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Superconvergence of a Galerkin FEM for Higher-Order Elements in Convection-Diffusion Problems

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Abstract. In this paper we present a first supercloseness analysis for higher-order Galerkin FEM applied to a singularly perturbed convection-diffusion problem. Using a solution decomposition and a special representation of our finite element space, we are able to prove a supercloseness property of $p+1/4$ in the energy norm where the polynomial order $p \geq 3$ is odd.

AMS subject classifications: 65N12, 65N30, 65N50

Key words: Singular perturbation, layer-adapted meshes, superconvergence, postprocessing.

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1. Introduction

Consider the convection dominated convection-diffusion problem

$$-\varepsilon\Delta u - (b\cdot\nabla)u + cu = f, \quad \text{in } \Omega = (0,1)^2, \tag{1.1a}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $c \in L_\infty(\Omega)$, $b \in W^1_\infty(\Omega)$, $f \in L_2(\Omega)$ and $0 < \varepsilon \ll 1$, assuming

$$c + \frac{1}{2}\text{div}\, b \geq \gamma > 0. \tag{1.2}$$

For a problem with exponential layers, i.e. in the case $b_1(x,y) \geq \beta_1 > 0$, $b_2(x,y) \geq \beta_2 > 0$, we have for linear or bilinear elements in the so called energy norm

$$|||v|||_\varepsilon^2 := \varepsilon\|\nabla v\|_0^2 + \|v\|_0^2,$$

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where $\|\cdot\|_0$ denotes the usual L_2-norm, on a Shishkin mesh (for the exact definition see Section 2)

$$\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon \lesssim N^{-1}\ln N.$$

We use the notation $a \lesssim b$, if a generic constant C independent of ε and N exists with $a \leq Cb$.

However, for bilinear elements Zhang [23] and Linß [13] observed a supercloseness property: the difference between the Galerkin solution u^N and the standard piecewise bilinear interpolant u^I of the exact solution u satisfies

$$\left|\left|\left|u^I - u^N\right|\right|\right|_\varepsilon \lesssim (N^{-1}\ln N)^2.$$

Supercloseness is a very important property. It allows optimal error estimates in L_2 (Nitsche's trick cannot be applied), improved error estimates in L_∞ inside the layer regions and recovery procedures for the gradient, important in a posteriori error estimation.

In the last ten years supercloseness for bilinear elements was also proved for problems with characteristic layers [6], for S-type meshes [13], for Bakhvalov meshes [15] and for several stabilisation methods, including streamline diffusion FEM (SDFEM), continuous interior penalty FEM (CIPFEM), local projection stabilisation FEM (LPS-FEM) and discontinuous Galerkin (see e.g. [3, 7–9, 17, 18, 21]). Recently, even corner singularities were included in the analysis [14].

For $\mathcal{Q}_p$-elements with $p \geq 2$ the situation is very different. Using the so-called vertex-edge-cell interpolant πu [11, 12] instead of the standard Lagrange-interpolant with equidistant interpolation points, Stynes and Tobiska [19] proved for SDFEM (but not for the Galerkin FEM)

$$\left|\left|\left|\pi u - \tilde{u}^N\right|\right|\right|_\varepsilon \lesssim N^{-(p+1/2)},$$

where $\tilde{u}^N$ denotes the SDFEM solution. It is not clear whether this estimate is optimal. The numerical results of [4, 5] indicate for the Galerkin FEM and $p \geq 3$ a supercloseness property of order $p+1$ for two different interpolation operators. One of them is the vertex-edge-cell interpolator πu, the other one is the Gauss-Lobatto interpolation operator $I^N u$. For SDFEM, the order $p+1$ is observed numerically for all $p \geq 2$.

In the present paper we study the Galerkin FEM for odd p. We shall prove some supercloseness properties, but the achieved order is probably not optimal.

The paper is organised as follows. In Section 2 we provide descriptions of the underlying mesh, the numerical method and a solution decomposition. The main part is Section 3 where the proof of our assertion can be found. As the proof is rather technical we provide it in full only for $p = 3$ and demonstrate its generalisation for arbitrary odd $p \geq 5$. In Section 4 we present some numerical simulations.

2. Mesh, method and a solution decomposition

We discretise the domain by a Shishkin mesh. Under the assumption

$$\varepsilon \leq \frac{\min\{\beta_1, \beta_2\}}{2\sigma \ln N}$$

we define the mesh-transition points by

$$\lambda_x := \frac{\sigma\varepsilon}{\beta_1} \ln N, \qquad \lambda_y := \frac{\sigma\varepsilon}{\beta_2} \ln N,$$

where $\sigma \geq p + 3/2$ is a user-chosen parameter. Let $\Omega_{11} = [\lambda_x, 1] \times [\lambda_y, 1]$, $\Omega_{12} = [0, \lambda_x] \times [\lambda_y, 1]$, $\Omega_{21} = [\lambda_x, 1] \times [0, \lambda_y]$, and $\Omega_{22} = [0, \lambda_x] \times [0, \lambda_y]$. The domain Ω is dissected by a tensor product mesh T^N, according to

$$x_i := \begin{cases} \frac{\sigma\varepsilon}{\beta_1} \ln N \frac{2i}{N}, & i = 0, \cdots, N/2, \\ 1 - 2(1-\lambda_x)(1-\frac{i}{N}), & i = N/2, \cdots, N, \end{cases}$$

$$y_j := \begin{cases} \frac{\sigma\varepsilon}{\beta_2} \ln N \frac{2j}{N}, & j = 0, \cdots, N/2, \\ 1 - 2(1-\lambda_y)(1-\frac{j}{N}), & j = N/2, \cdots, N. \end{cases}$$

Fig. 1 shows an example of T^N for (1.1). By h_i and k_j we denote the mesh sizes of a specific element $\tau_{ij} \in T^N$ in x- and y-direction, resp.

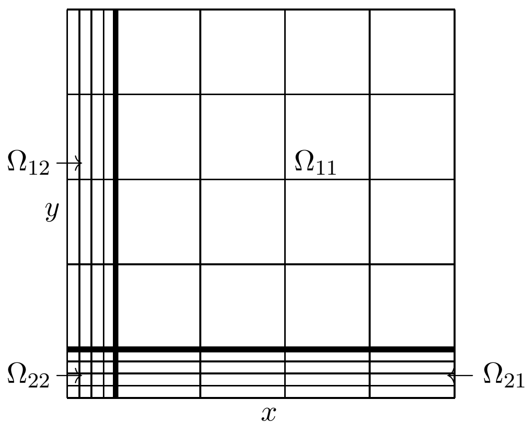

Figure 1: Shishkin mesh for Problem (1.1).

