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Numerical Method for Singularly Perturbed Third Order Ordinary Differential Equations of Convection-Diffusion Type

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Abstract. In this paper, we have proposed a numerical method for Singularly Perturbed Boundary Value Problems (SPBVPs) of convection-diffusion type of third order Ordinary Differential Equations (ODEs) in which the SPBVP is reduced into a weakly coupled system of two ODEs subject to suitable initial and boundary conditions. The numerical method combines boundary value technique, asymptotic expansion approximation, shooting method and finite difference scheme. In order to get a numerical solution for the derivative of the solution, the domain is divided into two regions namely inner region and outer region. The shooting method is applied to the inner region while standard finite difference scheme (FD) is applied for the outer region. Necessary error estimates are derived for the method. Computational efficiency and accuracy are verified through numerical examples. The method is easy to implement and suitable for parallel computing.

AMS subject classifications: 65L10

Key words: Singularly perturbed problems, third order ordinary differential equations, boundary value technique, asymptotic expansion approximation, shooting method, finite difference scheme, parallel computation.

1. Introduction

Singular Perturbation Problems (SPPs) arise frequently in many fields like geophysical dynamics, oceanic and atmospheric circulation, chemical reactions, etc. The presence of small parameter(s) in these problems prevent(s) us from obtaining satisfactory numerical solutions using classical numerical methods. It is a well known fact that the solutions of the SPPs have multi-scale character. That is there are thin transition layers where the solution can jump abruptly, while away from the layers the solution

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behave regularly and varies slowly. Such problems have been investigated by many researchers. The existence and uniqueness of such problems are discussed in [4, 20]. In recent years a variety of numerical methods are available in the literature to solve SPBVPs for second order ODEs. For details,one may refer the survey article [8], but for higher order equations only few results are reported in the literature.

Analytical treatment of SPBVPs for the higher order non-linear ODEs which have important applications in fluid dynamics is discussed in [3, 10, 16, 20, 29]. Niederdrenk and Yserentant [16] have considered convection-diffusion type problems and derived conditions for the uniform stability of the discrete and continuous problems. Gartland [3] has shown that the uniform stability of the discrete BVP follows from the uniform stability of the discrete IVP and uniform consistency of the scheme. In [20], an iterative method is described.

Feckan [10] has considered higher order problems and his works are based on the non-linear analysis involving fixed point theory, Leray-Schauder theory, etc. In fact, Howes [6] has considered the higher order problems and discussed the existence, uniqueness and asymptotic estimates of the solution. In [20,29], a FEM for convection-reaction type problems is described.

As far as author's knowledge goes only few results are reported in the case of third order differential equations. Zhao [27] has considered a more general class of third order non-linear SPBVPs and discussed the existence, uniqueness of the solution and obtained asymptotic estimates using the theory of differential inequalities. In fact Zhao [28] has derived results on third order non-linear SPPs using differential inequality theorems. Howes [5] has considered class of third order SPBVP and discussed the existence, uniqueness and asymptotic behaviour of the solution. Roberts [18] has suggested a method of finding approximate solutions for third order SPODEs. Valarmathi [21–24] have suggested methods of finding approximate solutions for third order SPBVPs.

Following the Boundary Value Technique (BVT) of Roberts [18], Vigo-Aguiar [26], Valarmathi [21] and using the basic idea underlying the method suggested in Jayakumar [7] and Natesan [12] we in the present paper, suggest a new computational method which makes use of the zero order asymptotic expansion approximation, BVT and shooting method to obtain a numerical solution for the derivative of SPBVPs for third order ODEs of convection-diffusion type of the form:

$$\varepsilon y'''(x) + a(x)y''(x) - b(x)y'(x) - c(x)y(x) = f(x), \quad x \in \Omega, \tag{1.1}$$

$$y(0) = p, \quad -y''(0) = q, \quad y'(1) - y''(1) = r, \tag{1.2}$$

where $0 < \varepsilon \ll 1$, $a(x)$, $b(x)$, $c(x)$ are sufficiently smooth functions satisfying the following conditions:

$$a(x) \geq \alpha, \quad \alpha > 0, \tag{1.3}$$

$$b(x) > 0, \tag{1.4}$$

$$0 \geq c(x) \geq -\gamma, \quad \gamma > 0, \tag{1.5}$$

$$\alpha > \gamma, \tag{1.6}$$

with $\Omega = (0,1),\ \Omega^0 = (0,1],\ \bar{\Omega} = [0,1]$ and $y \in C^{(3)}(\Omega) \cap C^{(2)}(\bar{\Omega})$.

In order to get a numerical solution for the derivative of the solution of SPBVP (1.1)-(1.2), we divide the interval $[0,1]$ into two subintervals $[0,\tau]$ and $[\tau,1]$ called inner and outer region respectively.

An Initial Value Problem (IVP) associated with inner region is solved by shooting method and Boundary Value Problem (BVP) corresponding to the outer region is solved based on the standard finite difference scheme. The problems defined in the intervals $[0,\tau]$ and $[\tau,1]$ are independent of each other. Therefore, these problems can be solved simultaneously, that is, more suitable for parallel architectures. By this the computation time is very much reduced in comparison with the existing sequential algorithms which are often used to solve SPBVPs.

Through out the paper we use C, with or without subscript to denote a generic positive constant which is independent of N and ε. We use h_1 for mesh size for inner region problem and h_2 for mesh size for outer region problem. Error estimates are derived. Numerical examples are presented to illustrate the method. We define $||\cdot||$ of $\bar{w} = (w_1, w_2)^T \in \mathbb{R}^2$ as $||\bar{w}|| = \max\{|w_1|, |w_2|\}$.

2. Preliminaries

The SPBVP (1.1)-(1.2) can be transformed into an equivalent weakly coupled system of the form:

$$\begin{cases} P_1\bar{y}(x) \equiv -y_1'(x) + y_2(x) = 0, & x \in \Omega^0, \\ P_2\bar{y}(x) \equiv \varepsilon y_2''(x) + a(x)y_2'(x) - b(x)y_2(x) - c(x)y_1(x) = f(x), & x \in \Omega, \end{cases} \tag{2.1}$$

$$\begin{cases} R_1\bar{y} \equiv y_1(0) = p, \\ R_2\bar{y} \equiv -y_2'(0) = q, \\ R_3\bar{y} \equiv y_2(1) - y_2'(1) = r, \end{cases} \tag{2.2}$$

where $\bar{y} = (y_1, y_2)^T$, the functions $a(x), b(x), c(x)$ and $f(x)$ are sufficiently smooth functions satisfying the following conditions:

$$a(x) \geq \alpha, \quad \alpha > 0, \tag{2.3}$$

$$b(x) > 0, \tag{2.4}$$

$$0 \geq c(x) \geq -\gamma, \quad \gamma > 0, \tag{2.5}$$

$$\alpha > \gamma. \tag{2.6}$$

In this section, we present a maximum principle for the above problem. Using this principle, a stability result is derived. Further, an asymptotic expansion approximation is constructed for the solution and a theorem is presented to establish its accuracy.

Remark 2.1. The solution of the problem (2.1)-(2.2) exhibits a boundary layer at $x = 0$ which is less severe because the boundary conditions are prescribed for the derivative of the solution [20]. The condition (2.3) says that (2.1)-(2.2) is a non turning point problem. The condition (2.5) is known as the quasi monotonicity condition [20]. The maximum principle for the above system (2.1)-(2.2) and for the corresponding discrete problem are established using the conditions (2.3)-(2.6) and using this principle, we can establish a stability result.

2.1. Maximum principle and stability result

Theorem 2.1 (Maximum Principle)**.** *Consider the BVP* (2.1)-(2.2)*. Let* $y_1(0) \geq 0$*,* $y_2'(0) \leq 0$ *and* $y_2(1) - y_2'(1) \geq 0$*. Then* $P_1\bar{y}(x) \leq 0$*, for* $x \in \Omega^0$ *and* $P_2\bar{y}(x) \leq 0$*, for* $x \in \Omega$*, implies that* $\bar{y}(x) \geq 0$*,* $\forall x \in \bar{\Omega}$*.*

Proof. Define the test functions $\bar{s}(x) = (s_1(x), s_2(x))^T$ by

$$s_1(x) = 1 + x + \eta, \quad s_2(x) = 1 - x, \quad x \in \bar{\Omega} \quad \text{and} \quad 0 < \eta \ll 1/2.$$

Clearly, $s_1(0) = 1 + \eta > 0$, $s_2'(0) = -1 < 0$, $s_2(1) - s_2'(1) = 1 > 0$.

