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On the Approximation of the Derivatives of Spline Quasi-Interpolation in Cubic Spline Space $S_3^{1,2}(\Delta_{mn}^{(2)})$

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Abstract. In this paper, based on the basis composed of two sets of splines with distinct local supports, cubic spline quasi-interpolating operators are reviewed on nonuniform type-2 triangulation. The variation diminishing operator is defined by discrete linear functionals based on a fixed number of triangular mesh-points, which can reproduce any polynomial of nearly best degrees. And by means of the modulus of continuity, the estimation of the operator approximating a real sufficiently smooth function is reviewed as well. Moreover, the derivatives of the nearly optimal variation diminishing operator can approximate that of the real sufficiently smooth function uniformly over quasi-uniform type-2 triangulation. And then the convergence results are worked out.

AMS subject classifications: 65D07, 41A25

Key words: Bivariate splines, conformality of smoothing cofactor method, nonuniform type-2 triangulation, quasi-interpolation, modulus of continuity.

1. Introduction

As is known, the nonuniform rational B-splines scheme has become a de facto standard in Computer Aided Geometric Design, which is a powerful tool for constructing free-form curves and surfaces [3, 7, 14, 16]. Due to its rational model, some new alternatives have been proposed for constructing fair-shape-preserving approximations recently [8–10, 15]. However, both B-spline surfaces and the new alternatives are constructed in the form of tensor product, which implies that the degrees of the surfaces are the addition of that of the parameters in two directions so that there may be some inflection points on the surface. Moreover, the bivariate function can not reproduce any polynomial of nearly best degree. Furthermore, it is restricted to construct surfaces over the rectangular mesh. Hence, to

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avoid the shortcomings, it is very important to study multivariate spline functions theoretically. Since multivariate spline functions are heavily dependent on the geometric property of the domain partitions, it is so complex that the non-Cartesian product multivariate spline functions has not been developed radically for a long time. But all is changed until the construction of the Conformality of Smoothing Cofactor Method [17, 18].

In a specific way, the smooth cofactor and conformality condition has been introduced to which the polynomials must satisfy by analysing the relation between the polynomials over two adjacent cells [17, 18]. The conformality condition establishes the equivalent conversion between multivariate spline functions and the corresponding algebraic problems. As a result, the Conformality of Smoothing Cofactor Method provides an algebraic approach to studying the multivariate spline functions, including the dimension and the locally supported basis functions in multivariate spline spaces [17, 18, 22, 23], etc., which are difficult but important. The dimension of the multivariate spline function space $S_k^\mu(\Delta)$ i.e., the multivariate spline space with degree k and smoothness μ over the domain D with respect to the partition Δ have been widely developed in [4, 13, 14, 17, 18, 24]. Recently, Liu, Hong, and Cao [6] determined the dimension and construct a local support basis of the space $S^{1,d}(\Delta(2))$, for $d = 0, 1$ of the spline functions over the type-2 nonuniform triangulation. The basis functions of bivariate cubic and quartic spline spaces on uniform type-2 triangulation have been derived in [5, 19], respectively, where spline quasi-interpolation has been also investigated thoroughly. These spline quasi-interpolating operators can reproduce any polynomial of (nearly) best degrees, respectively. Moreover, spline quasi-interpolation defined by discrete linear functionals based on a fixed number of triangular mesh-points has been investigated, which showed that they could approximate a real function and its partial derivatives up to an optimal order in [1, 2].

However, in view of the complexity in computation of the bases, the study on spline quasi-interpolation over nonuniform type-2 triangulations are almost restricted in bivariate quadratic B-splines, see [20, 21]. Since multivariate approximation over irregular triangulations may be more important than that over uniform triangulations, we have computed the cubic splines in [11], and have constructed the cubic spline quasi-interpolation in [12] by using the Conformality of Smoothing Cofactor Method [17, 18]. Now we shall make a further study on the approximation of the derivatives of the cubic spline quasi-interpolation in this paper.

A brief outline of this article is organized as follows. In Section 2, we review the dimension and the bases in $S_3^{1,2}(\Delta_{mn}^{(2)})$. Based on five mesh points or the center of the support of each spline B_{ij}^1 and five mesh points of the support of each spline B_{ij}^2, the representation of spline quasi-interpolation is investigated, which can reproduce any polynomial in $\mathbb{P}_2 \cup \{x^2y, xy^2\}$. Then in section 3, we make a further study of the derivatives of the cubic spline quasi-interpolation, which can approximate the derivatives of the real sufficiently smooth function uniformly over quasi-uniform triangulation.

2. Review of representation of spline quasi-interpolation in $S_3^{1,2}(\Delta_{mn}^{(2)})$

The domain $\Omega = [a, b] \times [c, d]$ is partitioned into mn rectangular cells $\Omega_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 0, \cdots, m-1$ and $j = 0, \cdots, n-1$, where m, n are given positive integers,

and $a = x_0 < x_1 < \cdots < x_m = b$ and $c = y_0 < y_1 < \cdots < y_n = d$. Thus, we obtain a type-2 triangulation $\Delta_{mn}^{(2)}$ by adding the two diagonals in each Ω_{ij}. Let $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$, respectively. In the particular case of $h_i = h_{i-1}$ and $k_j = k_{j-1}$, such a triangulation is called uniform, otherwise, nonuniform.

A nonuniform bivariate spline $s(x,y) \in S_3^{1,2}(\Delta_{mn}^{(2)})$ is a piecewise polynomial of degree three satisfying two continuous condition:

(a) $s(x,y)$ is C^1 continuous on the horizontal and vertical grid segments $x = x_i$ and $y = y_j$, where $i = 0, \cdots, m$ and $j = 0, \cdots, n$.

(b) $s(x,y)$ is C^2 continuous on the diagonal grid segments

$$y - y_j - \frac{k_{j+1}}{h_{i+1}}(x - x_i) = 0, \qquad y - y_j + \frac{k_{j+1}}{h_{i+1}}(x - x_{i+1}) = 0,$$

where $i = 0, \cdots, m-1$ and $j = 0, \cdots, n-1$.