Our finite-element space $V^N \subset H_0^1(\Omega)$ on T^N is given by

$$V^N := \{v \in H_0^1(\Omega) : v|_\tau \in \mathcal{Q}_p(\tau), \forall \tau \in T^N\},$$

where $H_0^1(\Omega)$ is the standard Sobolev space $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$ with $v|_{\partial\Omega} = 0$ being understood in the sense of traces and $\mathcal{Q}_p(\tau)$ is the space of polynomials of degree at most p in each coordinate direction.

Then the Galerkin method can be written as: Find $u^N \in V^N$ such that

$$a_{Gal}(u^N, v^N) = (f, v^N), \quad \text{for all } v^N \in V^N,$$

where the bilinear form $a(\cdot,\cdot)$ is given by

$$a_{Gal}(v, w) := \varepsilon(\nabla v, \nabla w) + (cv - b \cdot \nabla v, w), \quad \text{for all } v,\, w \in H_0^1(\Omega),$$

and $(\cdot,\cdot)$ is the standard L_2-product in Ω.

Our analysis is based on a solution decomposition of u, which we provide here.

Assumption 2.1. *The solution u of problem* (1.1) *can be decomposed as*

$$u = S + E_{12} + E_{21} + E_{22},$$

where we have for all $x, y \in [0,1]$ and $0 \le i + j \le p + 2$ the pointwise estimates

$$\left.\begin{aligned} &\left|\frac{\partial^{i+j} S}{\partial x^i \partial y^j}(x,y)\right| \le C, \quad \left|\frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^j}(x,y)\right| \lesssim \varepsilon^{-i} e^{-\beta_1 x/\varepsilon}, \\ &\left|\frac{\partial^{i+j} E_{21}}{\partial x^i \partial y^j}(x,y)\right| \lesssim \varepsilon^{-j} e^{-\beta_2 y/\varepsilon}, \\ &\left|\frac{\partial^{i+j} E_{22}}{\partial x^i \partial y^j}(x,y)\right| \lesssim \varepsilon^{-(i+j)} e^{-\beta_1 x/\varepsilon} e^{-\beta_2 y/\varepsilon}. \end{aligned}\right\} \tag{2.1}$$

Here E_{12} and E_{21} are exponential boundary layers, E_{22} is a the corner layer, and S is the regular part of the solution.

For conditions that guarantee the existence of such a decomposition, see [16, Theorem III.1.26].

Remark 2.1. With Assumption 2.1 for $i + j \le p + 1$ we immediately have for $\mathcal{P}_p$- or $\mathcal{Q}_p$-elements

$$\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon \lesssim (N^{-1} \ln N)^p.$$

For $\mathcal{Q}_p$-elements this result follows from the proof given in [19] for the streamline-diffusion FEM.

3. Supercloseness analysis

Before we start the analysis, let us define the two interpolation operators πu and $I^N u$ precisely. Let $\hat{a}_i$ and $\hat{e}_i$, $i = 1, \cdots, 4$, denote the vertices and edges of the reference element $\hat{\tau} = [-1, 1]^2$, respectively. We define the *vertex-edge-cell interpolation* operator, [11, 12], $\hat{\pi} : C(\hat{\tau}) \to \mathcal{Q}_p(\hat{\tau})$ by

$$\hat{\pi}\hat{v}(\hat{a}_i) = \hat{v}(\hat{a}_i), \quad i = 1, \cdots, 4, \tag{3.1a}$$

$$\int_{\hat{e}_i} (\hat{\pi}\hat{v})\hat{q} = \int_{\hat{e}_i} \hat{v}\hat{q}, \quad i = 1, \cdots, 4, \quad \hat{q} \in \mathcal{P}_{p-2}(\hat{e}_i), \tag{3.1b}$$

$$\iint_{\hat{\tau}} (\hat{\pi}\hat{v})\hat{q} = \iint_{\hat{\tau}} \hat{v}\hat{q}, \qquad \hat{q} \in \mathcal{Q}_{p-2}(\hat{\tau}). \tag{3.1c}$$

This operator is uniquely defined and can be extended to the globally defined interpolation operator $\pi^N : C(\overline{\Omega}) \to V^N$ by

$$(\pi^N v)|_\tau := \big(\hat{\pi}(v \circ F_\tau)\big) \circ F_\tau^{-1} \quad \forall \tau \in T^N,\ \ v \in C(\overline{\Omega}),$$

with the bijective reference mapping $F_\tau : \hat{\tau} \to \tau$.

Let $-1 = t_0 < t_1 < \cdots < t_{p-1} < t_p = +1$ be the zeros of

$$(1-t^2)L_p'(t) = 0, \quad t \in [-1,1],$$

where L_p is the Legendre polynomial of degree p, normalised to $L_p(1) = 1$. These points are also used in the Gauß-Lobatto quadrature rule of approximation order $2p-1$. Therefore, we refer to them as Gauß-Lobatto points. We define the *Gauß-Lobatto interpolation* operator $\mathcal{I} : C(\hat{\tau}) \to \mathcal{Q}_p(\hat{\tau})$ by values at

$$(\mathcal{I}\hat{v})(t_i, t_j) := \hat{v}(t_i, t_j) \tag{3.2}$$

and extend it to the operator $I^N : C(\overline{\Omega}) \to V^N$ in the same way as above, see also [22].

Lemma 3.1. *For the interpolation operators $\pi^N : C(\overline{\Omega}) \to V^N$ and $I^N : C(\overline{\Omega}) \to V^N$ holds the stability property*

$$\left\|\pi^N w\right\|_{L_\infty(\tau)} + \left\|I^N w\right\|_{L_\infty(\tau)} \lesssim \|w\|_{L_\infty(\tau)} \quad \forall w \in C(\tau),\ \ \forall \tau \subset \overline{\Omega}, \tag{3.3}$$

and for $\tau_{ij} \subset \overline{\Omega}$ and $q \in [1,\infty]$, $2 \le s \le p+1$, $1 \le t \le p$ hold the anisotropic error estimates

$$\left\|w - \pi^N w\right\|_{L_q(\tau_{ij})} + \left\|w - I^N w\right\|_{L_q(\tau_{ij})} \lesssim \sum_{r=0}^{s} h_i^{s-r} k_j^r \left\|\frac{\partial^s w}{\partial x^{s-r}\partial y^r}\right\|_{L_q(\tau_{ij})}, \tag{3.4a}$$

$$\left\|(w - \pi^N w)_x\right\|_{L_q(\tau_{ij})} + \left\|(w - I^N w)_x\right\|_{L_q(\tau_{ij})} \lesssim \sum_{r=0}^{t} h_i^{t-r} k_j^r \left\|\frac{\partial^{t+1} w}{\partial x^{t-r+1}\partial y^r}\right\|_{L_q(\tau_{ij})}, \tag{3.4b}$$

and similarly for the y-derivative.