We can easily prove that $P_1\bar{s} < 0$ in Ω^0 and $P_2\bar{s} < 0$ in Ω.

Assume that theorem is not true. We define

$$\xi = \max\left\{\max_{x\in\bar{\Omega}}\left(-\frac{y_1}{s_1}\right)(x),\ \max_{x\in\bar{\Omega}}\left(-\frac{y_2}{s_2}\right)(x)\right\}.$$

Then, $\xi > 0$. Also $(y_1 + \xi s_1)(x) \geq 0$ and $(y_2 + \xi s_2)(x) \geq 0$, $\forall x \in \bar{\Omega}$. Furthermore, there exists a point $x_0 \in \bar{\Omega}$ such that

$$(y_1 + \xi s_1)(x_0) = 0 \quad \text{for} \ \ x_0 \in \Omega^0 \quad \text{or} \quad (y_2 + \xi s_2)(x_0) = 0 \quad \text{for} \ \ x_0 \in \Omega.$$

Case 1: $(y_1 + \xi s_1)(x_0) = 0$, for $x_0 \in \Omega^0$.

This implies that $y_1 + \xi s_1$ attains its minimum at $x = x_0$. Therefore,

$$0 > P_1(\bar{y} + \xi\bar{s})(x_0) = -(y_1 + \xi s_1)'(x_0) + (y_2 + \xi s_2)(x_0) \geq 0,$$

which is a contradiction.

Case 2: $(y_2 + \xi s_2)(x_0) = 0$, for $x_0 \in \Omega$.

This implies that $y_2 + \xi s_2$ attains its minimum at $x = x_0$. Therefore,

$$\begin{aligned}
0 &> P_2(\bar{y} + \xi\bar{s})(x_0) \\
&= \varepsilon(y_2 + \xi s_2)''(x_0) + a(x)(y_2 + \xi s_2)'(x_0) - b(x)(y_2 + \xi s_2)(x_0) - c(x)(y_1 + \xi s_1)(x_0) \\
&\geq 0,
\end{aligned}$$

which is a contradiction. Hence it can be concluded that $\bar{y}(x) \geq 0$, $\forall x \in \bar{\Omega}$. □

Lemma 2.1 (Stability Result)**.** *If* $\bar{y}(x)$ *is the solution of the BVP* (2.1)-(2.2) *then*

$$||\bar{y}(x)|| \leq C\max\left\{|y_1(0)|,\ |y_2'(0)|,\ |y_2(1) - y_2'(1)|,\ \max_{x\in\bar{\Omega}}|P_1\bar{y}(x)|,\ \max_{x\in\bar{\Omega}}|P_2\bar{y}(x)|\right\},\ \ \forall x \in \bar{\Omega}.$$

Proof. Set

$$M = C \max \Big\{ |y_1(0)|,\ |y_2'(0)|,\ |y_2(1) - y_2'(1)|,\ \max_{x\in\bar{\Omega}} |P_1\bar{y}(x)|,\ \max_{x\in\bar{\Omega}} |P_2\bar{y}(x)| \Big\}.$$

Define two barrier functions $\bar{w}^{\pm}(x) = (w_1^{\pm}(x), w_2^{\pm}(x))^T$ by

$$w_1^{\pm}(x) = M\{1 + x + \eta\} \pm y_1(x) \quad \text{and} \quad w_2^{\pm}(x) = M(1 - x) \pm y_2(x).$$

We have

$$\begin{aligned} P_1\bar{w}^{\pm}(x) &= -w_1^{\pm\prime}(x) + w_2^{\pm}(x) < -Mx \pm P_1\bar{y}(x) \le 0, \\ P_2\bar{w}^{\pm}(x) &= \varepsilon w_2^{\pm\prime\prime}(x) + a(x)w_2^{\pm\prime}(x) - b(x)w_2^{\pm}(x) - c(x)w_1^{\pm}(x) \\ &\qquad < M(\gamma - \alpha) \pm P_2\bar{y}(x) \le 0, \end{aligned}$$

by a proper choice of C. Furthermore, we have

$$\begin{aligned} &w_1^{\pm}(0) = M(1 + \eta) \pm y_1(0) \ge 0, \quad w_2^{\pm\prime}(0) = -M \pm y_2'(0) \le 0, \\ &w_2^{\pm}(1) - w_2^{\pm\prime}(1) = M \pm (y_2(1) - y_2'(1)) \ge 0, \end{aligned}$$

by a proper choice of C.

Applying Theorem 2.1 to the barrier functions $\bar{w}^{\pm}(x)$, we get the desired result. □

2.2. Asymptotic expansion approximation

We look for an asymptotic expansion solution of the BVP (2.1)-(2.2) in the form

$$\bar{y}(x, \varepsilon) = (\bar{u}_0(x) + \bar{v}_0(x)) + \varepsilon(\bar{u}_1(x) + \bar{v}_1(x)) + \mathcal{O}(\varepsilon^2).$$

By the method of stretching variable [11] one can obtain a zero order asymptotic approximation as $\bar{y}_{as}(x) = \bar{u}_0(x) + \bar{v}_0(x)$, where $\bar{u}_0(x)$ is the solution of the reduced problem of the BVP (2.1)-(2.2) given by

$$\begin{cases} -u_{0_1}'(x) + u_{0_2}(x) = 0, \\ a(x)u_{0_2}'(x) - b(x)u_{0_2}(x) - c(x)u_{0_1}(x) = f(x), \\ u_{0_1}(0) = p, \quad u_{0_2}(1) - u_{0_2}'(1) = r, \end{cases} \tag{2.7}$$

and $\bar{v}_0(x) = (v_{0_1}(x), v_{0_2}(x))^T$ is a layer correction term satisfies

$$\begin{cases} -v_{0_1}'(x) + v_{0_2}(x) = 0, \\ \varepsilon v_{0_2}''(x) + a(0)v_{0_2}'(x) = 0, \\ v_{0_1}(0) = -(\varepsilon/a(0))v_{0_2}(0), \quad v_{0_2}(0) = (q + u_{0_2}'(0)), \\ v_{0_2}(1) = \exp(-(a(0)/\varepsilon))v_{0_2}(0), \end{cases} \tag{2.8}$$

and this $\bar{v}_0(x)$ given by

$$\begin{cases} v_{0_1}(x) = -(\varepsilon/a(0))(q + u'_{0_2}(0))\exp(-(a(0)/\varepsilon))(x), \\ v_{0_2}(x) = (q + u'_{0_2}(0))\exp(-(a(0)/\varepsilon))(x). \end{cases} \tag{2.9}$$

The following theorem gives the bound for the difference between the solution of the BVP (2.1)-(2.2) and its zero order asymptotic expansion approximation.

Theorem 2.2. *The zero order asymptotic approximation* $\bar{y}_{as} = \bar{u}_0(x) + \bar{v}_0(x)$ *of the solution* $\bar{y}(x)$ *of the BVP* (2.1)-(2.2) *defined by* (2.7)-(2.9) *satisfies the inequality*

$$||\bar{y}(x) - \bar{y}_{as}(x)|| \leq C\varepsilon, \quad \forall x \in \bar{\Omega}.$$

Proof. It is easy to prove that

$$|(y_1 - y_{1as})(0)| \leq C\varepsilon, \quad |(y_2 - y_{2as})'(0)| = 0, \quad |(y_2 - y_2')(1) - (y_{2as} - y'_{2as})(1)| \leq Ce^{-\alpha/\varepsilon}.$$

Applying the differential operator on $\bar{y} - \bar{y}_{as}$ and using the fact that

$$te^{-t} \leq e^{-t/2}, \quad \forall t \geq 0,$$

we have

$$|P_1(\bar{y} - \bar{y}_{as})(x)| = 0 \quad \text{and} \quad |P_2(\bar{y} - \bar{y}_{as})(x)| \leq C\varepsilon + Ce^{-\alpha x/2\varepsilon}.$$