The dimension of the nonuniform bivariate spline space $S_3^{1,2}(\Delta_{mn}^{(2)})$ in [11] is

$$\dim S_3^{1,2}(\Delta_{mn}^{(2)}) = 2mn + 3m + 3n + 4, \tag{2.1}$$

which is the same with the uniform case in [5].

And also the bases of $S_3^{1,2}(\Delta_{mn}^{(2)})$ has been calculated in [11] in term of B-net, which is composed of two sets of splines with two kind of distinct supports as shown in Fig. 1 and Fig. 2, respectively. Here, we shall provide the explicit representation of the basis over each triangular cell in the supports as shown in Appendix for the sake of illustration.

Denote by Δ_u^1 and Δ_v^2 the triangle cells in the supports, respectively, where the spline functions are expressed in the form of $B_{ij}^1(\Delta_u^1)$ $(u = 1, 2, \cdots, 16)$ and $B_{ij}^2(\Delta_v^2)$ $(v = 1, 2, \cdots, 24)$, respectively.

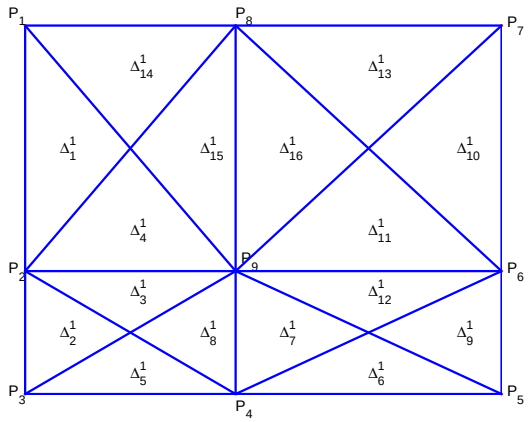

Figure 1: 16 triangular cells in the support of $B_{ij}^1(x,y)$.

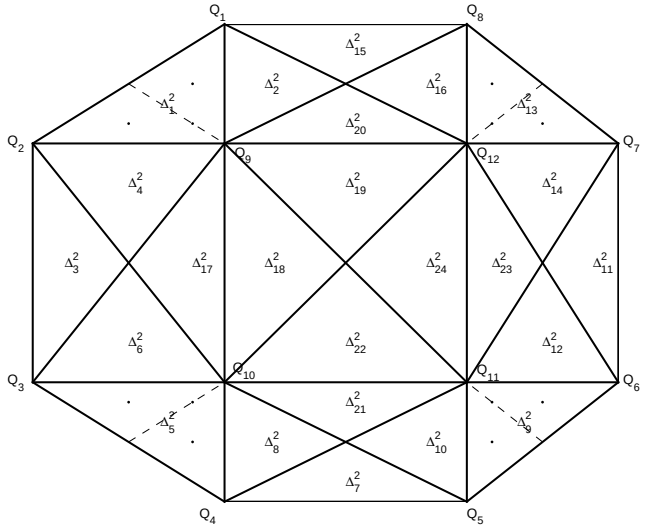

Figure 2: 24 triangular cells in the support of $B^2_{ij}(x,y)$.

Let the vertices

$$P_1(x_{i-1},y_{j+1}),\quad P_2(x_{i-1},y_j),\quad P_3(x_{i-1},y_{j-1}),\quad P_4(x_i,y_{j-1}),\quad P_5(x_{i+1},y_{j-1}),$$
$$P_6(x_{i+1},y_j),\quad P_7(x_{i+1},y_{j+1}),\quad P_8(x_i,y_{j+1}),\quad P_9(x_i,y_j)$$

in the spline $B^1_{ij}(x,y)$ with the center (x_i,y_j) as shown in Fig. 1, and the vertices

$$Q_1(x_i,y_{j+2}),\quad Q_2(x_{i-1},y_{j+1}),\quad Q_3(x_{i-1},y_j),\quad Q_4(x_i,y_{j-1}),$$
$$Q_5(x_{i+1},y_{j-1}),\quad Q_6(x_{i+2},y_j),\quad Q_7(x_{i+2},y_{j+1}),\quad Q_8(x_{i+1},y_{j+2}),$$
$$Q_9(x_i,y_{j+1}),\quad Q_{10}(x_i,y_j),\quad Q_{11}(x_{i+1},y_j),\quad Q_{12}(x_{i+1},y_{j+1})$$

in the spline $B^2_{ij}(x,y)$ with the center $((x_i+x_{i+1})/2,(y_j+y_{j+1})/2)$ as shown in Fig. 2.

In view of the translation of the bases, $B^1_{ij}(x,y)$'s and $B^2_{ij}(x,y)$'s do not vanish identically on the domain Ω defined in Section 2 for the index set

$$I_1=\{(i,j)=(\alpha,\beta):0\le\alpha\le m,\ 0\le\beta\le n\},$$
$$I_2=\{(i,j)=(\alpha,\beta):-1\le\alpha\le m,\ -1\le\beta\le n\},$$

respectively. Then it follows from the cardinality of I_1 and I_2 that the total number of the linear independent functions $B^1_{ij}(x,y)$'s and $B^2_{ij}(x,y)$'s which do not vanish identically on the domain Ω is

$$2mn+3m+3n+5.$$

By the formula (2.1), it is more than the dimension of $S_3^{1,2}(\Delta^{(2)}_{mn})$. As a result, we can construct the bases of $S_3^{1,2}(\Delta^{(2)}_{mn})$ by getting rid of arbitrary one function in either $B^1_{ij}(x,y)$ where $(i,j)\in I_1$, or $B^2_{ij}(x,y)$ where $(i,j)\in I_2$.

Theorem 2.1 (see [11]). *For arbitrary chosen* $(i_0, j_0) \in I_1$, $(i_1, j_1) \in I_2$, *let*

$$\mathbb{B}^1 = \{B_{ij}^1 : (i,j) \in I_1 \backslash \{(i_0, j_0)\}\} \bigcup \{B_{ij}^2 : (i,j) \in I_2\},$$
$$\mathbb{B}^2 = \{B_{ij}^2 : (i,j) \in I_2 \backslash \{(i_1, j_1)\}\} \bigcup \{B_{ij}^1 : (i,j) \in I_1\}.$$

Then, either $\mathbb{B}^1$ *or* $\mathbb{B}^2$ *is a basis of the non-uniform cubic spline space* $S_3^{1,2}(\Delta_{mn}^{(2)})$.