Proof. The proof can be found in [1, 10, 19].

Lemma 3.2. *For the interpolation operators $\pi^N : C(\overline{\Omega}) \to V^N$ and $I^N : C(\overline{\Omega}) \to V^N$ we have the interpolation error results*

$$\|u - \pi u\|_0 + \|u - I^N u\|_0 \lesssim (N^{-1}\ln N)^{p+1}, \tag{3.5a}$$

$$|||u - \pi u|||_\varepsilon + \left|\left|\left|u - I^N u\right|\right|\right|_\varepsilon \lesssim (N^{-1}\ln N)^p. \tag{3.5b}$$

Proof. The proof can be found in [1, 10, 19].

Let us come to the supercloseness analysis and denote by $J^N u \in V^N$ some interpolation of u. Then the analysis is based on a standard arguments involving coercivity and Galerkin orthogonality and yields

$$|||J^N u - u^N|||_\varepsilon^2 \lesssim a_{Gal}(J^N u - u^N, J^N u - u^N) = -a_{Gal}(u - J^N u, \chi), \tag{3.6}$$

where $\chi := J^N u - u^N \in V^N$. Thus one has to estimate

$$\varepsilon(\nabla(u - J^N u), \nabla\chi); \tag{3.7a}$$

$(b \cdot \nabla(u - J^N u), \chi)$ or equivalently using integration

$$\text{by parts } (u - J^N u, b \cdot \nabla\chi); \tag{3.7b}$$

$$(c(u - J^N u), \chi) \text{ or if integration by parts was used } ((c - \text{div} b)(u - J^N u), \chi). \tag{3.7c}$$

Lemma 3.3. *It holds that*

$$|(c(u - J^N u), \chi)| \lesssim (N^{-1} \ln N)^{p+1} \, |||\chi|||_\varepsilon \,. \tag{3.8}$$

Proof. Assuming J^N to be any of our two interpolation operators π^N or I^N, the L_2 interpolation-error estimate (3.5a) yields for the reaction term (3.7c)

$$|(c(u - J^N u), \chi)| \le \|c\|_{L_\infty(\Omega)} \|u - J^N u\|_0 \|\chi\|_0 \lesssim (N^{-1} \ln N)^{p+1} \, |||\chi|||_\varepsilon \,.$$

Similarly, the term involving $c - \text{div} b$ has the same bound. □

Lemma 3.4. *It holds that*

$$|\varepsilon(\nabla(u - \pi^N u), \nabla\chi)| \lesssim N^{-(p+1)} \, |||\chi|||_\varepsilon \,, \tag{3.9}$$

$$|\varepsilon(\nabla(u - I^N u), \nabla\chi)| \lesssim (N^{-1} \ln N)^{p+1} \, |||\chi|||_\varepsilon \,. \tag{3.10}$$

Proof. In the case of the vertex-edge-cell interpolation operator $\pi^N u$ we find in [19, Lemma 10] the estimate

$$|\varepsilon(\nabla(u - \pi^N u), \nabla\chi)| \lesssim N^{-(p+1/2)} \, |||\chi|||_\varepsilon \,.$$

A close inspection of the proof shows, that the only limiting term comes from [19, (3.16)]

$$\begin{aligned} N^{1/2} \|\pi^N E_{22}\|_{0,\Omega_{12} \cup \Omega_{21}} &\lesssim (\varepsilon(\varepsilon + N^{-1} \ln N))^{1/2} N^{-(\sigma - 1/2)} \\ &\lesssim (\varepsilon(\varepsilon + N^{-1} \ln N))^{1/2} N^{-(p+1/2)} \end{aligned}$$

because $\sigma \ge p + 1$ was chosen in [19]. All other terms involved are of order $p + 1$. In our paper we have $\sigma \ge p + 3/2$, and therefore (3.9) follows.

Let us denote by a subscript the polynomial order of the interpolation, i.e. we write I_p^N and π_p^N for the interpolation operators projecting into the FEM-spaces of order p.

In [5] we find the identity

$$I_p^N u = \pi_p^N u + \left(I_p^N(u - \pi_{p+1}^N u) - (u - \pi_{p+1}^N u))\right) + (u - \pi_{p+1}^N u)$$

also written as

$$I_p^N u = \pi_p^N u + Ru + (u - \pi_{p+1}^N u), \tag{3.11}$$

where $Ru := I_p^N(u - \pi_{p+1}^N u) - (u - \pi_{p+1}^N u))$. These are consequences of the basic identity

$$\pi_p^N = I_p^N \pi_{p+1}^N.$$

We apply (3.11) to the diffusion term (3.7a) and obtain

$$\varepsilon|(\nabla(u - I_p^N u), \nabla\chi)| \le \varepsilon|(\nabla(u - \pi_p^N u), \nabla\chi)| + \varepsilon|(\nabla(u - \pi_{p+1}^N u), \nabla\chi)| + \varepsilon|(\nabla Ru, \nabla\chi)|.$$

Now (3.9), the interpolation error result (3.5b) for $p+1$ and [5, Theorem 4.4], i.e.

$$\varepsilon^{1/2}\|\nabla Ru\|_0 \lesssim (N^{-1}\ln N)^{p+1}$$

prove (3.10). □

Now only the convective term (3.7b) has to be estimated. We will analyse it for the Gauß-Lobatto interpolation operator I^N. This estimate is the crucial point of the analysis. Stynes and Tobiska [19, Remark 16] state that the so called Lin-identities of [12, 20] do not yield bounds of order $p+1$. Instead, they use a fairly standard trick in the analysis of stabilised methods to obtain the order $p+1/2$ for the streamline-diffusion method and the vertex-edge-cell interpolation operator π^N.