Define the barrier functions $\bar{\phi}^{\pm}(x) = (\phi_1^{\pm}(x), \phi_2^{\pm}(x))^T$ for $x \in \bar{\Omega}$ by

$$\phi_1^{\pm}(x) = C_1(1 + x + \eta)\varepsilon + C_2\varepsilon(x + 1) \pm (y_1 - y_{1as})(x)$$

where $0 < \eta \ll 1/2$ and

$$\phi_2^{\pm}(x) = C_1(1 - x)\varepsilon + C_2\varepsilon \pm (y_2 - y_{2as})(x),$$

where C_1 and C_2 are positive constants to be chosen suitably, so that the following expressions are satisfied:

$$\begin{aligned} &\phi_1^{\pm}(0) \geq 0, \quad {\phi_2}^{\pm\prime}(0) < 0, \quad \phi_2^{\pm}(1) - \phi_2^{\pm\prime}(1) \geq 0, \\ &P_1\bar{\phi}^{\pm}(x) = -\phi_1^{\pm\prime}(x) + \phi_2^{\pm}(x) < 0 \quad \text{for } x \in \Omega^0, \\ &P_2\bar{\phi}^{\pm}(x) = \varepsilon\phi_2^{\pm\prime\prime}(x) + a(x)\phi_2^{\pm\prime}(x) - b(x)\phi_2^{\pm}(x) - c(x)\phi_1^{\pm}(x) \leq 0 \quad \text{for } x \in \Omega. \end{aligned}$$

Applying Theorem 2.1 to the functions $\bar{\phi}^{\pm}(x)$, it follows that

$$\bar{\phi}^{\pm}(x) \geq 0, \quad \forall x \in \bar{\Omega},$$

and consequently,

$$||\bar{y}(x) - \bar{y}_{as}(x)|| \leq C\varepsilon, \quad \forall x \in \bar{\Omega}.$$

This completes the proof. □

Corollary 2.1. *If* $y_1(x)$ *is the solution of the BVP* (2.1)-(2.2) *and* $u_{0_1}(x)$ *is the solution of the problem* (2.7) *then* $|y_1(x) - u_{0_1}(x)| \leq C\varepsilon, \ \forall x \in \bar{\Omega}$.

Proof. From the above theorem, $|y_1(x) - (u_{0_1}(x) + v_{0_1}(x))| \leq C_1\varepsilon$. Consider

$$\begin{aligned}|y_1(x) - u_{0_1}(x)| &= |y_1(x) - u_{0_1}(x) + v_{0_1}(x) - v_{0_1}(x)|,\\ &\leq |y_1(x) - (u_{0_1}(x) + v_{0_1}(x))| + |v_{0_1}(x)|,\\ &\leq C_1\varepsilon + C_2\varepsilon,\\ &\leq C\varepsilon.\end{aligned}$$

This completes the proof. □

3. Estimates of derivatives

Theorem 3.1. *Let* $\bar{y}(x)$ *be the solution of the BVP* (2.1)-(2.2). *Then* $y_2(x)$ *satisfy*

$$|y_2^{(k)}(x)| \leq C(1 + \varepsilon^{-(k-1)} \exp(-\alpha x/\varepsilon)), \tag{3.1}$$

for $0 \leq k \leq 3$, $x \in \bar{\Omega}$.

Proof. Consider the BVP

$$\begin{aligned}&\varepsilon y_2''(x) + a(x)y_2'(x) - b(x)y_2(x) - c(x)y_1(x) = f(x),\\ &y_2'(0) = -q, \quad y_2'(1) - y_2(1) = r.\end{aligned}$$

Rewrite this BVP as

$$\begin{aligned}&\varepsilon y_2''(x) + a(x)y_2'(x) - b(x)y_2(x) = f(x) + c(x)y_1(x),\\ &y_2'(0) = -q, \quad y_2'(1) - y_2(1) = r.\end{aligned}$$

Then, $y_1 \in C^{(2)}(\bar{\Omega})$ and using the procedure adopted in [13] we have

$$|y_2^{(k)}(x)| \leq C(1 + \varepsilon^{-(k-1)} \exp(-\alpha x/\varepsilon)),$$

as required. □

4. Some analytical and numerical results

We state some results for the following SPBVP which are needed for the rest of the paper. Consider the auxiliary second order SPBVP

$$Ly_2^\star(x) \equiv \varepsilon y_2^{\star''}(x) + a(x)y_2^{\star'}(x) - b(x)y_2^\star(x) = f(x) + c(x)u_{0_1}(x), \quad x \in \Omega, \tag{4.1}$$

$$B_0 y_2^\star(0) \equiv -y_2^{\star'}(0) = q, \quad B_1 y_2^\star(1) \equiv y_2^\star(1) - y_2^{\star'}(1) = r, \tag{4.2}$$

where $u_{0_1}(x)$ is defined as in (2.7), $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth and $a(x) \geq \alpha$ and $b(x) > 0$, $0 \geq c(x) \geq -\gamma, \gamma > 0$.

4.1. Analytical results

Theorem 4.1 (Maximum Principle). *Consider the BVP* (4.1)-(4.2). *Let* $y_2^\star(x)$ *be a smooth function satisfying* $B_0 y_2^\star(0) \geq 0$, $B_1 y_2^\star(1) \geq 0$ *and* $L y_2^\star(x) \leq 0$ *for* $x \in \Omega$. *Then,* $y_2^\star(x) \geq 0$, $\forall x \in \bar{\Omega}$.

Proof. Please refer to Ref. [1].

Lemma 4.1 (Stability Result). *If* $y_2^\star(x)$ *is the solution of the BVP* (4.1)-(4.2) *then*

$$|y_2^\star(x)| \leq C \max\left\{ |B_0 y_2^\star(0)|,\ |B_1 y_2^\star(1)|,\ \max_{x\in\bar{\Omega}} |L y_2^\star(x)| \right\}, \quad \forall x \in \bar{\Omega}.$$

Proof. Define a barrier function $\Psi^\pm(x)$ to be $\Psi^\pm(x) = A'(1-\eta' x) \pm y_2^\star(x),\ x \in \bar{\Omega}$, where

$$A' = C \max\left\{ |B_0 y_2^\star(0)|,\ |B_1 y_2^\star(1)|,\ \max_{x\in\bar{\Omega}} |L y_2^\star(x)| \right\} \quad \text{and} \quad 0 < \eta' < 1.$$

It is easy to check that $B_0 \Psi^\pm(0) \geq 0$, $B_1 \Psi^\pm(1) \geq 0$ and $L\Psi^\pm(x) \leq 0$, for a proper choice of the constant C. Then applying Theorem 4.1 to $\Psi^\pm(x)$, the required stability bound is obtained. □

Theorem 4.2. *If* $\bar{y}(x)$ *and* $y_2^\star(x)$ *are solutions of the BVPs* (2.1)-(2.2) *and* (4.1)-(4.2) *respectively, then*

$$|y_2(x) - y_2^\star(x)| \leq C\varepsilon, \quad \forall x \in \bar{\Omega}.$$

Proof. The second component y_2 of the solution $\bar{y}(x)$ of the BVP (2.1)-(2.2) satisfies the BVP

$$\begin{aligned} &\varepsilon y_2''(x) + a(x) y_2'(x) - b(x) y_2(x) = f(x) + c(x) y_1(x), \quad x \in \Omega, \\ &-y_2'(0) = q, \quad y_2(1) - y_2'(1) = r. \end{aligned}$$

Further, the function $w(x) = y_2(x) - y_2^\star(x)$ satisfies the BVP

$$\begin{aligned} &\varepsilon w''(x) + a(x) w'(x) - b(x) w(x) = c(x)[y_1(x) - u_{0_1}(x)], \quad x \in \Omega, \\ &w'(0) = 0, \quad w(1) - w'(1) = 0. \end{aligned}$$

From the stability result given in [1], we have

$$|w(x)| \leq C |y_1(x) - u_{0_1}(x)|.$$

From Theorem 2.2,

$$|y_1(x) - y_{1as}(x)| \leq C\varepsilon \quad \text{or} \quad |y_1(x) - u_{0_1}(x) - v_{0_1}(x)| \leq C\varepsilon.$$

Then

$$|y_1(x) - u_{0_1}(x)| - |v_{0_1}(x)| \leq |y_1(x) - u_{0_1}(x) - v_{0_1}(x)|$$

implies that

$$|y_1(x) - u_{0_1}(x)| \leq |v_{0_1}(x)| + C\varepsilon \leq C\varepsilon.$$

Therefore $|w(x)| \leq C\varepsilon$. Hence,

$$|y_2(x) - y_2^\star(x)| \ \leq C\varepsilon.$$

This completes the proof. □

4.2. Description of the method

Step 1: An asymptotic approximation is derived for the solution of (2.1)-(2.2) which is given by (2.7)-(2.8).