By means of translation of the bases and the values at the ten points on each triangle, it follows that the bases with minimal local support satisfy

Theorem 2.2 (see [11]). *For all* $(x, y) \in \Omega$

$$\sum_{(i,j)\in I_1} B_{ij}^1(x,y) = 1 + \lambda, \tag{2.2a}$$
$$\sum_{(i,j)\in I_2} B_{ij}^2(x,y) = -\lambda, \tag{2.2b}$$
$$\sum_{(i,j)\in I_1} B_{ij}^1(x,y) + \sum_{(i,j)\in I_2} B_{ij}^2(x,y) = 1, \tag{2.2c}$$

where $\lambda \neq 0, -1$.

Remark 2.1. With the choice of $\lambda = -0.25$ and -0.75, the splines $B_{ij}^1(x,y)$'s and $B_{ij}^2(x,y)$'s are shown in Fig. 3 to Fig. 6, respectively.

We have constructed the spline quasi-interpolation by discrete linear functionals based on some mesh-points either in the supports or close to them. These spline quasi-interpolating operators can reproduce polynomials with high degrees, and error estimation show that the variation diminishing operator $V_{mn}(f)$ can approximate a sufficiently smooth function uniformly.

Theorem 2.3 (see [12]). *Let* $V_{mn}(f)$ *be the variation diminishing operator that map* $C(\Omega)$ *into* $S_3^{1,2}(\Delta_{mn}^{(2)})$ *defined by*

$$V_{mn}(f) = \sum_{(i,j)\in I_1} \lambda_{ij}(f) B_{ij}^1(x,y) + \sum_{(i,j)\in I_2} \mu_{ij}(f) B_{ij}^2(x,y), \tag{2.3}$$

where $\lambda \neq -1, 0$, *and*

$$\lambda_{ij}(f) = \frac{1}{3(1+\lambda)} f(x_i, y_j), \tag{2.4a}$$
$$\mu_{ij}(f) = -\frac{4}{3\lambda} f\Big(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\Big) + \frac{1}{6\lambda}\Big[f(x_i, y_{j+1}) + f(x_{i+1}, y_{j+1}) + f(x_i, y_j) + f(x_{i+1}, y_j)\Big]. \tag{2.4b}$$

Then for all $(x, y) \in \Omega$,

$$V_{mn}(f) \equiv f(x,y), \quad \forall f(x,y) \in \mathbb{P}_2[x,y] \cup span\{x^2y, xy^2\}. \tag{2.5}$$

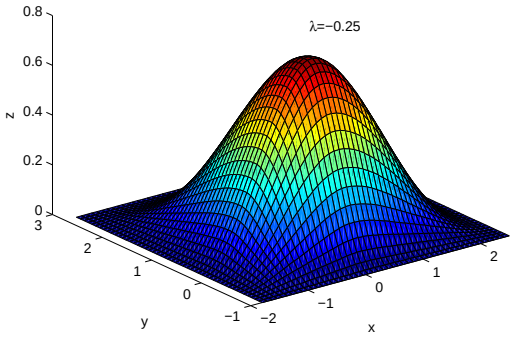

Figure 3: $B_{ij}^1(x,y)$ with $\lambda=-0.25$.

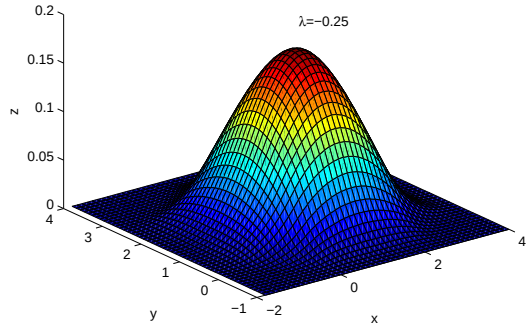

Figure 4: $B_{ij}^2(x,y)$ with $\lambda=-0.25$.

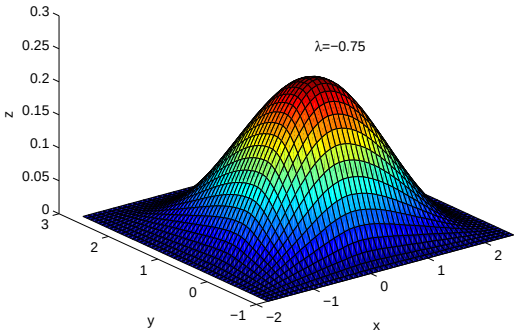

Figure 5: $B_{ij}^1(x,y)$ with $\lambda=-0.75$.

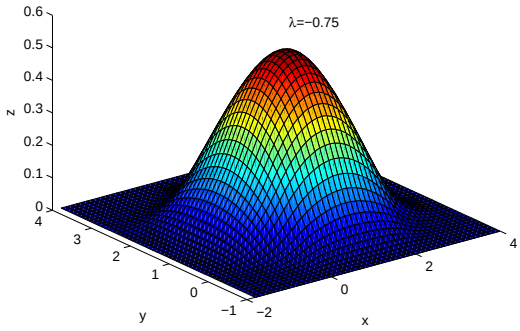

Figure 6: $B_{ij}^2(x,y)$ with $\lambda=-0.75$.