Lemma 3.5. *It holds for any boundary layer function E of our decomposition $u = S + E_1 + E_2 + E_{12}$ that*

$$|(E - I^N E, b\cdot\nabla\chi)| \lesssim (N^{-1}\ln N)^{p+1} \left|\!\left|\!\left|\chi\right|\!\right|\!\right|_\varepsilon. \tag{3.12}$$

Proof. We will make use of the anisotropic interpolation error bounds (3.4a) and derive

$$\begin{aligned}
\|E_{12} - I^N E_{12}\|_{0,\Omega_{12}\cup\Omega_{22}}^2 &\lesssim \sum_{\tau_{ij}\subset\Omega_{12}\cup\Omega_{22}} \sum_{r=0}^{p+1} h_i^{s-r} k_j^r \left\| \frac{\partial^s E_{12}}{\partial x^{s-r}\partial y^r} \right\|_{0,\tau_{ij}}^2 \\
&\lesssim \sum_{r=0}^{p+1} (\varepsilon N^{-1}\ln N)^{2(s-r)} N^{-2r} \varepsilon^{2(r-s)} \left\| e^{-\beta_1 x/\varepsilon} \right\|_{0,\Omega_{12}\cup\Omega_{22}}^2 \\
&\lesssim \varepsilon (N^{-1}\ln N)^{2(p+1)}
\end{aligned}$$

while ideas from [19, Lemma 9] can be applied to obtain

$$\begin{aligned}
\|E_{12} - I^N E_{12}\|_{0,\Omega_{11}} &\lesssim \|E_{12}\|_{0,\Omega_{11}} + \|I^N E_{12}\|_{0,\Omega_{11}} \\
&\lesssim \varepsilon^{1/2} N^{-\sigma} + (\varepsilon^{1/2} + N^{-1/2}) N^{-\sigma} \lesssim (\varepsilon^{1/2} + N^{-1/2}) N^{-\sigma}
\end{aligned}$$

and finally a Hölder inequality, stability (3.3) and $\mathrm{meas}(\Omega_{21}) \lesssim \varepsilon \ln N$ yields

$$\begin{aligned}\|E_{12} - I^N E_{12}\|_{0,\Omega_{21}} &\lesssim \mathrm{meas}(\Omega_{21})^{1/2} \left(\|E_{12}\|_{L_\infty(\Omega_{21})} + \|I^N E_{12}\|_{L_\infty(\Omega_{11})}\right) \\ &\lesssim \varepsilon^{1/2}(\ln N)^{1/2} N^{-\sigma}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&|(E_{12} - I^N E_{12}, b \cdot \nabla\chi)| \\ &\lesssim \varepsilon^{1/2}((N^{-1}\ln N)^{p+1} + N^{-\sigma}(\ln N)^{1/2})\|\nabla\chi\|_{0,\Omega} + N^{-\sigma-1/2}\|\nabla\chi\|_{0,\Omega_{11}} \\ &\lesssim (N^{-1}\ln N)^{p+1}\,|||\chi|||_\varepsilon + N^{-\sigma+1/2}\|\chi\|_{0,\Omega_{11}} \lesssim (N^{-1}\ln N)^{p+1}\,|||\chi|||_\varepsilon\,,\end{aligned}$$

where $\sigma \geq p + 3/2$ and an inverse inequality was used in estimating on Ω_{11}. Similarly the other two layer terms can be estimated. □

Surprisingly, the real difficulty lies in the estimation of the convective term (3.7b) for the smooth part S, see also [22]. The following estimates are rather technical. Therefore we split the analysis and start with the one-dimensional case and the polynomial order $p = 3$. The generalisation into arbitrary odd order p and 2d follows. Some ideas of our proof go back 30 years to Axelsson and Gustafsson [2].

The basic idea is to use a special representation of a piecewise cubic function v with a basis consisting almost completely of functions that are symmetric w.r.t. their domain of support.

They are defined on the reference intervals with Legendre polynomials L_k normalised to $L_k(1) = 1$. We define the standard piecewise linear hat-function

$$\hat{\phi}(t) := \frac{1 - L_1(2|t| - 1)}{2} = 1 - |t| \qquad \text{for } t \in [-1, 1],$$

a quadratic bubble function

$$\hat{\chi}_2(t) := \frac{1 - L_2(2t - 1)}{2} = 3t(1 - t) \qquad \text{for } t \in [0, 1],$$

and a piecewise cubic bubble function

$$\hat{\psi}_3(t) := \frac{L_1(2|t| - 1) - L_3(2|t| - 1)}{2} = 5|t|(2|t| - 1)(|t| - 1) \qquad \text{for } t \in [-1, 1].$$

Fig. 2 shows the three basis functions on their respective reference intervals. Let us denote by F_i the piecewise linear mapping of $[-1, 1]$ onto $[x_{i-1}, x_{i+1}]$, such that $[-1, 0]$ is mapped linearly onto $[x_{i-1}, x_i]$ and $[0, 1]$ is mapped linearly onto $[x_i, x_{i+1}]$. Note that in general the mapping F_i is non-linear.

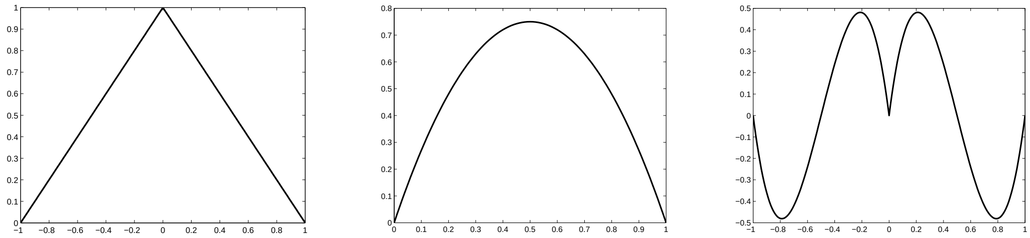

Figure 2: Basis function $\hat{\phi}$ (left), $\hat{\chi}_2$ (middle) and $\hat{\psi}_3$ (right) on their domains of support.

Above transformation and the functions on the reference intervals lead to the definition of the basis functions

$$\phi_i(x) = \begin{cases} \hat{\phi}(F_i^{-1}(x)), & x \in [x_{i-1}, x_{i+1}], \\ 0, & \text{otherwise}, \end{cases} \qquad i = 1, \cdots, N-1,$$
$$\chi_{2,i}(x) = \begin{cases} \hat{\chi}_2\left(\frac{x - x_{i-1}}{h_i}\right), & x \in [x_{i-1}, x_i], \\ 0, & \text{otherwise}, \end{cases} \qquad i = 1, \cdots, N,$$
$$\psi_{3,i}(x) = \begin{cases} \hat{\psi}_3(F_i^{-1}(x)), & x \in [x_{i-1}, x_{i+1}], \\ 0, & \text{otherwise}, \end{cases} \qquad i = 1, \cdots, N-1.$$

Finally, $\psi_{3,N}$ is the left part of $\hat{\psi}_3$ mapped onto $[x_{N-1}, 1]$.