Step 2: The first component of the solution $\bar{y}(x)$ of the BVP (2.1)-(2.2), namely y_1 is approximated by the first component of the solution of the reduced problem namely u_{0_1} given by (2.7). Then replacing y_1 appearing in the second equation of (2.1) by u_{0_1} and taking the same boundary values, one gets the auxiliary SPBVP (4.1)-(4.2). The solution of this problem is taken as an approximation to y_2 which is the second equation of (2.1) which has to be solved.

Step 3: In order to solve the auxiliary second order problem (4.1)-(4.2) numerically, we divide the interval $[0, 1]$ into two subintervals $[0, \tau]$ and $[\tau, 1]$ called inner and outer region respectively, where $\tau = \min\left\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln N\right\}$.

Step 4: In the inner region, the auxiliary problem (4.1)-(4.2) is solved by the shooting method using the initial conditions $\tilde{y}_2(0) = u_{0_2}(0) + v_{0_2}(0)$, $-\tilde{y}_2'(0) = q$.

Step 5: In the outer region, the DE (4.1) subject to boundary conditions $y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau)$, $y_2(1) - y_2'(1) = r$ is solved by the standard FD scheme.

Step 6: After solving both the inner region and outer region problems, we combine their solutions to obtain an approximate solution for the derivative of y_2 of the original problem (1.1)-(1.2) over the interval $\bar{\Omega}$.

4.3. Numerical schemes

4.3.1. Inner Region Problem

Consider the following IVP:

$$\begin{cases} \varepsilon\tilde{y}_2''(x) + a(x)\tilde{y}_2{}'(x) - b(x)\tilde{y}_2(x) = f(x) + c(x)u_{0_1}(x), \quad x \in (0, \tau], \\ \tilde{y}_2(0) = \bar{q} = u_{0_2}(0) + v_{0_2}(0), \quad -\tilde{y}_2'(0) = q. \end{cases} \tag{4.3}$$

This IVP is equivalent to the system

$$P^*\bar{y}^* = \begin{cases} P_1^*\bar{y}^* = -y_1^*{}'(x) + y_2^*(x) = 0, \\ P_2^*\bar{y}^* = \varepsilon y_2^*{}'(x) + a(x)y_2^*(x) - b(x)y_1^*(x) = f^*(x), \quad x \in (0, \tau], \\ y_1^*(0) = \bar{q}, \quad -y_2^*(0) = q, \end{cases} \tag{4.4}$$

where $f^*(x) = f(x) + c(x)u_{0_1}(x)$, $\bar{y}^* = (y_1^*, y_2^*)^T$, $a(x) \geq \alpha$, $\alpha > 0$, $b(x) > 0$, $0 \geq c(x) \geq -\gamma, \gamma > 0$.

Theorem 4.3 (Maximum Principle). *Consider the IVP* (4.4). *Let* $y_1^*(0) \geq 0$, $y_2^*(0) \leq 0$ *and* $P_1^*\bar{y}^*(x) \leq 0$, $P_2^*\bar{y}^*(x) \leq 0$, *for* $x \in (0, \tau]$. *Then,* $\bar{y}^*(x) \geq 0$, $\forall x \in [0, \tau]$.

Proof. Please refer to Ref. [30].

Lemma 4.2 (Stability Result). *If* $\bar{y}^*(x)$ *is the solution of the IVP* (4.4). *Then*

$$||\bar{y}^*(x)|| \leq C \max\Big\{|y_1^*(0)|,\ |y_2^*(0)|,\ \max_{x\in[0,\tau]} |P_1^*\bar{y}^*(x)|,\ \max_{x\in[0,\tau]} |P_2^*\bar{y}^*(x)|\Big\}, \quad \forall x \in \bar{\Omega}.$$

Proof. Defining two barrier functions $\bar{\chi}^{\pm}(x) = (\chi_1^{\pm}(x), \chi_2^{\pm}(x))^T$ by

$$\chi_1^{\pm}(x) = M'(1 + x) \pm y_1^*(x) \quad \text{and} \quad \chi_2^{\pm}(x) = M'(-1) \pm y_2^*(x),$$

where

$$M' = C \max\Big\{|y_1^*(0)|,\ |y_2^*(0)|,\ \max_{x\in[0,\tau]} |P_1^*\bar{y}^*(x)|,\ \max_{x\in[0,\tau]} |P_2^*\bar{y}^*(x)|\Big\}.$$

We have

$$\begin{aligned} P_1^*\bar{\chi}^{\pm}(x) &= -\chi_1^{\pm\prime}(x) + \chi_2^{\pm}(x) < -2M' \pm P_1^*\bar{y}^*(x) \leq 0, \\ P_2^*\bar{\chi}^{\pm}(x) &= \varepsilon\chi_2^{\pm\prime}(x) + a(x)\chi_2^{\pm}(x) - b(x)\chi_1^{\pm}(x) \pm P_2^*\bar{y}^*(x) \\ &\quad < -M'\alpha \pm P_2^*\bar{y}^*(x) \leq 0, \end{aligned}$$

by a proper choice of C. Furthermore, we have

$$\chi_1^{\pm}(0) = M' \pm y_1^*(0) \geq 0, \quad \chi_2^{\pm}(0) = -M' \pm y_2^*(0) \leq 0,$$

by a proper choice of C.

Applying Theorem 4.3 to the barrier functions $\bar{\chi}^{\pm}(x)$, we get the desired result. □

Theorem 4.4. *Let* $\bar{y}^*(x)$ *be the solution of the IVP* (4.4). *Then* $y_1^*(x)$ *and* $y_2^*(x)$ *satisfy*

$$|y_1^{*(k)}(x)| \leq C\varepsilon^{-(k-1)}, \quad |y_2^{*(k)}(x)| \leq C\varepsilon^{-(k)}, \quad \textit{for}\ \ 0 \leq k \leq 2, \ \ x \in (0, \tau].$$

Proof. For $k = 0$, the result follows from Lemma 4.2. From (4.4), it is evident that $|y_1^{*'}(x)| \leq C$ and $|y_2^{*'}(x)| \leq C\varepsilon^{-1}$. Differentiating the equations in (4.4) once and using the above estimates of $|y_1^{*'}(x)|$ and $|y_2^{*'}(x)|$, it is found that $|y_1^{*''}(x)| \leq C\varepsilon^{-1}$ and $|y_2^{*''}(x)| \leq C\varepsilon^{-2}$. □

Applying Shooting method for (4.4), we get

$$\begin{aligned} &P^{*N/2}\bar{y}_i^* \\ &= \begin{cases} P_1^{*N/2}\bar{y}_i^* = -D^-y_{1,i}^* + y_{2,i}^* = 0, \\ P_2^{*N/2}\bar{y}_i^* = \varepsilon D^-y_{2,i}^* + a(x_i)y_{2,i}^* - b(x_i)y_{1,i}^* = f^*(x_i), \quad i = 1:N, \\ y_{1,0}^* = \bar{q}, \quad -y_{2,0}^* = q, \end{cases} \end{aligned} \tag{4.5}$$

where

$$D^- y_{j,i}^* = (y_{j,i}^* - y_{j,i-1}^*)/h_1, \quad h_1 = \frac{2\tau}{N}, \quad x_i = x_{i-1} + ih_1, \quad j = 1,2, \quad i = 1(1)N.$$

Here, τ is the transition parameter given by $\tau = \min\left\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln N\right\}$. This fitted mesh is denoted by $\bar{\Omega}_\tau^{N/2}$.

Theorem 4.5 (Discrete Maximum Principle). *Consider the discrete IVP* (4.5). *Let* $y_{1,0}^* \geq 0$, $y_{2,0}^* \leq 0$. *Then* $P_1^{*N/2}\bar{y}_i^* \leq 0$, $P_2^{*N/2}\bar{y}_i^* \leq 0$, *for* $i = 1 : N$, *implies that* $\bar{y}_i^* \geq 0$, $\forall i = 0(1)N$.

Proof. Please refer to Ref. [30].