Let the compact set K be the closure of the open set containing Ω. The centers of the two distinct supports is located in the interior of K for sufficiently large m and n. Let the maximal radius of the two supports of B_{ij}^1 and B_{ij}^2 be

$$r_{mn,1}=\max_{0\le i\le m,0\le j\le n}\{|P_9P_1|,|P_9P_3|,|P_9P_5|,|P_9P_7|\}, \tag{2.6a}$$

$$r_{mn,2}=\max_{-1\le i\le m,-1\le j\le n}\{|QQ_1|,|QQ_2|,|QQ_5|,|QQ_6|\}, \tag{2.6b}$$

respectively, where $P_9(x_i,y_j)$ and $Q((x_i+x_{i+1})/2,(y_j+y_{j+1})/2)$ are the centers of the two supports of B_{ij}^1 and B_{ij}^2, respectively, and

$$|P_9P_1|=\sqrt{h_i^2+k_{j+1}^2}, \qquad |P_9P_7|=\sqrt{h_{i+1}^2+k_{j+1}^2}, \tag{2.7a}$$

$$|P_9P_5|=\sqrt{h_{i+1}^2+k_j^2}, \qquad |P_9P_3|=\sqrt{h_i^2+k_j^2}, \tag{2.7b}$$

$$|QQ_1|=\sqrt{\Big(\frac{h_{i+1}}{2}\Big)^2+\Big(k_{j+2}+\frac{k_{j+1}}{2}\Big)^2}, \quad |QQ_2|=\sqrt{\Big(h_i+\frac{h_{i+1}}{2}\Big)^2+\Big(\frac{k_{j+1}}{2}\Big)^2}, \tag{2.7c}$$

$$|QQ_5|=\sqrt{\Big(\frac{h_{i+1}}{2}\Big)^2+\Big(k_j+\frac{k_{j+1}}{2}\Big)^2}, \quad |QQ_6|=\sqrt{\Big(h_{i+2}+\frac{h_{i+1}}{2}\Big)^2+\Big(\frac{k_{j+1}}{2}\Big)^2}. \tag{2.7d}$$

Denote by

$$h_{mn} = \max_{-1\le i\le m+2}\{h_i\}, \qquad k_{mn} = \max_{-1\le j\le n+2}\{h_j\}, \tag{2.8a}$$

$$\delta_{mn} = \max\{h_{mn}, k_{mn}\}, \qquad \delta^*_{mn} = \max\{r_{mn,1}, r_{mn,2}\}, \tag{2.8b}$$

and

$$\omega^k(f) = \max_{l=0,\cdots,k}\left\{\omega_\Omega\Big(\frac{\partial^k f}{\partial^{k-l}x\partial^l y}, \frac{\delta_{mn}}{2}\Big)\right\}, \tag{2.9a}$$

$$\|D^k f\| = \max_{l=0,\cdots,k}\sup_{(x,y)\in\Omega}\left\{\Big|\frac{\partial^k f}{\partial^{k-l}x\partial^l y}\Big|\right\}, \tag{2.9b}$$

where ω_Ω is defined as the modulus of continuity, and $k = 1, 2, \cdots$. Let $\|\cdot\|_\Omega$ be the supremum over Ω, and we shall work out the estimation.

Theorem 2.4 (see [12]). *Let $f \in C(K)$, for sufficiently large positive zeals m and n,*

$$\|f - V_{mn}(f)\|_\Omega \le \frac{7}{3}\omega_\Omega(f, \delta^*_{mn}). \tag{2.10}$$

When $f \in C^1(\Omega)$, then

$$\|f - V_{mn}(f)\|_\Omega \le \frac{10}{3}\delta_{mn}\omega^1(f). \tag{2.11}$$

When $f \in C^2(\Omega)$, then

$$\|f - V_{mn}(f)\|_\Omega \le \frac{5}{3}\delta^2_{mn}\omega^2(f). \tag{2.12}$$

When $f \in C^3(\Omega)$, then

$$\|f - V_{mn}(f)\|_\Omega \le \frac{5}{9}\delta^3_{mn}\|D^3 f\|. \tag{2.13}$$

3. The approximation of the derivatives of the spline quasi-interpolating operator

In this section, we shall make further study of the nearly optimal cubic spline quasi-interpolating operator $V_{mn}(f)$, which indicates that its derivatives can approximate the derivatives of a smooth function uniformly based on the modulus of continuity. In much detail, we consider three cases as follows.

Case one: we shall consider for $f \in C^1$,

$$E_{st1}(\tilde{x}, \tilde{y}) = D^{st}V_{mn}(f)(\tilde{x}, \tilde{y}), \quad s + t = 1, \tag{3.1}$$

where $(\tilde{x}, \tilde{y}) \in \Omega$ and $D^{st} = \partial^{s+t}/\partial^s x\partial^t y$.

By using the reproduction of $V_{mn}(f)$, and the Taylor representation of f at the point $(\tilde{x}, \tilde{y}) \in \Omega$

$$f(x, y) = f(\tilde{x}, \tilde{y}) + f'_x(u_1, v_1)(x - \tilde{x}) + f'_y(u_1, v_1)(y - \tilde{y}), \tag{3.2}$$

where

$$(u_1, v_1) = \epsilon_1(\tilde{x}, \tilde{y}) + (1-\epsilon_1)(x, y), \quad \epsilon_1 \in (0,1),$$

we have

$$\begin{aligned} &|D^{st}(V_{mn}(f))| = |D^{st}V_{mn}(f - f(\tilde{x}, \tilde{y}))| \\ &\leq \sum_{(i,j)\in I_1} |\lambda_{ij}(f - f(\tilde{x}, \tilde{y}))||D^{st}B_{ij}^1| + \sum_{(i,j)\in I_2} |\mu_{ij}(f - f(\tilde{x}, \tilde{y}))||D^{st}B_{ij}^2|. \end{aligned} \tag{3.3}$$

Thus it is sufficient to derive the boundary of $|\lambda_{ij}(f - f(\tilde{x}, \tilde{y})|$, $|\mu_{ij}(f - f(\tilde{x}, \tilde{y}))|$, $|D^{st}B_{ij}^1|$, $|D^{st}B_{ij}^2|$. In fact, analogous to the proof of Theorem 2.4, it follows

$$|\lambda_{ij}(f - f(\tilde{x}, \tilde{y}))| \leq \frac{2}{3|1+\lambda|}\delta_{mn}\|D^1 f\|, \tag{3.4a}$$

$$|\mu_{ij}(f - f(\tilde{x}, \tilde{y}))| \leq \frac{2}{|\lambda|}\delta_{mn}\|D^1 f\|. \tag{3.4b}$$