Now we obtain for v the representation

$$v = \sum_{i=1}^{N-1} (v_i \phi_i + w_i \psi_{3,i}) + \sum_{i=1}^{N} y_j \chi_{2,j} + w_N \psi_{3,N}. \tag{3.13}$$

The functions ϕ_i, $\psi_{3,i}$ and $\chi_{2,j}$ are all symmetric w.r.t. their domain of support, with only a few exceptions. The last function $\psi_{3,N}$ is antisymmetric on $[x_{N-1}, 1]$, and $\phi_{N/2}$ and $\psi_{3,N/2}$ are in general not symmetric on a Shishkin mesh, as here two intervals with different sizes meet.

For a unique representation we still have to define the coefficients in (3.13). We use the following degrees of freedom

$$N_1^i v := v(x_i), \quad i = 1, \cdots, N-1, \tag{3.14a}$$
$$N_2^j v := \frac{\int_{x_{j-1}}^{x_j} L_2^j(x) v(x) dx}{\int_{x_{j-1}}^{x_j} L_2^j(x) \chi_{2,j}(x) dx}, \quad j = 1, \cdots, N, \tag{3.14b}$$
$$N_3^i v := \frac{\int_{x_{i-1}}^{x_i} L_3^i(x) v(x) dx}{\int_{x_{i-1}}^{x_i} L_3^i(x) \psi_{3,i}(x) dx}, \quad i = 1, \cdots, N, \tag{3.14c}$$

where L_k^i is the k-th Legendre polynomial L_k mapped onto $[x_{i-1}, x_i]$. Then it follows

$$v_i = N_1^i v, \quad y_j = N_2^j v, \quad w_i = \int_0^{x_i} \widetilde{L}_3 v,$$

where

$$\widetilde{L}_3\big|_{x_{k-1}}^{x_k} = \frac{L_3^k(x)}{\int_{x_{k-1}}^{x_k} L_3^k(x)\psi_{3,k}(x)dx}.$$

With the representation (3.13) we can write the L_2-norm of v as

$$\begin{aligned}\|v\|_0^2 = &\left\| \sum_{i=1}^{N-1} (v_i\phi_i + w_i\psi_{3,i}) \right\|_0^2 + \left\| \sum_{j=1}^{N} y_j\chi_{2,j} \right\|_0^2 + \|w_N\psi_{3,N}\|_0^2 \\ &+ 2\left(\sum_{i=1}^{N-1} v_i\phi_i, \sum_{j=1}^{N} y_j\chi_{2,j} \right) + 2\left(\sum_{i=1}^{N-1} v_i\phi_i, w_N\psi_{3,N} \right).\end{aligned}$$

All other scalar products involve the even functions $\chi_{2,j}$ and the functions $\psi_{3,i}$ that are either zero or odd on the support of $\chi_{2,j}$. Thus, those scalar products are zero. The two remaining scalar products can be rewritten as

$$\begin{aligned}\left(\sum_{i=1}^{N-1} v_i\phi_i, \sum_{j=1}^{N} y_j\chi_{2,j} \right) &= \sum_{i=1}^{N-1} v_i \left[y_i \int_{x_{i-1}}^{x_i} \phi_i\chi_{2,i} + y_{i+1} \int_{x_i}^{x_{i+1}} \phi_i\chi_{2,i+1} \right] \\ &= \frac{1}{4} \sum_{i=1}^{N-1} v_i \left[h_i y_i + h_{i+1} y_{i+1} \right], \\ \left(\sum_{i=1}^{N-1} v_i\phi_i, w_N\psi_{3,N} \right) &= v_{N-1}w_N \int_{x_{N-1}}^{x_N} \phi_{N-1}\psi_{3,N} = \frac{1}{12} v_{N-1}w_N h_N.\end{aligned}$$

Lemma 3.6. *Let $p = 3$ and consider the one-dimensional case. Then we obtain for the convective term in the smooth part S*

$$\left| \int_0^1 b(S - \hat{S})'v \right| \lesssim N^{-(3+1/4)} |||v|||_\varepsilon. \tag{3.15}$$

Proof. Let $\{x_i\}$ be a Shishkin mesh on $[0, 1]$, i.e.

$$x_i := \begin{cases} \frac{\sigma\varepsilon}{\beta_1} \ln N \frac{2i}{N}, & i = 0, \cdots, N/2, \\ 1 - 2(1 - \lambda_x)(1 - \frac{i}{N}), & i = N/2, \cdots, N, \end{cases}$$

and $h_i = x_i - x_{i-1}$ the local mesh size. We have to estimate

$$\int_0^1 b(S - \hat{S})'v, \tag{3.16}$$

where v is piecewise polynomial of degree $p = 3$ and $\hat{S}$ some Lagrange interpolant of S with $\hat{S} \in H_0^1(0,1)$. Later we will see that the estimates require some properties of the interior interpolation points that are fulfilled e.g. for the Gauß-Lobatto interpolation operator.

Now, using (3.13) and setting $\eta = S - \hat{S}$ we can rewrite (3.16) as

$$\int_0^1 b(S-\hat{S})'v$$
$$= \sum_{i=1}^{N-1} \int_{x_{i-1}}^{x_{i+1}} b\eta'(v_i\phi_i + w_i\psi_{3,i}) + \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} y_j b\eta'\chi_{2,j} + \int_{x_{N-1}}^{1} w_N b\eta'\psi_{3,N}. \tag{3.17}$$

In the two sums we will replace $b\eta'$ by

$$b\eta' = b_i\tilde{\eta}_i' + (b-b_i)\eta' + b_i(\eta - \tilde{\eta}_i)'$$

with constant $b_i = b(x_i)$ and $\tilde{\eta}_i$ defined in such a way that

- $\int\limits_{x_{i-1}}^{x_{i+1}} \tilde{\eta}_i'\phi_i = 0, \quad \int\limits_{x_{i-1}}^{x_{i+1}} \tilde{\eta}_i'\psi_{3,i} = 0,$ for $i \in \{1, \cdots, N-1\} \setminus \{N/2\}$,
- $\int\limits_{x_{i-1}}^{x_i} \tilde{\eta}_i'\chi_{2,i} = 0$ for $i = 1, \cdots, N$,
- $\|(\eta - \tilde{\eta}_i)'\|_{L_\infty(x_{i-1},x_{i+1})}$ is of order 4 in $h_i + h_{i+1}$ (compared to $\|\eta'\|_{L_\infty(x_{i-1},x_{i+1})}$ being of order 3).

