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A Method for Solving the Inverse Scattering Problem for Shape and Impedance of Crack

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Abstract. The inverse problem considered in this paper is to determine the shape and the impedance of crack from a knowledge of the time-harmonic incident field and the corresponding far field pattern of the scattered waves in two-dimension. The combined single- and double-layer potential is used to approach the scattered waves. As an important feature, this method does not require the solution of u and $\partial u/\partial \nu$ at each iteration. An approximate method is presented and the convergence of this method is proven. Numerical examples are given to show that this method is both accurate and simple to use.

AMS subject classifications: 34L25, 81U40

Key words: Impedance boundary condition, Helmholtz equation, inverse scattering problem.

1. Introduction

The inverse scattering problem for electromagnetic time-harmonic plane wave by very thin obstacles has been considered in a series of papers [1–3]. Among these papers, the Dirichlet and Neumann crack problem has been solved. In the paper [4], the inverse problem considered is to determine the shape and the impedance of an obstacle from a knowledge of the time-harmonic incident field and the phase and amplitude of the far field pattern of the scattered wave in two-dimension. In this paper we are interested in numerical methods for determining the shape and impedance for crack from the knowledge of the incident field and the scattered field of the far field pattern. The difference is that the closed boundary curve is considered in paper [4] but a non-intersecting arc is considered in this paper. And the combined potential is put forward.

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In comparison with [1], our method considers the impedance problem. In paper [5], the same problem is considered. But in this paper, using the combined single and double layer potentials to approach the scattered field u^s, the problem is changed to a minimization problem. Furthermore, our reconstructions do not require the solution of the function u and its normal derivative $\partial u/\partial \nu$ at each iteration step and only require the nonzero initials of φ, Γ, λ. We approximate the functions ψ, z and λ by finite trigonometric series.

Let $\Gamma \subset \mathbb{R}^2$ be a non-intersecting C^3-smooth arc, i.e.,

$$\Gamma = \big\{z(t) : t \in [-1,1],\ z \in C^3[-1,1] \ \text{ and } \ |z'(t)| \neq 0, \ \ \forall t \in [-1,1]\big\}.$$

By z_1, z_{-1} we denote the two end points $z_1 := z(1)$ and $z_{-1} := z(-1)$ of Γ and set $\Gamma_0 := \Gamma \setminus \{z_{-1}, z_1\}$. Assuming an orientation for Γ from z_{-1} to z_1, by Γ_+ and Γ_- we denote the left- and right-hand sides of Γ, respectively, and by ν the unit normal vector to Γ directed towards Γ_+. Let the incident field u^i be given by $u^i(x) = \exp[ikx \cdot d]$, where $k > 0$ is the wave number and d is a fixed unit vector. The direct scattering problem consists of finding the total field $u = u^i + u^s$ such that both the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \ \mathbb{R}^2 \backslash \Gamma \tag{1.1}$$

and the impedance boundary condition

$$\frac{\partial u_\pm}{\partial \nu} \pm \mathrm{i}k\lambda u_\pm = 0 \quad \text{on } \ \Gamma_0 \tag{1.2}$$

are satisfied. To ensure uniqueness, the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left\{\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s\right\} = 0, \quad r = |x| \tag{1.3}$$

is imposed uniformly for all directions.

The radiation condition (1.3) ensures an behavior of the form

$$u^s(x) = \frac{e^{\mathrm{i}k|x|}}{\sqrt{|x|}}\left(u_\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right)\right), \quad |x| \to \infty \tag{1.4}$$

uniformly for all directions $\hat{x} = x/|x|$ (see [6]). The amplitude factor u_∞ is known as the far field pattern of the scattered wave u^s and defined in the unit circle $\Omega \subset \mathbb{R}^2$. The inverse problem we are concerned with is to determine the impedance and the shape of crack Γ from a knowledge of the far field pattern u_∞ for the incident wave u^i.

For the problem (1.1)-(1.3), there exists the following theorem.