Moreover, by computing directly, one can obtain the partial derivatives of the two sets of splines B_{ij}^1 and B_{ij}^2 which depend on the values at three vertices and three midpoints on three edges over each triangular cell as shown in Figs. 7 and 8, respectively. The values of the partial derivatives $D^{10}B_{ij}^1$ and $D^{10}B_{ij}^2$ are listed in Tables 1 and 2, respectively, where

$$A_i = \frac{h_i}{h_i + h_{i+1}}, \quad A_i^{'} = \frac{h_{i+1}}{h_i + h_{i+1}}, \quad B_j = \frac{k_j}{k_j + k_{j+1}}, \quad B_j^{'} = \frac{k_{j+1}}{k_j + k_{j+1}},$$

and $\forall\lambda(\neq 0, -1) \in \mathbb{R}$ throughout this paper. And the values of $D^{01}B_{ij}^1$ and $D^{01}B_{ij}^2$ can be calculated as well, which are omitted here. It should be noted that all the values of $D^{st}B_{ij}^1$

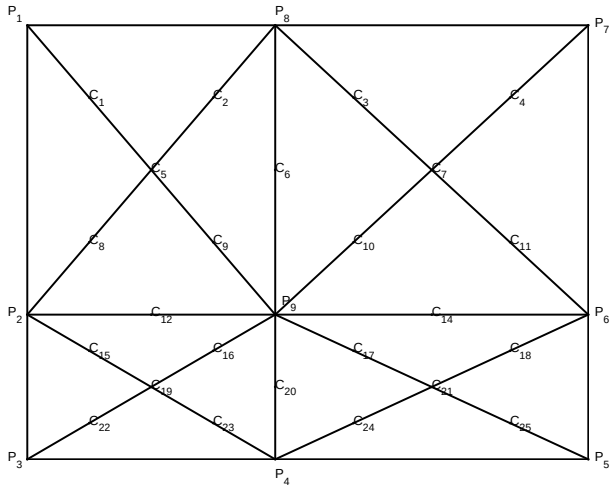

Figure 7: Middle points on the edges in the support of B_{ij}^1.

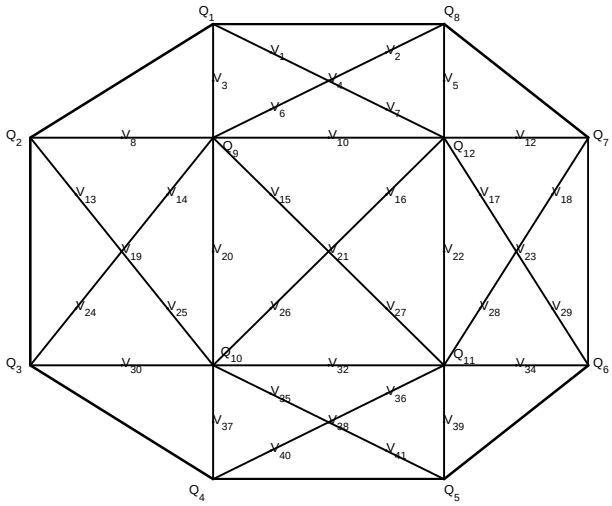

Figure 8: Middle points on the edges in the support of B_{ij}^2.

and $D^{st}B_{ij}^2$ are equal to zero on the boundary of the supports for $s+t=1$. After resolving the maximal values of $D^{st}B_{ij}^1$ and $D^{st}B_{ij}^2$ on each triangular cell, we have

Theorem 3.1. *Let (x,y) in any triangular cell in the supports of B_{ij}^1 and the supports of B_{ij}^2. Then*

$$|D^{st}B_{ij}^1| \le \frac{a_1|1+\lambda|}{\tilde{h}_i^s \tilde{k}_j^t}, \quad |D^{st}B_{ij}^2| \le \frac{b_1|\lambda|}{\tilde{h}_i^s \tilde{k}_j^t}, \tag{3.5}$$

where $s+t=1$, $\tilde{h_i} = \min\{h_i, h_{i+1}, h_{i+2}\}$, $\tilde{k_j} = \min\{k_j, k_{j+1}, k_{j+2}\}$, and a_1, b_1 are real constant.

Hence, by means of (3.4) and (3.5), it follows

Theorem 3.2. *For $f(x,y) \in C^1$, $(x,y) \in \Omega$,*

$$|D^{st}(V_{mn}(f))| \le \Big(\frac{2a_1}{3} + 4b_1\Big)\frac{\delta_{mn}}{\tilde{\delta}_{mn}}\|D^1 f\|, \tag{3.6}$$

where $\tilde{\delta}_{mn} = \min_{ij}\{\tilde{h}_i, \tilde{k}_j\}$.

Case two: we shall consider for $f \in C^2$,

$$E_{st2}(\tilde{x},\tilde{y}) = \begin{cases} D^{st}(f - V_{mn}(f))(\tilde{x},\tilde{y}), & s+t=1, \\ D^{st}V_{mn}(f)(\tilde{x},\tilde{y}), & s+t=2. \end{cases} \tag{3.7}$$

When $s+t=1$, by using the reproduction of $V_{mn}(f)$, Theorem 3.1, and the Taylor representation of f (3.2), that is,

$$\begin{aligned} f(x,y) =& q_1(x,y) + [f_x^{'}(u_1,v_1) - f_x^{'}(x,y)](x-\tilde{x}) \\ &+ [f_y^{'}(u_1,v_1) - f_y^{'}(x,y)](y-\tilde{y}), \end{aligned} \tag{3.8}$$

Table 1: The values of $D^{10}B_{ij}^1(x,y)$ $(\lambda \neq -1, 0)$ in the support.