Lemma 4.3 (Discrete Stability Result). *If* $\bar{y}_i^*$ *is any mesh function, then*

$$||\bar{y}_i^*|| \leq C \max\left\{|y_{1,0}^*|, |y_{2,0}^*|, \max_{1\leq i\leq N}|P_1^{*N/2}\bar{y}_i^*|, \max_{1\leq i\leq N}|P_2^{*N/2}\bar{y}_i^*|\right\}, \quad i = 0(1)N.$$

Proof. Set

$$M' = C \max\left\{|y_{1,0}^*|, |y_{2,0}^*|, \max_{1\leq i\leq N}|P_1^{*N/2}\bar{y}_i^*|, \max_{1\leq i\leq N}|P_2^{*N/2}\bar{y}_i^*|\right\}.$$

Define the barrier functions

$$\bar{\chi}_i^{\pm} = (\chi_{1,i}^{\pm}, \chi_{2,i}^{\pm})^T$$

by

$$\chi_{1,i}^{\pm} = M'\{1 + x_i\} \pm y_{1,i}^* \quad \text{and} \quad \chi_{2,i}^{\pm}(x) = M'(-1) \pm y_{2,i}^*, \quad i = 0(1)N.$$

Then for a proper selection of the constant C, applying Theorem 4.5 to the barrier functions $\bar{\chi}_i^{\pm}$, we can obtain the desired bounds for $\bar{y}_i^*$. ☐

4.3.2. Outer region problem

The outer region problem for (4.1)-(4.2) is given by

$$\begin{cases} Ly_2(x) := \varepsilon y_2{}''(x) + a(x)y_2{}'(x) - b(x)y_2(x) = f(x) + c(x)u_{0_1}(x), \ x \in (\tau, 1), \\ B_0 y_2(0) = y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau) = \bar{r}, \quad B_1 y_2(1) = y_2(1) - y_2'(1) = r, \end{cases} \tag{4.6}$$

where $u_{0_1}(x)$ is defined as in (2.7), $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth and $a(x) \geq \alpha$ and $b(x) > 0$, $0 \geq c(x) \geq -\gamma, \gamma > 0$.

Theorem 4.6 (Maximum Principle). *Consider the BVP* (4.6). *Let* $y_2(x)$ *be a smooth function satisfying* $B_0 y_2(0) \geq 0$, $B_1 y_2(1) \geq 0$ *and* $Ly_2(x) \geq 0$ *for* $x \in (\tau, 1)$. *Then,* $y_2(x) \geq 0$, $\forall x \in [\tau, 1]$.

Proof. Please refer to Ref. [1].

Lemma 4.4 (Stability Result). *If $y_2(x)$ is the solution of the BVP* (4.6) *then*

$$|y_2(x)| \leq C \max\Big\{|B_0 y_2(0)| + |B_1 y_2(1)| + \max_{x\in[\tau,1]} |Ly_2(x)|\Big\}, \quad \forall x \in [\tau, 1].$$

Proof. Please refer to Ref. [1].

To solve this BVP (4.6), we apply the CFD scheme defined by

$$\begin{cases} L^{N/2} y_{2,i} := \varepsilon \delta^2 y_{2,i} + a(x_i) D^+ y_{2,i} - b(x_i) y_{2,i} \\ \qquad\qquad = f(x_i) + c(x_i) u_{0_1}(x_i), \quad i = 1(1)N-1, \\ B_0^{N/2} y_{2,0} = \bar{r}, \quad B_1^{N/2} y_{2,N} = y_{2,N} - (y_{2,N} - y_{2,N-1})/h_2 = r, \end{cases} \tag{4.7}$$

where $D^+ y_{2,i} = (y_{2,i+1} - y_{2,i})/h_2$, $\delta^2 y_{2,i} = (y_{2,i+1} - 2y_{2,i} + y_{2,i-1})/h_2^2$,

$$x_i = x_{i-1} + ih_2, \quad \text{and} \quad h_2 = \frac{2(1-\tau)}{N}, \quad i = 1(1)N.$$

Here, τ is defined as before in Section 4.3.1. This fitted mesh is denoted by $\bar{\Omega}_\tau^{N/2}$.

Theorem 4.7 (Discrete Maximum Principle). *Consider the discrete BVP* (4.7). *If $B_0^{N/2} y_{2,0} \geq 0$, $B_1^{N/2} y_{2,N} \geq 0$ and $L^{N/2} y_{2,i} \geq 0$, $\forall i = 1(1)N-1$. Then $y_{2,i} \geq 0$, $\forall i = 0(1)N$.*

Proof. Please refer to Ref. [1].

Lemma 4.5 (Discrete Stability Result). *If $y_{2,i}$ is the solution of the BVP* (4.7) *then*

$$|y_{2,i}| \leq C \max\Big\{|B_0^{N/2} y_{2,0}| + |B_1^{N/2} y_{2,N}| + \max_{1\leq i\leq N} |L^{N/2} y_{2,i}|\Big\}, \quad \textit{for} \ \ 0 \leq i \leq N.$$

Proof. Please refer to Ref. [1].

5. Error estimates

In this section, we derive an error estimates for the solution of (4.1)-(4.2).

5.1. Inner region problem

If we adopt exactly the same analysis presented as in [9] we can derive the following error estimate. In order to derive an error estimate for the solution of the inner region problem we prove the following theorems.

Theorem 5.1. *Let $\bar{y}^* = (y_1^*, y_2^*)^T$ and $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ be respectively, the solutions of* (4.4) *and* (4.5). *Then*

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq CN^{-1} \ln N, \quad \textit{for} \ \ i = 0(1)N, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

.

Proof. From Lemma 4.1 in [9] and Theorem 4.4 it is clear that for each $i,$ the consistency errors due to $\bar{y}^*(x)$ with $P_1^{*N/2}$ and $P_2^{*N/2}$ are bounded as given below:

$$\begin{aligned}|P_1^{*N/2}(\bar{y}^*(x_i)-\bar{y}_i^*)| &= |(D^- - D)y_1^*(x_i)| \\ &= \frac{h_1}{2}|y_1^{*''}(t)| \\ &= \frac{h_1}{2\varepsilon},\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}|P_2^{*N/2}(\bar{y}^*(x_i)-\bar{y}_i^*)| &= \varepsilon|(D^- - D)y_2^*(x_i)| \\ &= \frac{\varepsilon h_1}{2}|y_2^{*''}(t)| \\ &= \frac{h_1}{2\varepsilon},\end{aligned} \tag{5.2}$$

for some point t satisfying $x_{i-1} \le t \le x_i$.

Since $\tau = \min\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln N\}$, the argument is considered for two cases $\tau = \frac{1}{2}$ and $\tau = \frac{2\varepsilon}{\alpha}\ln N$ separately.

Case 1: $\tau = \frac{1}{2}$. Note that $\frac{1}{2} \le \frac{2\varepsilon}{\alpha}\ln N$ implies $\varepsilon^{-1} \le C\ln N$.

From (5.1) and (5.2) and using $h_1 \le CN^{-1}$, we have

$$\begin{cases}|P_1^{*N/2}(\bar{y}^*(x_i)-\bar{y}_i^*)| \le CN^{-1}\ln N, \\ |P_2^{*N/2}(\bar{y}^*(x_i)-\bar{y}_i^*)| \le CN^{-1}\ln N.\end{cases} \tag{5.3}$$

Case 2: $\tau = \frac{2\varepsilon}{\alpha}\ln N$.