Theorem 1.1 (see [7]). *The impedance crack problem has at most one solution.*

2. Mathematical analysis of the inverse scattering problem

In the paper [7], the direct problem is solved by the combined single- and double-layer potential. We are now in a position to present our method through the same potential. Let the single- and double-layer potential

$$v(x)=\int_\Gamma \Phi(x,y)\varphi_1(y)\mathrm{d}s(y)+\int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi_2(y)\mathrm{d}s(y), \quad x\in\mathbb{R}^2\setminus\Gamma \tag{2.1}$$

with densities $\varphi_1\in C(\Gamma)$ and $\varphi_2\in C^{1,\alpha}_{0,loc}(\Gamma)$ approach the scattered field u^s, where $\Phi(x,y)=\frac{\mathrm{i}}{4}H_0^{(1)}(k|x-y|)$ $(x\neq y)$ denotes the fundamental solution to the Helmholtz equation in two-dimension. From the asymptotic for $u^s(x)$, we see that the far-field pattern of the potential (2.1) is given by

$$u_\infty(\hat{x})=\frac{e^{-\mathrm{i}\pi/4}}{\sqrt{8\pi k}}\int_\Gamma \{\mathrm{i}\varphi_1(y)+k\hat{x}\cdot\nu(y)\varphi_2(y)\}e^{-\mathrm{i}k\hat{x}\cdot y}\mathrm{d}s(y). \tag{2.2}$$

For solving inverse problem, we would focus on the method as a numerical method for shape and impedance. Hence, for the given far-field pattern, we should solve the integral equation

$$(F\varphi_{1,2})(\hat{x})=u_\infty(\hat{x}), \tag{2.3}$$

where $F: L^2(\Gamma)\times L^2(\Gamma)\to L^2(\Omega)$ is defined by the form (2.2).

Then one tries to find the boundary Γ as the location where the boundary condition (1.2) is satisfied.

Now we have the following theorem:

Theorem 2.1. *The far-field patterns corresponding to an infinite number of plane waves with distinct directions uniquely determine the shape and location of the scatterer Γ and the impedance function λ.*

Proof. See [1, Theorem 3.1] and [3, Theorem 2]. □

Eq. (2.3) is an ill-posed problem, so we use the Tikhonov regularization method to solve this problem, that is for the regularization parameters α, $\beta>0$, find the solution $\varphi_{1,2;\alpha,\beta}\in L^2(\Gamma)$ satisfying

$$\begin{aligned}&\left\|F\varphi_{1,2;\alpha,\beta}-u_\infty\right\|^2_{L^2(\Omega)}+\alpha\left\|\varphi_{1;\alpha}\right\|^2_{L^2(\Gamma)}+\beta\left\|\varphi_{2;\beta}\right\|^2_{L^2(\Gamma)}\\ =&\inf_{\varphi_{1,2}\in L^2(\Gamma)\times L^2(\Gamma)}\left\{\left\|F\varphi_{1,2}-u_\infty\right\|^2_{L^2(\Omega)}+\alpha\left\|\varphi_1\right\|^2_{L^2(\Gamma)}+\beta\left\|\varphi_2\right\|^2_{L^2(\Gamma)}\right\}.\end{aligned} \tag{2.4}$$

Define

$$U:=\{\lambda: 0\leqslant\lambda\leqslant M_1,\ \ |x(t)-y(t)|\leqslant M_2,\ \ x,y\in\Gamma\},$$

where M_1 and M_2 are positive constants. From theorem Arzela-Ascoli, U is compact in $C(\Gamma)$. By the approach of the scattered wave

$$u^s_{\alpha,\beta}(x)=\int_\Gamma \Phi(x,y)\varphi_{1;\alpha}(y)\mathrm{d}s(y)+\int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi_{2;\beta}(y)\mathrm{d}s(y), \tag{2.5}$$

we should find Γ and λ, which minimize the impedance boundary condition

$$\inf_{(\Gamma,\lambda)\in C^3[-1,1]\times U}\left\|\frac{\partial}{\partial\nu}\big(u^i(\Gamma)+u^s_{\alpha,\beta}(\Gamma)\big)\pm \mathrm{i}k\lambda\big(u^i(\Gamma)+u^s_{\alpha,\beta}(\Gamma)\big)\right\|_{L^2(\Gamma_0)}.$$

Define operators

$$\begin{aligned}
&(S\varphi)(x):=2\int_\Gamma \varphi(y)\Phi(x,y)\mathrm{d}s(y), &&(K\varphi)(x):=2\int_\Gamma \varphi(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}\mathrm{d}s(y),\\
&(K'\varphi)(x)=2\int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(x)}\varphi(y)\mathrm{d}s(y), &&(T\varphi)(x)=2\frac{\partial}{\partial\nu(x)}\int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)\mathrm{d}s(y),
\end{aligned}$$

for $x\in\Gamma_0$. As in the case of boundary curves it can be obtained by partial integration using $\varphi(x_1)=\varphi(x_{-1})=0$.

