)(\mathbf{z}) =& A\Big(u_h - u_I - \frac{h^2}{24} w_h, g_h^z\Big) \\ =& \mathcal{O}(h^3) \|u\|_4 |g_h^z|_1 = \mathcal{O}(h^3 |\ln h|^{\frac{1}{2}}) \|u\|_4. \end{aligned}$$

By Lemma 3.9,

$$(u_h - u_I)(\mathbf{z}) = \frac{h^2}{24} w(\mathbf{z}) + \mathcal{O}(h^{3-\epsilon}) \|u\|_4.$$

This completes the proof of Theorem 3.1. □

4. Superconvergence

We choose any inner node $P(x_1^i,x_2^i)$ whose four neighboring nodes are $P_1(x_1^i,x_2^i-h)$, $P_2(x_1^i+h,x_2^i)$, $P_3(x_1^i,x_2^i+h)$, $P_4(x_1^i-h,x_2^i)$ (see Fig. 3). Let $u_h^l=u_h(P_l)$, $(l=1,2,3,4)$ and define

$$\overline{\partial}_{x_1}^h u(P):=\frac{u_h^2-u_h^1}{2h},\quad \overline{\partial}_{x_2}^h u(P):=\frac{u_h^4-u_h^3}{2h}, \tag{4.1}$$

to be the average partial derivative values at point P along x_1 and x_2 directions,respectively. Then we define the discrete average gradient operator

$$\overline{\nabla}_h u(P)=(\overline{\partial}_{x_1}^h u,\overline{\partial}_{x_2}^h u)(P),\quad \mathit{and}\quad |\overline{\nabla}_h u(P)|=((\overline{\partial}_{x_1}^h u(P))^2+(\overline{\partial}_{x_2}^h u(P))^2)^{\frac{1}{2}}. \tag{4.2}$$

Similar to the proof of Lemma 3.8 or that in [22], we can prove that the following result holds true.

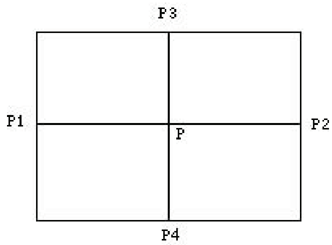

Figure 3: The four nodes and elements neighboring P.

Lemma 4.1. *Assume that* $u\in H^4(\Omega)\cap H_0^1(\Omega)$ *is the solution of* (2.1)*, then, for any inner node* $P(x_1^i,x_2^i)$*, we have*

$$|\overline{\nabla}_h(u-u_I)(P)|=\mathcal{O}(h^{2-\epsilon})\|u\|_4. \tag{4.3}$$

Lemma 4.2. *Assume that* $u\in H^4(\Omega)\cap H_0^1(\Omega)$ *is the solution of* (2.1) *and* $u_h\in V_h$ *is the isoparametric bilinear finite volume element solution. Then we have*

$$|\overline{\nabla}_h(u_h-u_I)(P)|=\mathcal{O}(h^{2-\epsilon})\|u\|_4, \tag{4.4}$$

where

$$\overline{\nabla}_h(u_h-u_I)(P)=\big(\overline{\partial}_{x_1}^h(u_h-u_I)(P),\overline{\partial}_{x_1}^h(u_h-u_I)(P)\big).$$

Proof. By Theorem 3.1,

$$\overline{\nabla}_h(u_h-u_I)(P)=\frac{h^2}{24}\overline{\nabla}_h w(P)+\mathcal{O}(h^{2-\epsilon})\|u\|_4. \tag{4.5}$$

Noting the fact $H^2(\Omega)\hookrightarrow C^{1-\epsilon}(\Omega)$ when $w\in H^2(\Omega)$, and using (3.30), we can obtain

$$\left|\overline{\partial}_{x_1}^h w(P)\right|=\left|\frac{w_2-w_1}{2h}\right|\lesssim h^{-\epsilon}\|w\|_2\lesssim h^{-\epsilon}|u|_4.$$

In the same way, we have

$$|\overline{\partial}_{x_2}^h w(P)| \lesssim h^{-\epsilon}|u|_4.$$

Hence,

$$|\overline{\nabla}_h w(P)| = \mathcal{O}(h^{-\epsilon})|u|_4. \tag{4.6}$$

Substituting (4.6) into (4.5), one can easily see that (4.4) holds. This completes the proof of Lemma 4.2. □

From Lemmas 4.1 and 4.2, and noting

$$|(\overline{\nabla}_h u - \nabla u)(P)| \lesssim h^2|u|_4,$$

one can obtain the following superconvergence result.