From (5.1) and (5.2), we have

$$\begin{cases}|P_1^{*N/2}(\bar{y}^*(x_i)-\bar{y}_i^*)| \le CN^{-1}\ln N, \\ |P_2^{*N/2}(\bar{y}^*(x_i)-\bar{y}_i^*)| \le CN^{-1}\ln N.\end{cases} \tag{5.4}$$

Hence

$$|P_1^{*N/2}(\bar{y}^*(x_i)-\bar{y}_i^*)| \le CN^{-1}\ln N,$$
$$|P_2^{*N/2}(\bar{y}^*(x_i)-\bar{y}_i^*)| \le CN^{-1}\ln N.$$

Since $y_1^*(0) = y_{1,0}^*$, $y_2^*(0) = y_{2,0}^*$ by the discrete stability result given by Lemma 4.3 it follows that

$$||\bar{y}^*(x_i)-\bar{y}_i^*|| \le CN^{-1}\ln N.$$

This completes the proof. $\square$

Theorem 5.2. *Let* $\bar{y}^* = (y_1^*, y_2^*)^T$ *and* $\bar{y}^{*1} = (y_1^{*1}, y_2^{*1})^T$ *be respectively, the solutions of the IVPs*

$$\begin{cases} y_1^{*'} - y_2^* = 0, \\ \varepsilon y_2^{*'} + a(x) y_2^* - b(x) y_1^* = f(x) + c(x) u_{0_1}, \quad x \in \Omega, \\ y_1^*(0) = \alpha, \quad y_2^*(0) = \beta, \end{cases} \tag{5.5}$$

and

$$\begin{cases} y_1^{*'} - y_2^* = 0, \\ \varepsilon y_2^{*'} + a(x) y_2^* - b(x) y_1^* = f(x) + c(x) u_{0_1}, \quad x \in \Omega, \\ y_1^*(0) = \alpha + \mathcal{O}(\varepsilon), \quad y_2^*(0) = \beta. \end{cases} \tag{5.6}$$

Then, $||\bar{y}^*(x) - \bar{y}^{*1}(x)|| \leq C\varepsilon$.

Proof. Let $\bar{w} = \bar{y}^* - \bar{y}^{*1}$. Then $\bar{w}$ satisfies

$$\begin{cases} w_1' - w_2 = 0, \\ \varepsilon w_2' + a(x) w_2 - b(x) w_1 = 0, \quad x \in \Omega, \\ w_1(0) = \mathcal{O}(\varepsilon), \quad w_2(0) = 0. \end{cases} \tag{5.7}$$

Using the maximum principle for the system (5.7)as in [1], we have

$$||\bar{y}^*(x) - \bar{y}^{*1}(x)|| \leq C\varepsilon, \quad x \in \Omega.$$

This completes the proof. □

Theorem 5.3. *Let* $\bar{y}^* = (y_1^*, y_2^*)^T$ *be the solution of the IVP* (5.5). *Further, let* $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ *be the numerical solution of the IVP* (5.6) *after applying the shooting method as given in* (4.5). *Then,*

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq C\varepsilon + CN^{-1}\ln N, \quad \textit{for } \ i = 0(1)N \quad \textit{and} \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. From Theorem 5.2, $||\bar{y}^*(x_i) - \bar{y}^{*1}(x_i)|| \leq C\varepsilon$. And from Theorem 5.1, $||\bar{y}^{*1}(x_i) - \bar{y}_i^*|| \leq CN^{-1}\ln N$.

Using these estimates in the inequality,

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq ||\bar{y}^*(x_i) - \bar{y}^{*1}(x_i)|| + ||\bar{y}^{*1}(x_i) - \bar{y}_i^*||,$$

where $\bar{y}^{*1}(x)$ is the solution of the system (5.6), this theorem gets proved. □

The BVP (4.1)-(4.2) is equivalent to the following IVP

$$\begin{cases} \varepsilon y_2''(x) + a(x) y_2'(x) - b(x) y_2(x) = f^*(x), \quad x \in \Omega, \\ y_2(0) = q^*, \quad y_2'(0) = -q, \end{cases} \tag{5.8}$$

where q^* is the exact value of the solution of the BVP (4.1)-(4.2) at $x = 0$. Because of uniqueness of the solutions of the IVP (5.8) and the BVP (4.1)-(4.2), we have the following result on the error estimate for the inner region problem.

Theorem 5.4. *Let $y_2^\star(x_i)$ be the solution of the BVP* (4.1)-(4.2)*. Further, let* $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ *be the numerical solution of the IVP* (4.5)*. Then,*

$$|y_2^\star(x_i) - y_{1,i}^*| \leq C\varepsilon + CN^{-1}\ln N, \quad \textit{for} \ \ i = 0(1)N, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. Consider the inequality,

$$|y_2^\star(x_i) - y_{1,i}^*| \leq |y_2^\star(x_i) - y_1^{*1}(x_i)| + |y_1^{*1}(x_i) - y_{1,i}^*|,$$

where $y_1^{*1}(x)$ is the solution of the system (5.6). The proof follows from Theorems 5.2 and 5.3. □

Theorem 5.5. *Let $\bar{y}(x)$ be the solution of the BVP* (2.1)-(2.2) *and let* $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ *be the numerical solution of the IVP* (4.5)*. Then,*

$$|y_2(x_i) - y_{1,i}^*| \leq C\varepsilon + CN^{-1}\ln N, \quad \textit{for} \ \ i = 0(1)N, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. Consider the inequality,

$$|y_2(x_i) - y_{1,i}^*| \leq |y_2(x_i) - y_2^\star(x_i)| + |y_2^\star(x_i) - y_{1,i}^*|,$$

where $y_2^\star(x)$ is the solution of the BVP (4.1)-(4.2). The proof follows from Theorems 3.1 and 5.4. □

5.2. Outer region problem

In order to derive an error estimate for the solution of the outer region problem we prove the following theorems.

Theorem 5.6. *Let $y_2(x_i)$ be the solution of the BVP* (4.6) *and $y_{2,i}$ be its numerical solution given by* (4.7)*. Then,*

$$|y_2(x_i)) - y_{2,i}| \leq CN^{-1}\ln N, \quad \textit{for} \ \ i = 0(1)N, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. From Lemma 4.1 in [9] and Theorem 3.1 it is clear that for each i, the consistency errors due to $y_2(x_i)$ with $L^{N/2}$ is bounded as given below.

$$\begin{aligned}
|L^{N/2}(y_2(x_i) - y_{2,i})| &= |(L^{N/2} - L)y_2(x_i)| \\
&\leq C\varepsilon h_2 |y_2^{(3)}(x_i)| + Ch_2|y_2^{(2)}(x_i)| \\
&\leq Ch_2\varepsilon(1 + \varepsilon^{-2}e^{-\alpha\tau/\varepsilon}) + Ch_2(1 + \varepsilon^{-1}e^{-\alpha\tau/\varepsilon}) \\
&\leq Ch_2 + Ch_2\varepsilon^{-1}e^{-\alpha\tau/\varepsilon}.
\end{aligned} \tag{5.9}$$

Since $\tau = \min\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln N\}$, the argument is considered for two cases $\tau = \frac{1}{2}$ and $\tau = \frac{2\varepsilon}{\alpha}\ln N$ separately.

Case 1: $\tau = \frac{1}{2}$. Note that $\varepsilon^{-1} \leq C\ln N$ and $h_2 \leq CN^{-1}$.

From (5.9), we have

$$\begin{aligned}
|L^{N/2}(y_2(x_i)-y_{2,i})| &\le CN^{-1}+CN^{-1}\ln N\\
&\le CN^{-1}\ln N,\\
|B_0^{N/2}(y_2(0)-y_{2,0})| &= |(B_0^{N/2}-B_0)y_2(0)|=0,\\
|B_1^{N/2}(y_2(1)-y_{2,N})| &= |(B_1^{N/2}-B_1)y_2(1)|\\
&\le |(D^+-D)y_2(1)|\\
&\le Ch_2|y^{(2)}(1)|\\
&\le Ch_2(1+\varepsilon^{-1}e^{-\alpha/\varepsilon})\\
&\le CN^{-1}\ln N.
\end{aligned}$$

Then by the discrete stability result given by Lemma 4.5, we have

$$|y_2(x_i)-y_{2,i}|\le CN^{-1}\ln N,\quad \text{for}\ \ i=0(1)N.$$

Case 2: $\tau=\frac{2\varepsilon}{\alpha}\ln N$.