Clearly, u^s satisfies the radiation condition. After rewriting the two boundary conditions (1.2) in the equivalent form of their difference and their sum, the jump relations also imply that u satisfies the boundary condition provided the densities $\varphi_{1;\alpha}$ and $\varphi_{2;\beta}$ solve the system of integral equations

$$2u^i-\varphi_{1;\alpha}+\mathrm{i}k\lambda S\varphi_{1;\alpha}+\mathrm{i}k\lambda K\varphi_{2;\beta}=0, \tag{2.6a}$$

$$2\frac{\partial u^i}{\partial\nu}+T\varphi_{2;\beta}+K'\varphi_{1;\alpha}+\mathrm{i}k\lambda\varphi_{2;\beta}=0. \tag{2.6b}$$

Then for the boundary Γ and impedance λ, we can define the minimization problem

$$\begin{aligned}
&\mu(\varphi_{1,2;\alpha,\beta},z,\lambda;\alpha,\beta)\\
=&\min_{(\varphi_{1,2;\alpha,\beta},z,\lambda)\in L^2(\Gamma)\times L^2(\Gamma)\times C^3[-1,1]\times U}\Big\{\big\|F\varphi_{1,2;\alpha,\beta}-u_\infty\big\|^2_{L^2(\Gamma)}+\alpha\big\|\varphi_{1;\alpha}\big\|^2_{L^2(\Gamma)}\\
&\quad+\beta\big\|\varphi_{2;\beta}\big\|^2_{L^2(\Gamma)}+\big\|(-\varphi_{1;\alpha}+\mathrm{i}k\lambda S\varphi_{1;\alpha}+\mathrm{i}k\lambda K\varphi_{2;\beta}+2\mathrm{i}k\lambda u^i)\big\|^2_{L^2(\Gamma_0)}\\
&\quad+\Big\|\Big(2\frac{\partial u^i}{\partial\nu}+T\varphi_{2;\beta}+K'\varphi_{1;\alpha}+\mathrm{i}k\lambda\varphi_{2;\beta}\Big)\Big\|^2_{L^2(\Gamma_0)}\Big\}.
\end{aligned}\tag{2.7}$$

3. Convergence analysis

In this section, we will consider a minimization problem that is related to the method presented in the previous section. To this end, we choose a closed and bounded subset $V\subset C^3[-1,1]$, which makes each $\xi\in V$ be an injective mapping $\xi:[-1,1]\to\mathbb{R}^2$ representing a crack Γ.

Definition 3.1. *Given the incident field u^i, a far field pattern u_∞ and regularization parameters $\alpha, \beta > 0$, a pair $(z_0, \lambda_0) \in V \times U$ will be called admissible if there exists $\varphi_{1,2;0} \in L^2(\Gamma) \times L^2(\Gamma)$ such that $(\varphi_{1,2;0}, z_0, \lambda_0)$ minimizes the expression in (2.7) over all $\varphi_{1,2} \in L^2(\Gamma) \times L^2(\Gamma)$, $z \in V$ and $\lambda \in U$, that is, we have*

$$\mu(z_0, \lambda_0, \varphi_{1,2;0}; \alpha, \beta) = m(\alpha, \beta),$$

where

$$m(\alpha, \beta) := \inf_{(\varphi_{1,2}, z, \lambda) \in L^2(\Gamma) \times L^2(\Gamma) \times V \times U} \mu(\varphi_{1,2}, z, \lambda; \alpha, \beta).$$

Theorem 3.1. *For each α, $\beta > 0$ there exists an optimal pair $(z_0, \lambda_0) \in V \times U$.*

Proof. We will follow [8], but have to take into account the additional regularization term on the length of Γ. Consider a minimizing sequence $(\varphi_{1,2;n}, z_n, \lambda_n)$ in $L^2(\Gamma) \times L^2(\Gamma) \times V \times U$, i.e.,

$$\lim_{n\to\infty} \mu(\varphi_{1,2;n}, z_n, \lambda_n; \alpha, \beta) = m(\alpha, \beta),$$

where

$$m(\alpha, \beta) = \inf_{(\varphi_{1,2}, z, \lambda) \in L^2(\Gamma) \times L^2(\Gamma) \times V \times U} \mu(\varphi_{1,2}, z, \lambda; \alpha, \beta).$$

As $V \times U$ is assumed to be closed and bounded, we can assume convergence $z_n \to z \in V$ as $n \to \infty$ with respect to the C^2-norm and $\lambda_n \to \lambda \in U$ as $n \to \infty$ with respect to the C-norm.