Points	Values	Points	Values
C_1	$\frac{3(1+\lambda)}{16h_i}$	C_{14}	$-\frac{3(1+\lambda)}{2h_{i+1}}$
C_2	$\frac{3(1+\lambda)}{16h_i}$	C_{15}	$\frac{15(1+\lambda)}{16h_i}$
C_3	$-\frac{3(1+\lambda)}{16h_{i+1}}$	C_{16}	$\frac{15(1+\lambda)}{16h_i}$
C_4	$-\frac{3(1+\lambda)}{16h_{i+1}}$	C_{17}	$-\frac{15(1+\lambda)}{16h_{i+1}}$
C_5	$\frac{3(1+\lambda)}{4h_i}$	C_{18}	$-\frac{15(1+\lambda)}{16h_{i+1}}$
C_6	0	C_{19}	$\frac{3(1+\lambda)}{4h_i}$
C_7	$-\frac{3(1+\lambda)}{4h_{i+1}}$	C_{20}	0
C_8	$\frac{15(1+\lambda)}{16h_i}$	C_{21}	$-\frac{3(1+\lambda)}{4h_{i+1}}$
C_9	$\frac{15(1+\lambda)}{16h_i}$	C_{22}	$\frac{3(1+\lambda)}{16h_i}$
C_{10}	$-\frac{15(1+\lambda)}{16h_{i+1}}$	C_{23}	$\frac{3(1+\lambda)}{16h_i}$
C_{11}	$-\frac{15(1+\lambda)}{16h_{i+1}}$	C_{24}	$-\frac{3(1+\lambda)}{16h_{i+1}}$
C_{12}	$\frac{3(1+\lambda)}{2h_i}$	C_{25}	$-\frac{3(1+\lambda)}{16h_{i+1}}$
P_9	0		

we have

$$\begin{aligned}
&|D^{st}(f - V_{mn}(f))| = |D^{st}V_{mn}(f - q_1)| \\
&\leq \sum_{(i,j)\in I_1} |\lambda_{ij}(f - q_1)||D^{st}B_{ij}^1| + \sum_{(i,j)\in I_2} |\mu_{ij}(f - q_1)||D^{st}B_{ij}^2| \\
&\leq \Big(\frac{2a_1}{3} + 4b_1\Big)\frac{\delta_{mn}}{\tilde{\delta}_{mn}}\|\tilde{\omega}^1 f\|, \qquad (3.9)
\end{aligned}$$

where

$$\tilde{\omega}^k(f) = \max_{l=0,\cdots,k}\left\{\omega_\Omega\Big(\frac{\partial^k f}{\partial^{k-l}x\partial^l y}, \delta_{mn}\Big)\right\}, \quad k = 1, 2, \cdots. \qquad (3.10)$$

When $s + t = 2$, by using the reproduction of $V_{mn}(f)$, and the Taylor representation of f at the point $(\tilde{x}, \tilde{y}) \in \Omega$

$$f(x,y) = q_1(x,y) + \frac{1}{2}\Big[(x-\tilde{x})\frac{\partial}{\partial x} + (y-\tilde{y})\frac{\partial}{\partial y}\Big]^2 f(u_2, v_2), \qquad (3.11)$$

where

$$q_1(x,y) = f(\tilde{x},\tilde{y}) + f_x^{'}(\tilde{x},\tilde{y})(x-\tilde{x}) + f_y^{'}(\tilde{x},\tilde{y})(y-\tilde{y}), \qquad (3.12)$$

and

$$(u_2, v_2) = \epsilon_2(x,y) + (1-\epsilon_2)(\tilde{x},\tilde{y}), \quad \epsilon_2 \in (0,1),$$

Table 2: The values of $D^{10}B_{ij}^2(x,y)$ $(\lambda \neq -1, 0)$ in the support.

Points	Values	Points	Values
V_1	0	V_{22}	$\frac{\lambda}{h_{i+2}}A_{i+1}^{'}[\frac{27}{16}-\frac{3}{4}(B_j^{'}+B_{j+1})]$
V_2	0	V_{23}	$\frac{3\lambda}{4h_{i+2}}A_{i+1}^{'}$
V_3	$-\frac{3\lambda}{4h_i}A_iB_{j+1}^{'}$	V_{24}	$-\frac{3\lambda}{16h_i}A_i$
V_4	0	V_{25}	$\frac{\lambda}{h_i}A_i(-\frac{27}{16}+\frac{3}{4}B_j^{'})$
V_5	$\frac{3\lambda}{4h_{i+2}}A_{i+1}^{'}B_{j+1}^{'}$	V_{26}	$\frac{3\lambda}{16h_{i+1}}A_{i+1}-\frac{\lambda}{h_{i+1}}A_i^{'}(\frac{15}{16}+\frac{3}{4}B_j)$
V_6	$-\frac{3\lambda}{4h_{i+1}}A_i^{'}B_{j+1}^{'}$	V_{27}	$-\frac{3\lambda}{16h_{i+1}}A_i^{'}+\frac{\lambda}{h_{i+1}}A_{i+1}(\frac{15}{16}+\frac{3}{4}B_j)$
V_7	$\frac{3\lambda}{4h_{i+1}}A_{i+1}B_{j+1}^{'}$	V_{28}	$\frac{\lambda}{h_{i+2}}A_{i+1}^{'}(\frac{27}{16}-\frac{3}{4}B_j^{'})$
V_8	$-\frac{3\lambda}{4h_i}A_iB_{j+1}^{'}$	V_{29}	$\frac{3\lambda}{16h_{i+2}}A_{i+1}^{'}$
Q_9	$-\frac{3\lambda}{h_i}A_iB_{j+1}^{'}$	V_{30}	$-\frac{3\lambda}{4h_i}A_iB_j$
V_{10}	$\frac{3\lambda}{4h_{i+1}}(A_{i+1}-A_i^{'})B_{j+1}^{'}$	Q_{10}	$-\frac{3\lambda}{h_i}A_iB_j$
Q_{12}	$\frac{3\lambda}{h_{i+2}}A_{i+1}^{'}B_{j+1}^{'}$	V_{32}	$\frac{3\lambda}{4h_{i+1}}(A_{i+1}-A_i^{'})B_j$
V_{12}	$\frac{3\lambda}{4h_{i+2}}A_{i+1}^{'}B_{j+1}^{'}$	Q_{11}	$\frac{3\lambda}{h_{i+1}}A_{i+1}B_j$
V_{13}	$-\frac{3\lambda}{16h_i}A_i$	V_{34}	$\frac{3\lambda}{4h_{i+2}}A_{i+1}^{'}B_j$
V_{14}	$\frac{\lambda}{h_i}A_i(-\frac{27}{16}+\frac{3}{4}B_{j+1})$	V_{35}	$-\frac{3\lambda}{4h_{i+1}}A_i^{'}B_j$
V_{15}	$\frac{3\lambda}{16h_{i+1}}A_{i+1}-\frac{\lambda}{h_{i+1}}A_i^{'}(\frac{15}{16}+\frac{3}{4}B_{j+1}^{'})$	V_{36}	$\frac{3\lambda}{4h_{i+1}}A_{i+1}B_j$
V_{16}	$-\frac{3\lambda}{16h_{i+1}}A_i^{'}+\frac{\lambda}{h_{i+1}}A_{i+1}(\frac{15}{16}+\frac{3}{4}B_{j+1}^{'})$	V_{37}	$-\frac{3\lambda}{4h_i}A_iB_j$
V_{17}	$\frac{\lambda}{h_{i+2}}A_{i+1}^{'}(\frac{27}{16}-\frac{3}{4}B_{j+1})$	V_{38}	0
V_{18}	$\frac{3\lambda}{16h_{i+2}}A_{i+1}^{'}$	V_{39}	$\frac{3\lambda}{4h_{i+2}}A_{i+1}^{'}B_j$
V_{19}	$-\frac{3\lambda}{4h_i}A_i$	V_{40}	0
V_{20}	$\frac{\lambda}{h_i}A_i[-\frac{27}{16}+\frac{3}{4}(B_j^{'}+B_{j+1})]$	V_{41}	0
V_{21}	$\frac{3\lambda}{4h_{i+1}}(A_{i+1}-A_i^{'})$		