From (5.9) and using $h\le CN^{-1}$, we have

$$\begin{aligned}
|L^{N/2}(y_2(x_i)-y_{2,i})| &\le CN^{-1}+CN^{-1}\\
&\le CN^{-1},
\end{aligned}$$

since $\varepsilon e^{-\alpha\tau/\varepsilon}\le C$, when $x_i\ge\tau$. Also,

$$\begin{aligned}
|B_0^{N/2}(y_2(0)-y_{2,0})| &= |(B_0^{N/2}-B_0)y_2(0)|=0,\\
|B_1^{N/2}(y_2(1)-y_{2,N})| &= |(B_1^{N/2}-B_1)y_2(1)|\\
&\le |(D^+-D)y_2(1)|\\
&\le Ch_2|y_2^{(2)}(1)|\\
&\le Ch_2(1+\varepsilon^{-1}e^{-\alpha/\varepsilon})\\
&\le CN^{-1}.
\end{aligned}$$

Then by the discrete stability result given by Lemma 4.5, we have

$$|y_2(x_i)-y_{2,i}|\le CN^{-1}\ln N,\quad \text{for}\ \ i=0(1)N.$$

Hence in both the cases,

$$|y_2(x_i)-y_{2,i}|\le CN^{-1}\ln N,\quad \text{for}\ \ i=0(1)N.$$

This completes the proof. $\square$

Theorem 5.7. *Let $y_2^\star(x_i)$ be the solution of the BVP* (4.1)-(4.2) *and $y_{2,i}$ be the numerical solution of the BVP* (4.6) *after applying the FD scheme as given in* (4.7)*. Then,*

$$|y_2^\star(x_i) - y_{2,i}| \le C\varepsilon + CN^{-1}\ln N, \quad \textit{for} \;\; i = 0(1)N, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. From Theorem 3.1, $|y_2^\star(x_i) - y_2(x_i)| \le C\varepsilon$. And from Theorem 5.6, $|y_2(x_i) - y_{2,i}| \le CN^{-1}\ln N$.

Using these estimates in the inequality,

$$|y_2^\star(x_i) - y_{2,i}| \le |y_2^\star(x_i) - y_2(x_i)| + |y_2(x_i) - y_{2,i}|,$$

where $y_2(x_i)$ is the solution of the BVP (4.6), this theorem gets proved. □

Theorem 5.8. *Let $\bar{y}(x)$ be the solution of the BVP* (2.1)-(2.2) *and $y_{2,i}$ be the numerical approximation obtained for $y_2(x_i)$ for the BVP* (4.6) *after applying the FD scheme as given in* (4.7)*. Then,*

$$|y_2(x_i) - y_{2,i}| \le C\varepsilon + CN^{-1}\ln N, \quad \textit{for} \;\; i = 0(1)N, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. From Theorem 3.1, $|y_2(x_i) - y_2^\star(x_i)| \le C\varepsilon$. And from Theorem 5.7, $|y_2^\star(x_i) - y_{2,i}| \le CN^{-1}\ln N$.

Using these estimates in the inequality,

$$|y_2(x_i) - y_{2,i}| \le |y_2(x_i) - y_2^\star(x_i)| + |y_2^\star(x_i) - y_{2,i}|,$$

where $y_2^\star(x_i)$ is the solution of the BVP (4.1)-(4.2), this theorem gets proved. □

6. Non-linear problem

Consider the quasi-linear BVP

$$\varepsilon y'''(x) = F(x, y, y', y''), \quad x \in \Omega, \tag{6.1}$$
$$y(0) = p, \quad -y''(0) = q, \quad y'(1) - y''(1) = r, \tag{6.2}$$

where $F(x, y, y', y'')$ is a smooth function such that

$$\begin{cases} F_{y''}(x, y, y', y'') \ge \alpha, \quad \alpha > 0, \\ F_{y'}(x, y, y', y'') > 0, \\ 0 \ge F_y(x, y, y', y'') \ge -\gamma, \quad \gamma > 0, \\ \alpha > \gamma. \end{cases} \tag{6.3}$$

Assume that the reduced problem

$$F(x, y, y', y'') = 0, \quad y(0) = p, \quad y'(1) - y''(1) = r$$

has a solution $y_0 \in C^{(3)}(\bar{\Omega})$. Then the BVP (6.1)-(6.2) has a unique solution and has a less severe boundary layer of width $\mathcal{O}(\varepsilon)$ near $x = 0$ [17, 28]. In order to obtain a numerical solution of (6.1)-(6.2), Newton's method of quasi-linearisation is applied [1]. Consequently, we get a sequence $\{y^{[m]}\}_0^\infty$ of successive approximations with a proper choice of initial guess $y^{[0]}$.

We define $y^{[m+1]}$ for each fixed non-negative integer m, to be the solution of the following linear problem:

$$\begin{cases}\varepsilon(y''')^{[m+1]} + a^m(x)(y'')^{[m+1]} - b^m(x)(y')^{[m+1]} - c^m(x)y^{[m+1]} = F^{[m]}(x),\\ y^{[m+1]}(0) = p, \quad (y'')^{[m+1]}(0) = -q, \quad (y')^{[m+1]}(1) - (y'')^{[m+1]}(1) = r,\end{cases} \tag{6.4}$$

where

$$\begin{cases}a^m(x) = F_{y''}(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]}),\\ b^m(x) = F_{y'}(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]}),\\ c^m(x) = F_y(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]}),\\ F^{[m]}(x) = F(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]}) - (y'')^{[m]}F_{y''}(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]})\\ \qquad +(y')^{[m]}F_{y'}(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]}) + (y)^{[m]}F_y(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]}).\end{cases} \tag{6.5}$$

Remark 6.1. If the initial guess $y^{[0]}$ is sufficiently close to the solution $y(x)$ of (6.1)-(6.2), then, following the method of proof given in [1], one can prove that the sequence $\{y^{[m]}\}_0^\infty$ converges to $y(x)$. From (6.3),it follows that for each fixed m:

$$\begin{aligned}&a^m(x) = F_{y''}(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]}) \geq \alpha, \quad \alpha > 0,\\ &b^m(x) = F_{y'}(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]}) > 0,\\ &0 \geq c^m(x) = F_y(x, y^{[m]}, (y')^{[m]}, (y'')^{[m]}) \geq -\gamma, \quad \gamma > 0,\\ &\alpha > \gamma.\end{aligned}$$

Remark 6.2. Problem (6.4)-(6.5) for each fixed m is a linear BVP of third order and is of the form (1.1)-(1.2).

Remark 6.3. The solution of the reduced problem of (6.1)-(6.2) or a suitable approximation will be taken as the initial guess $y^{[0]}$ to generate the successive approximations $\{y^{[m]}\}_0^\infty$.

Remark 6.4. For the above Newton's quasi-linearisation process the following convergence criterion is used.

$$|y^{[m+1]}(x_j) - y^{[m]}(x_j)| \leq \delta, \quad x_j \in \bar{\Omega}, \quad m \geq 0.$$

7. Illustrations

In this section, we present three examples to illustrate the method described in this paper. Let $\bar{Y}_j^{2N}$ be the piecewise linear interpolants of the numerical solution Y_j^N on the mesh Ω^{2N}, where N, $2N$ are the number of mesh points. The nodal errors and order of convergence are estimated using the fitted mesh principle [2]. Define the fitted mesh differences as E_ε^N which denote, respectively, the numerical solutions obtained using N and $2N$ mesh intervals

$$\max_{x\in\bar{\Omega}^N} \mid Y^N(x_j) - \bar{Y}^{2N}(x_j) \mid \quad \text{and} \quad E^N = \max_{\varepsilon} E_\varepsilon^N.$$

Then, the order of convergence is given by

$$p^* = \min_N p^N, \quad \text{where} \quad p^N = \log_2 \left\{ \frac{E^N}{E^{2N}} \right\}.$$

The computed maximum pointwise errors E_ε^N, E^N and order of convergence p^* are tabulated in Tables 1, 2 and 3.

Example 7.1. Consider the BVP

$$\begin{aligned} &\varepsilon y'''(x) + 2y''(x) - 3y'(x) + y(x) = -(1+2x), \\ &y(0) = 0, \quad y''(0) = -1, \quad y'(1) - y''(1) = 0. \end{aligned}$$

The numerical results are presented in Table 1.

Table 1: Maximum pointwise errors E_ε^N, E^N and p^N of Example 7.1.