One also has that

$$\alpha \|\varphi_1\|^2_{L^2(\Gamma)} + \beta \|\varphi_2\|^2_{L^2(\Gamma)} \le \lim_{n\to\infty} \mu(\varphi_{1,2;n}, z_n, \lambda_n; \alpha, \beta) = m(\alpha, \beta)$$

as $n \to \infty$, so $\varphi_{1,2}$ is bounded and by a similar argument using the compact V one can assume that $\varphi_{1,2;n} \to \varphi_{1,2}$. By continuity of the function μ in all its variables, one has the result, since

$$\mu(\varphi_{1,2}, z, \lambda; \alpha, \beta) = \lim_{n\to\infty} \mu(\varphi_{1,2;n}, z_n, \lambda_n; \alpha, \beta) = m(\alpha, \beta).$$

The proof is complete. □

Theorem 3.2. *Let u_∞ be the far field pattern corresponding to the incident field u^i, $(z(t), \lambda(t)) \in V \times U$, then we have convergence of the cost functional*

$$\lim_{\alpha,\beta\to 0} m(\alpha, \beta) = 0.$$

Proof. From Theorem 1.1, the solution to the direct problem has unique solution. One knows that the solution can be represented by a combined single and double layer potentials. So there is $\|F\varphi_{1,2;\alpha,\beta} - u_\infty\|^2_{L^2(\Gamma)} = 0$. Since we can represent the solution for the boundary curve Γ via (1.2) through the unique solution φ of the integral equations (2.6a) and (2.6b), there are

$$\left\|(-\varphi_{1;\alpha} + ik\lambda S\varphi_{1;\alpha} + ik\lambda K\varphi_{2;\beta} + 2ik\lambda u^i)\right\|^2_{L^2(\Gamma_0)} = 0$$

and

$$\left\|\left(2\frac{\partial u^i}{\partial \nu}+T\varphi_{2;\beta}+K'\varphi_{1;\alpha}+\mathrm{i}k\lambda\varphi_{2;\beta}\right)\right\|^2_{L^2(\Gamma_0)}=0.$$

Therefore one has that

$$\lim_{\alpha,\beta\to 0} m(\alpha,\beta)=0,$$

since

$$m(\alpha,\beta)\le \mu(\varphi_{1,2},z,\lambda;\alpha,\beta)=\alpha\|\varphi_1\|^2_{L^2(\Gamma)}+\beta\|\varphi_2\|^2_{L^2(\Gamma)}. \tag{3.1}$$

The proof is complete. □

Theorem 3.3. *If the conditions of Theorem* 3.2 *is satisfies,* α_n, $\beta_n>0$, $n=1,2,\cdots$, *are sequences converging to zero,* $\{z_n,\lambda_n\}$ *are the admissible solutions corresponding to them, then* $z_n(t)\to z(t)$, $\lambda_n(t)\to\lambda(t)$ *as* $n\to\infty$.

Proof. There exits a convergent subsequence of (z_n,λ_n) by the proof of Theorem 3.1. We again denote by (z_n,λ_n), and $z_n\to z^*\in V$ and $\lambda_n\to\lambda^*\in U$. We want to show that $z^*=z$ and $\lambda^*=\lambda$.

Let u^* be the scattering waves corresponding to the boundary z^* and impedance λ^*, that is, it satisfies the boundary condition

$$\frac{\partial}{\partial\nu}\big(u^*(z^*)+u^i(z^*)\big)\pm \mathrm{i}k\lambda^*\big(u^*(z^*)+u^i(z^*)\big)=0\quad\text{on }\ \Gamma_0.$$

z_n, λ_n are the admissible solutions corresponding to α_n, β_n, and by Definition 3.1, there exists $\varphi_{1,2;n}\in L^2(\Gamma)$ such that

$$\mu(\varphi_{1,2;n},z_n,\lambda_n;\alpha,\beta)=m(\alpha_n,\beta_n).$$

By Theorem 3.2, these boundary data satisfy

$$\left\|F_{z_n,\lambda_n}\varphi_{1,2;n}-u_\infty\right\|^2_{L^2(\Gamma)}\to 0 \tag{3.2}$$

and

$$\left\|\frac{\partial}{\partial\nu}(u^s(z_n)+u^i(z_n))\pm \mathrm{i}k\lambda_n(u^s(z_n)+u^i(z_n))\right\|^2_{L^2(\Gamma_0)}\to 0, \tag{3.3}$$

both as $n\to\infty$.