we have

$$\begin{aligned}&|D^{st}(V_{mn}(f))| = |D^{st}V_{mn}(f-q_1)|\\ &\leq \sum_{(i,j)\in I_1}|\lambda_{ij}(f-q_1)||D^{st}B_{ij}^1| + \sum_{(i,j)\in I_2}|\mu_{ij}(f-q_1)||D^{st}B_{ij}^2|. \end{aligned} \tag{3.13}$$

Analogous to Case one, we have

$$|\lambda_{ij}(f-q_1)| \leq \frac{2}{3|1+\lambda|}\delta_{mn}^2\|D^1f\|, \tag{3.14a}$$

$$|\mu_{ij}(f-f(\tilde{x},\tilde{y}))| \leq \frac{4}{|\lambda|}\delta_{mn}^2\|D^1f\|. \tag{3.14b}$$

Moreover, as $D^{st}B^1_{ij}$ and $D^{st}B^2_{ij}$ are linear polynomial in each triangular cell in Ω for $s+t=2$, the maximal values on the triangular cell depend on their values at the three vertices, respectively. After simple computation, it follows:

Theorem 3.3. *Let (x,y) in any triangular cell in the supports of B^1_{ij} and the supports of B^2_{ij}. Then for $s+t=2$*

$$|D^{st}B^1_{ij}| \le \frac{a_2|1+\lambda|}{\tilde{h}_i^s\tilde{k}_j^t}, \quad |D^{st}B^2_{ij}| \le \frac{b_2|\lambda|}{\tilde{h}_i^s\tilde{k}_j^t}, \tag{3.15}$$

where a_2, b_2 are real constant.

Hence, by means of (3.14) and (3.15), it follows

$$\begin{aligned} &|D^{st}(V_{mn}(f))| = |D^{st}V_{mn}(f-q_1)| \\ \le &\sum_{(i,j)\in I_1} |\lambda_{ij}(f-q_1)||D^{st}B^1_{ij}| + \sum_{(i,j)\in I_2} |\mu_{ij}(f-q_1)||D^{st}B^2_{ij}| \\ \le &\Big(\frac{2a_2}{3}+4b_2\Big)\frac{\delta^2_{mn}}{\tilde{\delta}^2_{mn}}\|D^2 f\|. \end{aligned} \tag{3.16}$$

As a result, it follows

Theorem 3.4. *For $f(x,y)\in C^2$, $(\tilde{x},\tilde{y})\in\Omega$,*

$$|D^{st}((f-V_{mn}(f))| \le \Big(\frac{2a_1}{3}+4b_1\Big)\frac{\delta_{mn}}{\tilde{\delta}_{mn}}\|\tilde{\omega}^1 f\|, \qquad s+t=1, \tag{3.17a}$$

$$|D^{st}(V_{mn}(f))| \le \Big(\frac{2a_2}{3}+4b_2\Big)\frac{\delta^2_{mn}}{\tilde{\delta}^2_{mn}}\|D^2 f\|, \qquad s+t=2. \tag{3.17b}$$

Case three: we shall consider for $f\in C^3$,

$$E_{st2}(\tilde{x},\tilde{y}) = \begin{cases} D^{st}(f-V_{mn}(f))(\tilde{x},\tilde{y}), & s+t=1,2, \\ D^{st}V_{mn}(f)(\tilde{x},\tilde{y}), & s+t=3. \end{cases} \tag{3.18}$$

When $s+t=1$, by means of the reproduction of $V_{mn}(f)$, Theorem 3.1, and the Taylor representation of f at the point $(\tilde{x},\tilde{y})\in\Omega$

$$\begin{aligned} f(x,y) =& q_2(x,y) + \frac{1}{2}\big\{[f''_{xx}(u_2,v_2) - f''_{xx}(\tilde{x},\tilde{y})](x-\tilde{x})^2 + 2[f''_{xy}(u_2,v_2) \\ &- f''_{xy}(\tilde{x},\tilde{y})]\cdot(x-\tilde{x})(y-\tilde{y}) + [f''_{yy}(u_2,v_2) - f''_{yy}(\tilde{x},\tilde{y})](y-\tilde{y})^2\big\}, \end{aligned} \tag{3.19}$$

where

$$q_2(x,y) = q_1(x,y) + \frac{1}{2}\Big[(x-\tilde{x})\frac{\partial}{\partial x} + (y-\tilde{y})\frac{\partial}{\partial y}\Big]^2 f(u_2,v_2), \tag{3.20}$$

and

$$(u_2,v_2) = \epsilon_2(x,y) + (1-\epsilon_2)(\tilde{x},\tilde{y}), \quad \epsilon_2\in(0,1),$$