Theorem 4.1. *Assume that $u \in H^4(\Omega)\cap H_0^1(\Omega)$ is the solution of problem* (2.1) *and $u_h \in V_h$ is solution of* (2.6)*. Then, for any inner node P and any constant $0 < \epsilon < 1$, we have*

$$|(\overline{\nabla}_h u_h - \nabla u)(P)| = \mathcal{O}(h^{2-\epsilon})\|u\|_4. \tag{4.7}$$

5. Numerical experiment

Example 5.1. We consider the problem (2.1) and take

$$\Omega = (0,1)^2, \quad \kappa(\mathbf{x}) = 1 + x_1 + x_2, \quad f(\mathbf{x}) = 4\pi^2 \sin(\pi x_1)\sin(\pi x_2).$$

Let Ω^h be an uniform quadrilateral partition, N_1 and N_2 be, respectively, the partition numbers along x_1 and x_2 axis directions. The scheme (2.6) is employed. We display some results on four typical inner points $P_1(0.125,0.125)$, $P_2(0.625,0.125)$, $P_3(0.75,0.25)$, $P_4(0.25,0.75)$ in Tables 1 and 2, where $e_1^h = |(\partial_{x_1}u - \overline{\partial}_{x_1}^h u_h)(P_i)|$, $e_2^h = |(\partial_{x_2}u - \overline{\partial}_{x_2}^h u_h)(P_i)|$, $(1 \le i \le 4)$ are, respectively, the errors of partial derivatives about variables x_1 and x_2 at point P_i, and γ is the rate.

From Tables 1 and 2, one can see that the approximations are convergent of order two about the partial derivatives. It confirms the result in Theorem 4.1.

Table 1: Numerical results about P_1, P_2.

	P_1(0.125,0.125)				P_2(0.625,0.125)			
$N_1 \times N_2$	e_1^h	γ	e_2^h	γ	e_1^h	γ	e_2^h	γ
8×16	2.41e-3		2.37e-2		2.80e-5		6.59e-2	
16×32	7.04e-4	3.42	5.89e-3	4.02	3.78e-6	7.41	1.72e-2	3.83
32×64	1.82e-4	3.87	1.47e-3	4.01	7.46e-7	5.06	4.29e-3	4.01
64×128	4.59e-5	3.97	3.67e-4	4.01	1.51e-7	4.94	1.07e-3	4.01

Table 2: Numerical results about P_3, P_4.

	$P_3(0.75,0.25)$				$P_4(0.25,0.75)$			
$N_1 \times N_2$	e_1^h	γ	e_2^h	γ	e_1^h	γ	e_2^h	γ
8×16	7.74e-3		4.37e-2		1.21e-2		3.75e-2	
16×32	1.92e-3	4.03	1.08e-2	4.05	3.06e-3	3.95	9.28e-3	4.04
32×64	4.79e-4	4.01	2.69e-3	4.01	7.68e-4	3.98	2.32e-3	4.00
64×128	1.19e-4	4.02	6.72e-4	4.01	1.92e-4	4.00	5.77e-4	4.02

Acknowledgments This work was partially supported by NSFC Project (Grant No. 11031006, 91130002, 11171281), the Key Project of Scientific Research Fund of Hunan Provincial Science and Technology Department (Grant No. 2011FJ2011), and Specialized research Fund for the Doctoral Program of Higher Education (Grant No. 20124301110003), and Program for Changjiang Scholars and Innovative Research Team in University of China (No. IRT1179), and Hunan Provincial Natural Science Foundation of China (Grant No. 12JJ3010). The authors also thank the referees for their valuable suggestions.

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