From Theorem 1.1 and the expressions (2.2) and (3.3), the far field pattern $F\varphi_{1,2;n}$ of the combined double- and single-layer potential converges to the far field pattern u^*_∞ of u^*. By (3.2) we conclude that $u_\infty=u^*_\infty$ and so $u^s=u^*$ follows. Since it satisfies the condition of [9, Theorems 6.12, 6.13], this completes the proof of the theorem. □

4. Numerical examples

In this section, we shall discuss the numerical implementation of the algorithm presented in the previous section. The data for the inverse problem is the far field pattern for a variety of incoming waves and choices of the wave number k. In order to get the better result, the incoming waves are written as $u_N^i(x) = \sum_{p=1}^{N} e^{ikx\cdot d_p}$ and $u_\infty^N(\hat{x})$ is the far field pattern corresponding to it. For our examples, this data is generated by approximately solving the direct scattering problem [7].

In order to discretize the inverse problem (2.7), define $t = \cos(s)$ and the integrals are approximated using the trapezium rule with $s_j = j\pi/n$, $j = 0, 1, \cdots, n$ and $\psi_{1,2}(s) = \varphi_{1,2}(y)$. For $-1 < t < 1$ and the Maue's identity for operator T we can write

$$
\begin{aligned}
(S\varphi)(z(t)) &= \int_{-1}^{1} L_1(t,\tau)\varphi(z(\tau))\mathrm{d}\tau,\\
(K\varphi)(z(t)) &= \int_{-1}^{1} L_2(t,\tau)\varphi(z(\tau))\mathrm{d}\tau,\\
(K'\varphi)(z(t)) &= \frac{1}{|z'(t)|}\int_{-1}^{1} L_3(t,\tau)\varphi(z(\tau))\mathrm{d}\tau,\\
(T\varphi)(z(t)) &= \frac{1}{|z'(t)|}\int_{-1}^{1} \left\{\frac{1}{\pi}\frac{1}{\tau - t}\frac{\mathrm{d}}{\mathrm{d}\tau}\varphi(z(\tau)) + L_4(t,\tau)\varphi(z(\tau))\mathrm{d}\tau\right\},
\end{aligned}
$$

with the kernels

$$
\begin{aligned}
L_1(t,\tau) &:= \frac{\mathrm{i}}{2}H_0^{(1)}(k|z(t)-z(\tau)|)|z'(\tau)|,\\
L_2(t,\tau) &:= \frac{\mathrm{i}k}{2}H_1^{(1)}(k|z(t)-z(\tau)|)\frac{\{z(t)-z(\tau)\}(z_2'(\tau),-z_1'(\tau))}{|z(t)-z(\tau)|},\\
L_3(t,\tau) &:= |z'(\tau)|L_2(\tau,t),
\end{aligned}
$$

and

$$
\begin{aligned}
L_4(t,\tau) :=& -\frac{\mathrm{i}}{2}\frac{z'(t)\{z(t)-z(\tau)\}z'(\tau)\{z(t)-z(\tau)\}}{|z(t)-z(\tau)|^2}\\
&\times\left\{k^2H_0^{(1)}(k|z(t)-z(\tau)|) - \frac{2kH_1^{(1)}(k|z(t)-z(\tau)|)}{|z(t)-z(\tau)|}\right\}\\
&-\frac{\mathrm{i}k}{2}\frac{z'(t)z'(\tau)}{|z(t)-z(\tau)|}H_1^{(1)}(k|z(t)-z(\tau)|)\\
&-\frac{1}{\pi}\frac{1}{(\tau-t)^2} + \frac{\mathrm{i}k^2}{2}H_0^{(1)}(k|z(t)-z(\tau)|)z'(t)z'(\tau).
\end{aligned}
$$