We will show now, that such an $\tilde{\eta}_i$ exists. It is well known that the interpolation error $S - \hat{S} = \eta$ can be represented as

$$(S-\hat{S})(x) = \frac{S^{(4)}(\xi(x))}{4!}(x-x_{i-1})(x-\alpha_i)(x-\beta_i)(x-x_i)$$

if interpolated in x_{i-1}, α_i, β_i and x_i, where α_i and β_i are the interior interpolation points. Consequently,

$$(S-\hat{S})(x) = \frac{S^{(4)}(x_i)}{4!}(x-x_{i-1})(x-\alpha_i)(x-\beta_i)(x-x_i) + \mathcal{O}\left(h_i^5\right) \tag{3.18}$$

on $[x_{i-1}, x_i]$. Thus we set

$$\tilde{\eta}_i = \begin{cases} \frac{S^{(4)}(x_i)}{4!}(x-x_{i-1})(x-\alpha_i)(x-\beta_i)(x-x_i), & x \in [x_{i-1}, x_i], \\ \frac{S^{(4)}(x_i)}{4!}(x-x_i)(x-\alpha_{i+1})(x-\beta_{i+1})(x-x_{i+1}), & x \in [x_i, x_{i+1}]. \end{cases}$$

By the choice of the symmetric interior interpolation points of the Gauß-Lobatto interpolation, our approximation $\tilde{\eta}_i$ is an even function on the three intervals $[x_{i-1}, x_{i+1}]$,

$[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$. Therefore, $\tilde{\eta}_i'$ is an odd function on these intervals. Together with ϕ_i and $\psi_{3,i}$ being even on $[x_{i-1}, x_{i+1}]$ for $i \in \{1, \cdots, N-1\} \setminus \{N/2\}$ and $\chi_{2,i}$ being even on $[x_{i-1}, x_i]$ for any i, we obtain the first two wanted properties. The last property is due to (3.18).

Thus (3.17) can be rewritten as

$$\begin{aligned}\int_0^1 b(S-\hat{S})'v = & \int_{x_{N/2-1}}^{x_{N/2+1}} b_{N/2}\tilde{\eta}_i'(v_{N/2}\phi_{N/2} + w_{N/2}\psi_{3,N/2}) \\ & + \sum_{i=1}^{N-1}\int_{x_{i-1}}^{x_{i+1}} [(b-b_i)\eta' + b_i(\eta-\tilde{\eta}_i)'](v_i\phi_i + w_i\psi_{3,i}) \\ & + \sum_{j=1}^{N}\int_{x_{j-1}}^{x_j} [(b-b_j)\eta' + b_j(\eta-\tilde{\eta}_j)']y_j\chi_{2,j} \\ & + \int_{x_{N-1}}^{1} w_N b\eta'\psi_{3,N} =: I + II + III + IV. \end{aligned} \tag{3.19}$$

I: For the first term of (3.19) we obtain

$$\begin{aligned}|I| &= \left|\int_{x_{N/2-1}}^{x_{N/2+1}} b_{N/2}\tilde{\eta}_i'(v_{N/2}\phi_{N/2} + w_{N/2}\psi_{3,N/2})\right| \\ &\lesssim N^{-3}(x_{N/2+1} - x_{N/2-1})(|v_{N/2}| + |w_{N/2}|).\end{aligned}$$

A Cauchy-Schwarz inequality gives

$$v_{N/2} = \int_0^{\lambda_x} v' \lesssim \|v'\|_{L_1(0,\lambda_x)}. \tag{3.20}$$

For $w_{N/2}$ we recall

$$w_{N/2} = \int_0^{\lambda_x} \widetilde{L}_3 v = \sum_{k=1}^{N/2} \frac{\int_{x_{k-1}}^{x_k} L_3^k(x)v(x)dx}{\int_{x_{k-1}}^{x_k} L_3^k(x)\psi_{3,k}(x)dx}.$$

With L_3^k being odd on $[x_{k-1}, x_k]$ it holds

$$\begin{aligned}\int_{x_{k-1}}^{x_k} L_3^k(x)v(x)dx &= \int_{x_{k-1}}^{x_k} L_3^k(x)\frac{v(x) - v(x_k - (x - x_{k-1}))}{2}dx \\ &= \frac{1}{2}\int_{x_{k-1}}^{x_k} L_3^k(x)\int_{x_k-(x-x_{k-1})}^{x} v'(t)dtdx \\ &\le \frac{1}{2}\|L_3^k\|_{L_1[x_{k-1},x_k]}\|v'\|_{L_1[x_{k-1},x_k]}.\end{aligned}$$

Thus we have for $w_{N/2}$

$$|w_{N/2}| \lesssim \sum_{k=1}^{N/2} \frac{\|L_3^k\|_{L_1[x_{k-1},x_k]}}{\|L_3^k\|^2_{0,[x_{k-1},x_k]}}\|v'\|_{L_1[x_{k-1},x_k]} \lesssim \|v'\|_{L_1[0,\lambda_x]}. \tag{3.21}$$

Combining the estimates for the two coefficients yields

$$|I| \lesssim N^{-4}\|v'\|_{L_1(0,\lambda_x)} \lesssim N^{-4}(\varepsilon \ln N)^{1/2}\|v'\|_0 \lesssim N^{-4}(\ln N)^{1/2} |||v|||_\varepsilon . \tag{3.22}$$

II+III: It holds with the interpolation properties of $b - b_i$, η' and $(\eta - \tilde{\eta}_i)'$

$$\begin{aligned}(II + III)^2 &\le 2(II^2 + III^2)\\ &\lesssim N^{-8}\left[\|v\|_0^2 - \frac{1}{2}\sum_{i=1}^{N-1} v_i\left[h_i y_i + h_{i+1}y_{i+1}\right] - \frac{1}{6}v_{N-1}w_N h_N\right].\end{aligned}$$