Number of mesh points N						
ε	128	256	512	1024	2048	4096
2^{-13}	1.8320e-006	1.1259e-006	6.6088e-007	3.7689e-007	2.1038e-007	1.1571e-007
2^{-14}	9.1601e-007	5.6294e-007	3.3044e-007	1.8844e-007	1.0519e-007	5.7854e-008
2^{-15}	4.5800e-007	2.8147e-007	1.6522e-007	9.4221e-008	5.2594e-008	2.8927e-008
2^{-16}	2.2900e-007	1.4073e-007	8.2609e-008	4.7110e-008	2.6297e-008	1.4464e-008
2^{-17}	1.1450e-007	7.0367e-008	4.1305e-008	2.3555e-008	1.3148e-008	7.2318e-009
2^{-18}	5.7250e-008	3.5183e-008	2.0652e-008	1.1778e-008	6.5742e-009	3.6159e-009
2^{-19}	2.8625e-008	1.7592e-008	1.0326e-008	5.8888e-009	3.2871e-009	1.8079e-009
2^{-20}	1.4313e-008	8.7959e-009	5.1631e-009	2.9444e-009	1.6436e-009	9.0397e-010
2^{-13}	1.2877e-001	6.6463e-002	3.3757e-002	1.7010e-002	8.5383e-003	4.2775e-003
2^{-14}	1.2877e-001	6.6463e-002	3.3757e-002	1.7010e-002	8.5383e-003	4.2774e-003
2^{-15}	1.2877e-001	6.6463e-002	3.3757e-002	1.7010e-002	8.5383e-003	4.2774e-003
2^{-16}	1.2877e-001	6.6463e-002	3.3757e-002	1.7010e-002	8.5383e-003	4.2774e-003
2^{-17}	1.2877e-001	6.6463e-002	3.3757e-002	1.7010e-002	8.5383e-003	4.2774e-003
2^{-18}	1.2877e-001	6.6463e-002	3.3757e-002	1.7010e-002	8.5383e-003	4.2774e-003
2^{-19}	1.2877e-001	6.6463e-002	3.3757e-002	1.7010e-002	8.5383e-003	4.2774e-003
2^{-20}	1.2877e-001	6.6463e-002	3.3757e-002	1.7010e-002	8.5383e-003	4.2774e-003
E^N	1.2877e-001	6.6463e-002	3.3757e-002	1.7010e-002	8.5383e-003	4.2775e-003
p^N	p^* 9.5417e-001	9.7736e-01	9.8880e-001	9.9436e-001	9.9718e-001	
The order of convergence = 9.5417e-001						

Example 7.2. Consider the BVP

$$\varepsilon y'''(x) + (2x+3)^2 y''(x) - 2(2x+3)y'(x) + y(x) = 6x,$$
$$y(0) = 0, \quad y''(0) = -1, \quad y'(1) - y''(1) = 1.$$

The numerical results are presented in Table 2.

Table 2: Maximum pointwise errors E_ε^N, E^N and p^N of Example 7.2.

	Number of mesh points N					
ε	128	256	512	1024	2048	4096
2^{-13}	7.8605e-011	5.3932e-011	3.5654e-011	2.1894e-011	1.2857e-011	7.2664e-012
2^{-14}	1.9651e-011	1.3483e-011	8.9138e-012	5.4738e-012	3.2148e-012	1.8165e-012
2^{-15}	4.9127e-012	3.3706e-012	2.2284e-012	1.3678e-012	8.0336e-013	4.5430e-013
2^{-16}	1.2284e-012	8.4244e-013	5.5689e-013	4.1722e-013	4.1722e-013	1.1369e-013
2^{-17}	3.0709e-013	2.1072e-013	1.3922e-013	1.0281e-013	4.9960e-014	4.2100e-013
2^{-18}	4.2100e-013	4.2100e-013	4.2100e-013	4.2100e-013	4.2100e-013	4.2477e-013
2^{-19}	4.2477e-013	4.2477e-013	4.2477e-013	4.2477e-013	4.2477e-013	1.5543e-015
2^{-20}	1.1768e-014	2.5535e-014	2.6645e-015	1.7764e-015	2.1427e-013	1.3323e-015
2^{-13}	1.9122e-003	8.9488e-004	4.3219e-004	2.1262e-004	1.0583e-004	5.3223e-005
2^{-14}	1.9111e-003	8.9401e-004	4.3148e-004	2.1200e-004	1.0525e-004	5.2634e-005
2^{-15}	1.9105e-003	8.9358e-004	4.3112e-004	2.1170e-004	1.0497e-004	5.2359e-005
2^{-16}	1.9102e-003	8.9336e-004	4.3095e-004	2.1154e-004	1.0483e-004	5.2225e-005
2^{-17}	1.9101e-003	8.9325e-004	4.3086e-004	2.1147e-004	1.0476e-004	5.2160e-005
2^{-18}	1.9100e-003	8.9320e-004	4.3082e-004	2.1143e-004	1.0473e-004	5.2127e-005
2^{-19}	1.9100e-003	8.9317e-004	4.3080e-004	2.1141e-004	1.0471e-004	5.2111e-005
2^{-20}	1.9100e-003	8.9316e-004	4.3078e-004	2.1140e-004	1.0470e-004	5.2103e-005
E^N	1.9122e-003	8.9488e-004	4.3219e-004	2.1262e-004	1.0583e-004	5.3223e-005
p^N	1.0955e+000	1.0500e+000	1.0234e+000	1.0065e+000	p^*9.9163e-001	
The order of convergence = 9.9163e-001						

Example 7.3. Consider the BVP

$$\varepsilon y'''(x) + 2y''(x) - 2(y')^2(x) + y(x) = 1 + x^2,$$
$$y(0) = 1, \quad y''(0) = 0, \quad y'(1) - y''(1) = 0.$$

The numerical results are presented in Table 3.

8. Conclusions

In this paper, we presented a numerical method to solve third-order SPBVPs for ODEs subject to particular type of boundary conditions by adopting the techniques of [7, 19, 21, 26] and [12, 14, 15, 25] who used to solve second and third order SPBVPs for ODEs. The boundary conditions help us to reduce the given third order ordinary

Table 3: Maximum pointwise errors E_ε^N, E^N and p^N of Example 7.3.

	Number of mesh points N					
ε	128	256	512	1024	2048	4096
2^{-13}	4.6286e-006	2.6451e-006	1.4879e-006	8.2666e-007	4.5469e-007	2.4802e-007
2^{-14}	2.3140e-006	1.3223e-006	7.4381e-007	4.1324e-007	2.4802e-007	1.2398e-007
2^{-15}	1.1569e-006	6.6109e-007	3.7187e-007	2.0660e-007	1.2398e-007	6.1981e-008
2^{-16}	5.7843e-007	3.3053e-007	1.8593e-007	1.0329e-007	6.1981e-008	3.0988e-008
2^{-17}	2.8921e-007	1.6526e-007	9.2961e-008	5.1645e-008	3.0988e-008	1.5494e-008
2^{-18}	1.4460e-007	8.2630e-008	4.6480e-008	2.5822e-008	1.5494e-008	7.7467e-009
2^{-19}	7.2301e-008	4.1315e-008	2.3240e-008	1.2911e-008	7.7467e-009	3.8733e-009
2^{-20}	3.6151e-008	2.0657e-008	1.1620e-008	6.4555e-009	3.8733e-009	1.9366e-009
2^{-13}	7.9764e-002	4.0464e-002	2.0634e-002	1.0693e-002	5.7582e-003	3.4093e-003
2^{-14}	7.9634e-002	4.0241e-002	2.0352e-002	1.0363e-002	5.3668e-003	2.8892e-003
2^{-15}	7.9570e-002	4.0131e-002	2.0215e-002	1.0207e-002	5.1937e-003	2.6889e-003
2^{-16}	7.9538e-002	4.0077e-002	2.0148e-002	1.0132e-002	5.1121e-003	2.6001e-003
2^{-17}	7.9523e-002	4.0050e-002	2.0114e-002	1.0095e-002	5.0724e-003	2.5582e-003
2^{-18}	7.9515e-002	4.0037e-002	2.0098e-002	1.0077e-002	5.0529e-003	2.5379e-003
2^{-19}	7.9511e-002	4.0030e-002	2.0090e-002	1.0067e-002	5.0432e-003	2.5278e-003
2^{-20}	7.9509e-002	4.0027e-002	2.0085e-002	1.0063e-002	5.0384e-003	2.5228e-003
E^N	7.9764e-002	4.0464e-002	2.0634e-002	1.0693e-002	5.7582e-003	3.4093e-003
p^N	9.7910e-001	9.7162e-001	9.4836e-001	8.9298e-001	p^*7.5614e-001	
The order of convergence = 7.5641e-001						

differential equation into a system of one first order and one second order equation subject to initial and boundary conditions, respectively. Error estimates derived in Section 5 show first order convergence. Our numerical experiments show that this method gives good approximate solutions especially with in the layer region. This can be seen from the numerical results presented in Table 1, Table 2 and Table 3. In all the three tables the numerical results appearing in the rows 1-8 correspond to the left boundary layer region and the rest of the rows namely 9-16 correspond to the outer region.

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