By $t = \cos(s)$, the operators S, K, K', T are parameterized see [7]. Now the representation

(2.7) will be written to

$$
\begin{aligned}
\mu(\psi_{1,2},z,\lambda;\alpha,\beta) =& \sum_{q=0}^{L-1}\Bigg\| \frac{e^{-\mathrm{i}\pi/4}}{\sqrt{8\pi k}} \sum_{j=0}^{n} \mathrm{i}|z'(s_j)| e^{-\mathrm{i}k\hat{x}_q\cdot(z_1(s_j),z_2(s_j))} w_j \sin s_j \psi_1(s_j) \\
&+ \sum_{j=1}^{n-1} \frac{\pi}{n} k\hat{x}_q\cdot(z_1(s_j),z_2(s_j)) e^{-\mathrm{i}k\hat{x}_q\cdot(z_1(s_j),z_2(s_j))} \sin s_j \psi_2(s_j) - u_\infty(\hat{x}_q)\Bigg\|^2 \\
&+ \alpha \sum_{j=0}^{n} \|\psi_1(s_j)\|^2 + \beta \sum_{j=1}^{n-1} \|\psi_2(s_j)\|^2 \\
&+ \Bigg\| \sum_{j=0}^{n} \Big\{ -\psi_1(s_j) + \mathrm{i}k\lambda(s_j)S(s_j)\psi_1(s_j) + \mathrm{i}k\lambda(s_j)K(s_j)\psi_2(s_j) \\
&+ 2\mathrm{i}k\lambda \sum_{p=0}^{N-1} e^{\mathrm{i}kz(s_j)\cdot d_p} \Big\} \Bigg\|^2 + \Bigg\| \sum_{j=1}^{n-1} \Big\{ 2\sum_{p=0}^{N-1} \frac{\partial e^{\mathrm{i}kz(s_j)\cdot d_p}}{\partial \nu} + T(s_j)\psi_2(s_j) \\
&+ K'(s_j)\psi_1(s_j) + \mathrm{i}k\lambda(s_j)\psi_2(s_j) \Big\} \Bigg\|^2.
\end{aligned}
\tag{4.1}
$$

In order to discrete the inverse problem (4.1), we approximate the functions ψ, z and λ by finite trigonometric series

$$
\begin{aligned}
\psi_{1,2;\alpha,\beta}(s) &= \sum_{j=-n_1}^{n_1} g_{1,2;j} e^{\mathrm{i}js}, && g_{1,2;j} \in \mathbb{C}, \\
z_{\alpha,\beta}(s) &= a_0^{(n_2)} + \sum_{j=1}^{n_2} \big(a_j^{(n_2)} \cos js + b_j^{(n_2)} \sin js\big), && a_j^{(n_2)}, b_j^{(n_2)} \in \mathbb{R}, \\
\lambda_{\alpha,\beta}(s) &= a_0^{(n_3)} + \sum_{j=1}^{n_3} \big(a_j^{(n_3)} \cos js + b_j^{(n_3)} \sin js\big), && a_j^{(n_3)}, b_j^{(n_3)} \in \mathbb{R}.
\end{aligned}
$$

We now report on the examples we have computed. The approximate minimum of μ occurred at $k = 1.0$ and the fixed unit vector

$$
d_p = \begin{pmatrix} \cos(2\pi p/3), \\ \sin(2\pi p/3), \end{pmatrix}, \quad (p = 0,1,2).
$$

In our examples, the full line denotes graph of Γ or λ, and the broken line denotes graph of $\Gamma_{\alpha,\beta}$ or $\lambda_{\alpha,\beta}$.

Example 4.1. Exact figure: The curve:

$$
\Gamma = \Big(t, 0.5\cos\frac{\pi t}{2} + 0.2\sin\frac{\pi t}{2} - 0.1\cos\frac{3\pi t}{2}\Big).
$$

The impedance:

$$
\lambda = (1-t^2)^2.
$$

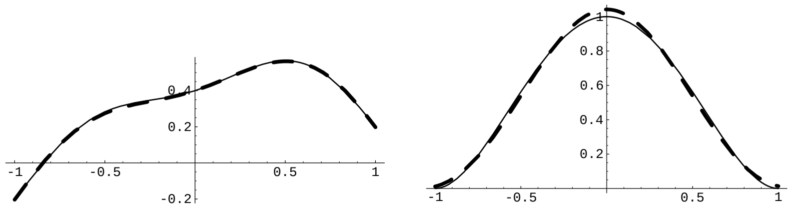

Figure 1: The numerical results for Example 4.1.