The coefficients v_i, y_i and w_N can be bound by

$$\begin{aligned}&|h_i y_i| = |h_i N_2^i v| = h_i \left|\frac{\int_{x_{i-1}}^{x_i} L_2^i v}{\int_{x_{i-1}}^{x_i} L_2^i \chi_{2,i}}\right| \lesssim \|v\|_{L_1(x_{i-1},x_i)},\\ &|w_N| \le |w_{N/2}| + \|v'\|_{L_1(\lambda_x,1)} \lesssim (\ln N)^{1/2} |||v|||_\varepsilon + N\|v\|_0,\\ &|v_i| \lesssim \begin{cases}(\ln N)^{1/2} |||v|||_\varepsilon , & i \le N/2,\\ N\|v\|_{L_1(x_{i-1},x_{i+1})}, & j > N/2,\end{cases}\end{aligned}$$

where we have used (3.21) and an inverse inequality in the second line, and a similar reasoning to (3.20) and an inverse inequality in the last line. Thus, we obtain

$$\begin{aligned}(II + III)^2 &\lesssim N^{-8}\Bigg[\|v\|_0^2 + (\ln N)^{1/2} |||v|||_\varepsilon \|v\|_{L_1(0,x_{N/2+1})} + \sum_{i=N/2+1}^{N-1} N\|v\|^2_{L_1(x_{i-1},x_{i+1})}\\ &\qquad + \|v\|_{L_1(x_{N-2},x_N)}\big((\ln N)^{1/2} |||v|||_\varepsilon + N\|v\|_0\big)\Bigg]\\ &\lesssim N^{-8}\left[(\ln N)^{1/2} |||v|||^2_\varepsilon + N^{1/2}\|v\|_0^2\right] \lesssim N^{-(8-1/2)} |||v|||^2_\varepsilon . \end{aligned}\tag{3.23}$$

Therefore, we can conclude

$$|II + III| \lesssim N^{-(4-1/4)} |||v|||_\varepsilon . \tag{3.24}$$

IV: Finally, integration by parts, the bound on $|w_N|$ and the interpolation properties of η give

$$IV = -\int_{x_{N-1}}^1 w_N b'\eta\psi_{3,N} - \int_{x_{N-1}}^1 w_N b\eta\psi'_{3,N} \lesssim N^{-4}(|||v|||_\varepsilon + \|w_N\psi'_{3,N}\|_{L_1(x_{N-1},1)}).$$

For $\|w_N\psi'_{3,N}\|_{L_1(x_{N-1},1)}$ an inverse inequality gives

$$\begin{aligned}\|w_N\psi'_{3,N}\|_{L_1(x_{N-1},1)} &\lesssim N\|w_N\psi_{3,N}\|_{L_1(x_{N-1},1)} \lesssim N^{1/2}\|w_N\psi_{3,N}\|_{0,(x_{N-1},1)}\\ &\lesssim N^{1/2}\left(\|v\|_0^2 - \frac{1}{2}\sum_{i=1}^{N-1} v_i\left[h_i y_i + h_{i+1}y_{i+1}\right] - \frac{1}{6}v_{N-1}w_N h_N\right)^{1/2}\\ &\lesssim N^{1/2}N^{1/4} |||v|||_\varepsilon ,\end{aligned}$$

where the estimation of the scalar products in (3.23) was used. Together we obtain

$$|IV| \lesssim N^{-(4-3/4)} \, |||v|||_{\varepsilon} \,. \tag{3.25}$$

Combining (3.22), (3.24) and (3.25) finishes the proof. □

Lemma 3.7. *It holds for any odd $p \geq 3$ in the two-dimensional setting for the smooth part S that*

$$|(b \cdot \nabla(S - \hat{S}), v)| \lesssim N^{-(p+1/4)} \, |||v|||_{\varepsilon} \,. \tag{3.26}$$

Proof. For any odd polynomial degree p larger than three, we simply extend the approach of Lemma 3.6. On each interval $[x_{i-1}, x_i]$ we add even-order bubble functions $\chi_{2k,i}$, $k = 2, \cdots, (p-1)/2$. They are defined on $[0,1]$ by

$$\hat{\chi}_{2k}(t) := \frac{1 - L_{2k}(2t-1)}{2}$$

and mapped linearly onto $[x_{i-1}, x_i]$. On each double interval $[x_{i-1}, x_{i+1}]$ we add piecewise polynomial bubble functions $\psi_{2k+1,i}$, $k = 2, \cdots, (p-1)/2$, defined on the reference interval $[-1,1]$ by

$$\hat{\psi}_{2k+1}(t) := \frac{L_1(2|t|-1) - L_{2k+1}(2|t|-1)}{2}$$

and mapped by F_i. Fig. 3 shows in the case of $p = 5$ the two additional functions. Thus we obtain the representation

$$v = \sum_{i=1}^{N-1} v_i \phi_i + \sum_{k=1}^{(p-1)/2} \sum_{i=1}^{N-1} w_i^{2k+1} \psi_{2k+1,i} + \sum_{k=1}^{(p-1)/2} \sum_{i=1}^{N} y_j^{2k} \chi_{2k,j} + \sum_{k=1}^{(p-1)/2} w_n \psi_{2k+1,n}.$$

The new coefficients can be defined by using the degrees of freedom

$$N_{2k}^j v := \frac{\int_{x_{j-1}}^{x_j} L_{2k}^j(x) v(x) dx}{\int_{x_{j-1}}^{x_j} L_{2k}^j(x) \chi_{2k,j}(x) dx}, \qquad j = 1, \cdots, N,$$

$$N_{2k+1}^i v := \frac{\int_{x_{i-1}}^{x_i} L_{2k+1}^i(x) v(x) dx}{\int_{x_{i-1}}^{x_i} L_{2k+1}^i(x) \psi_{2k+1,i}(x) dx}, \quad i = 1, \cdots, N.$$

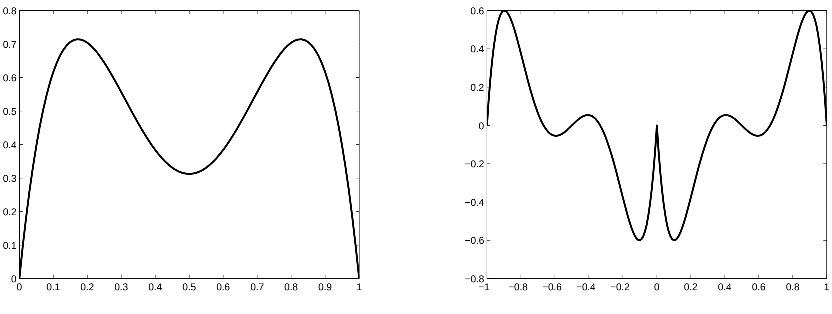

Figure 3: Additional basis functions $\hat{\chi}_4$ (left) and $\hat{\psi}_5$ (right) on their domains of support.

If we compare the new basis functions with the old ones $\chi_{2,i}$ and $\psi_{3,i}$, we notice a very similar behaviour. Thus, the same analytical steps can be applied and it follows for the convective term in S and any odd degree p

$$\int_0^1 b(S-\hat{S})'v \lesssim N^{-(p+1/4)} \left|\left|\left|v\right|\right|\right|_\varepsilon . \tag{3.27}$$

The extension to the two-dimensional problem is fairly easy. By the tensor-product structure of our problem, the mesh and the definitions of the norms, we obtain immediately from (3.27)

$$(b\cdot\nabla(S-\hat{S}),v) = (b_1(S-\hat{S})_x,v) + (b_2(S-\hat{S})_y,v) \lesssim N^{-(p+1/4)} \left|\left|\left|v\right|\right|\right|_\varepsilon .$$

This completes the proof. □

Consequently, by combining (3.6) and Lemmas 3.3-3.5 and 3.7 we have the main result of this paper.

Theorem 3.1. *For the Galerkin solution u^N of a finite element method of odd degree p holds*

$$\left|\left|\left|u^N - J^N u\right|\right|\right|_\varepsilon \lesssim (N^{-1}\ln N)^{p+1} + N^{-(p+1/4)},$$

where J^N is either the vertex-edge-cell interpolation operator π^N or the Gauß-Lobatto interpolation operator I^N.

Proof. By combining the previous Lemmas we have the main result for the Gauß-Lobatto interpolation operator immediately. For the vertex-edge-cell interpolation operator π^N we use the identity (3.11) and the ideas presented at the end of the proof of Lemma 3.4. □

Corollary 3.1. *With a suitable postprocessing operator P^N that maps the piecewise $\mathcal{Q}_p$-solution into a piecewise $\mathcal{Q}_{p+1}$-solution on a macro-mesh, a superconvergence property of the numerical solution $P^N u^N$*

$$\left|\left|\left|P^N u^N - u\right|\right|\right|_\varepsilon \lesssim (N^{-1}\ln N)^{p+1} + N^{-(p+1/4)}$$

can be deduced easily. For details and examples of suitable operators, see e.g. [5].

4. Numerical simulations

We consider for the numerical simulations the problem

$$-\varepsilon\Delta u - (2+x)u_x - (3+y^3)u_y + u = f, \quad \text{in } \Omega = (0,1)^2, \tag{4.1a}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{4.1b}$$

where the right-hand side f is chosen such that

$$u(x,y) = \cos(\pi x/2)(1-e^{-2x/\varepsilon})(1-y)^3(1-e^{-3y/\varepsilon}) \tag{4.1c}$$

Table 1: Convergence and supercloseness errors for $\mathcal{Q}_1$- and $\mathcal{Q}_3$-elements on a Shishkin mesh for $\varepsilon = 10^{-6}$ with corresponding rates r_N.

	$p=1$				$p=3$					
N	$\|\|\|u-u^N\|\|\|_\varepsilon$		$\|\|\|I^N u-u^N\|\|\|_\varepsilon$		$\|\|\|u-u^N\|\|\|_\varepsilon$		$\|\|\|\pi^N u-u^N\|\|\|_\varepsilon$		$\|\|\|I^N u-u^N\|\|\|_\varepsilon$	
8	3.39e-01	0.94	9.25e-02	2.01	2.85e-02	2.63	5.28e-03	3.55	7.37e-03	3.55
16	2.31e-01	0.97	4.10e-02	1.97	9.80e-03	2.79	1.25e-03	3.75	1.75e-03	3.74
24	1.78e-01	0.98	2.41e-02	1.97	4.62e-03	2.86	4.57e-04	3.83	6.39e-04	3.83
32	1.47e-01	0.99	1.62e-02	1.98	2.60e-03	2.91	2.12e-04	3.89	2.96e-04	3.89
48	1.10e-01	0.99	9.04e-03	1.99	1.10e-03	2.95	6.71e-05	3.94	9.39e-05	3.94
64	8.84e-02	1.00	5.88e-03	2.00	5.83e-04	2.97	2.87e-05	3.96	4.01e-05	3.96
96	6.47e-02	1.00	3.16e-03	2.00	2.30e-04	2.98	8.32e-06	3.98	1.16e-05	3.98
128	5.16e-02	1.00	2.01e-03	2.00	1.17e-04	2.99	3.38e-06	3.99	4.73e-06	3.99
192	3.73e-02	1.00	1.05e-03	2.00	4.44e-05	2.99	9.24e-07	3.99	1.29e-06	3.99
256	2.95e-02	1.00	6.55e-04	2.00	2.20e-05	3.00	3.62e-07	4.00	5.07e-07	4.00
384	2.11e-02	1.00	3.35e-04	2.00	8.06e-06		9.50e-08		1.33e-07	
512	1.66e-02	1.00	2.07e-04	2.00						
768	1.18e-02	1.00	1.04e-04	2.00						
1024	9.23e-03	1.00	6.39e-05	2.00						
1536	6.52e-03	1.00	3.18e-05	2.00						
2048	5.08e-03		1.93e-05							

is the exact solution. We use a fixed perturbation parameter $\varepsilon = 10^{-6}$. In Table 1 we present results for polynomial degrees $p = 1$ and $p = 3$ thus covering the lower order bilinear case analysed in [6] and the biqubic case covered by our analysis in Theorem 3.1. The experimental rates of convergence for given measured errors e_N are calculated by

$$r_N = \frac{\ln(e_N/e_{2N})}{\ln(2\ln(N)/\ln(2N))},$$

assuming $e_N = C(N^{-1}\ln N)^{r_N}$. All calculations were done in MATLAB using the backslash solver to solve the resulting linear systems.

As it can be seen, we observe in both cases convergence of $\mathcal{O}\left((N^{-1}\ln N)^p\right)$ and supercloseness of $\mathcal{O}\left((N^{-1}\ln N)^{p+1}\right)$ which is for $p = 1$ proved in [6] while for $p = 3$ it is better than the predicted rate of $\mathcal{O}\left(N^{-(p+1/4)}\right)$. Thus our analysis might not be sharp. For further numerical results we refer to [4, 5] that show numerically for *any* $p \geq 3$ a supercloseness property for the Galerkin method of order $p + 1$.

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