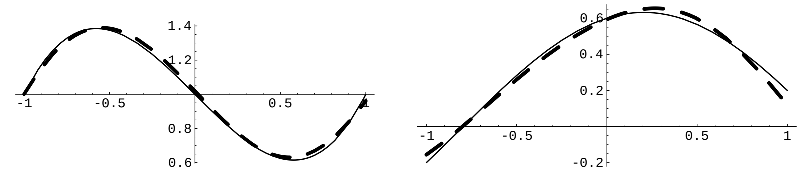

Figure 2: The numerical results for Example 4.2.

Parameters: Number of incoming waves: 3.
Inverse problem: $n_1 = 6$, $n_2 = 4$, $n_3 = 4$, $L = 6$, $\alpha = \beta = 1.0E - 10$.
The numerical results of $\Gamma_{\alpha,\beta}$ and $\lambda_{\alpha,\beta}$ are shown in Fig. 1.

Example 4.2. Exact figure: The curve:

$$\Gamma = (t, t^3 - t + 1).$$

The impedance:

$$\lambda = 0.6\cos\left(\frac{\pi}{2}t\right) + 0.2\sin\left(\frac{\pi}{2}t\right).$$

Parameters: Number of incoming waves: 3.
Inverse problem: $n_1 = 6$, $n_2 = 4$, $n_3 = 4$, $L = 6$, $\alpha = \beta = 1.0E - 10$.
The numerical results of $\Gamma_{\alpha,\beta}$ and $\lambda_{\alpha,\beta}$ are shown in Fig. 2.

Example 4.3. For this example, we consider the noisy data for the prior two example. The far field pattern is taken to be $u^\delta_\infty(\hat{x}_i, d) := (1 + ((-1)^i + 0.4)\delta)u_\infty(\hat{x}_i, d)$ with $\delta = 5\%$. The results (Figs. 3 and 4) show that our method is stable.

We want to point out that this method can be carried over the parametric type boundary value with

$$z_{1;\alpha,\beta}(s) = a_0^{(n_2^1)} + \sum_{j=1}^{n_2^1}\left(a_j^{(n_2^1)}\cos js + b_j^{(n_2^1)}\sin js\right), \quad a_j^{(n_2^1)}, b_j^{(n_2^1)} \in \mathbb{R},$$

$$z_{2;\alpha,\beta}(s) = a_0^{(n_2^2)} + \sum_{j=1}^{n_2^2}\left(a_j^{(n_2^2)}\cos js + b_j^{(n_2^2)}\sin js\right), \quad a_j^{(n_2^2)}, b_j^{(n_2^2)} \in \mathbb{R}.$$

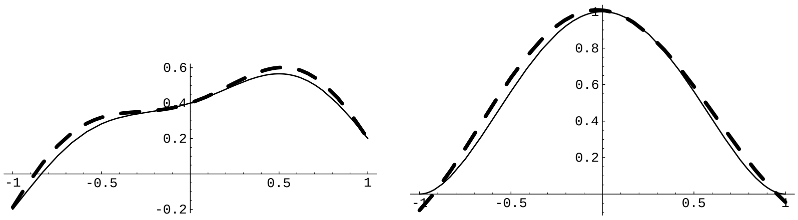

Figure 3: The noisy results for Example 4.1.

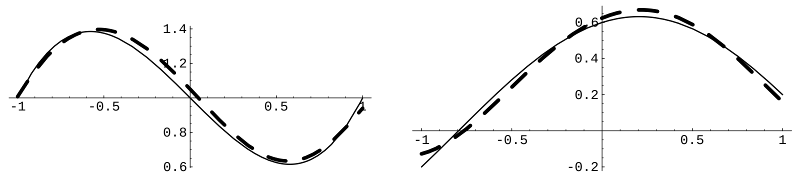

Figure 4: The noisy results for Example 4.2.

Our reconstructions do not require the solution of the direct scattering problem at each iteration step. For both examples we used as initial guess a curve at $z = 1.0$ and a constant impedance $\lambda = 1.0$. The examples imply that, as for the arbitrary form of the arc Γ and the impedance λ, our method is simple, accurate and fast.

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