it follows

$$\begin{aligned}
&|D^{st}(f - V_{mn}(f))| = |D^{st}V_{mn}(f - q_2)| \\
\leq &\sum_{(i,j)\in I_1} |\lambda_{ij}(f - q_2)||D^{st}B^1_{ij}| + \sum_{(i,j)\in I_2} |\mu_{ij}(f - q_2)||D^{st}B^2_{ij}| \\
\leq &\Big(\frac{2a_1}{3} + 4b_1\Big)\frac{\delta^2_{mn}}{\tilde{\delta}_{mn}}\tilde{\omega}^2 f,
\end{aligned} \tag{3.21}$$

where $\tilde{\omega}^2 f$ is defined in (3.10). It should be noted that the formula

$$D^{st}(f - V_{mn}(f)) = D^{st}(f - q_2 - V_{mn}(f - q_2)) = -D^{st}V_{mn}(f - q_2)$$

at $(\tilde{x}, \tilde{y}) \in \Omega$ holds for $s + t = 0, 1, 2$, for the formula $f - q_2$ is equal to 0 at the point $(\tilde{x}, \tilde{y})$, by means of the Taylor representation

$$f(x, y) = q_2(x, y) + \frac{1}{3!}\Big[(x - \tilde{x})\frac{\partial}{\partial x} + (y - \tilde{y})\frac{\partial}{\partial y}\Big]^3 f(u_3, v_3), \tag{3.22}$$

where

$$(u_3, v_3) = \epsilon_3(x, y) + (1 - \epsilon_3)(\tilde{x}, \tilde{y}), \quad \epsilon_3 \in (0, 1).$$

When $s + t = 2$, analogous to the case of $s + t = 1$, it follows

$$\begin{aligned}
&|D^{st}(f - V_{mn}(f))| = |D^{st}V_{mn}(f - q_2)| \\
\leq &\sum_{(i,j)\in I_1} |\lambda_{ij}(f - q_2)||D^{st}B^1_{ij}| + \sum_{(i,j)\in I_2} |\mu_{ij}(f - q_2)||D^{st}B^2_{ij}| \\
\leq &\Big(\frac{2a_1}{3} + 4b_1\Big)\frac{\delta^2_{mn}}{\tilde{\delta}^2_{mn}}\tilde{\omega}^2 f.
\end{aligned} \tag{3.23}$$

When $s + t = 3$, the values of $D^{st}B^1_{ij}$ and $D^{st}B^2_{ij}$ are constant on each triangular cell in Ω, respectively. We would like to make a list of the values of $D^{30}B^1_{ij}$ and $D^{21}B^1_{ij}$ on each triangular cell in the support in Table 3, while $D^{30}B^2_{ij}$ and $D^{21}B^2_{ij}$ in Tables 4 and 5. It is apparent that one can obtain the values of $D^{03}B^1_{ij}$, $D^{03}B^2_{ij}$, $D^{12}B^1_{ij}$ and $D^{12}B^2_{ij}$ on each triangular cell in the supports, respectively, which are omitted here. Thus we have

Theorem 3.5. *Let (x, y) in any triangular cell in the supports of B^1_{ij} and the supports of B^2_{ij}. Then for $s + t = 3$*

$$|D^{st}B^1_{ij}| \leq \frac{a_3|1 + \lambda|}{\tilde{h}^s_i \tilde{k}^t_j}, \quad |D^{st}B^2_{ij}| \leq \frac{b_3|\lambda|}{\tilde{h}^s_i \tilde{k}^t_j}, \tag{3.24}$$

where a_3, b_3 are real constant.

Table 3: The values of $D^{30}B^1_{ij}(x,y)$ and $D^{21}B^1_{ij}(x,y)$ on triangular cells in the support.

Triangular cells	$D^{30}B^1_{ij}(x,y)$	$D^{21}B^1_{ij}(x,y)$
Δ^1_1	$-\frac{6(1+\lambda)}{h_i^3}$	$-\frac{6(1+\lambda)}{h_i^2k_{j+1}}$
Δ^1_2	$-\frac{6(1+\lambda)}{h_i^3}$	$\frac{6(1+\lambda)}{h_i^2k_j}$
Δ^1_3	$-\frac{12(1+\lambda)}{h_i^3}$	0
Δ^1_4	$-\frac{12(1+\lambda)}{h_i^3}$	0
Δ^1_5	0	0
Δ^1_6	0	0
Δ^1_7	$\frac{6(1+\lambda)}{h_{i+1}^3}$	$-\frac{6(1+\lambda)}{h_{i+1}^2k_j}$
Δ^1_8	$-\frac{6(1+\lambda)}{h_i^3}$	$-\frac{6(1+\lambda)}{h_i^2k_j}$
Δ^1_9	$\frac{6(1+\lambda)}{h_{i+1}^3}$	$\frac{6(1+\lambda)}{h_{i+1}^2k_j}$
Δ^1_{10}	$\frac{6(1+\lambda)}{h_{i+1}^3}$	$-\frac{6(1+\lambda)}{h_{i+1}^2k_{j+1}}$
Δ^1_{11}	$\frac{12(1+\lambda)}{h_{i+1}^3}$	0
Δ^1_{12}	$\frac{12(1+\lambda)}{h_{i+1}^3}$	0
Δ^1_{13}	0	0
Δ^1_{14}	0	0
Δ^1_{15}	$-\frac{6(1+\lambda)}{h_i^3}$	$\frac{6(1+\lambda)}{h_i^2k_{j+1}}$
Δ^1_{16}	$\frac{6(1+\lambda)}{h_{i+1}^3}$	$\frac{6(1+\lambda)}{h_{i+1}^2k_{j+1}}$

By Theorem 3.5, the reproduction of $V_{mn}(f)$, and the Taylor representation (3.22), we have

$$\begin{aligned}
&|D^{st}V_{mn}(f)| = |D^{st}V_{mn}(f-q_2)| \\
\le &\sum_{(i,j)\in I_1} |\lambda_{ij}(f-q_2)||D^{st}B^1_{ij}| + \sum_{(i,j)\in I_2} |\mu_{ij}(f-q_2)||D^{st}B^2_{ij}| \\
\le &\frac{4}{3}\Big(\frac{a_3}{3}+2b_3\Big)\frac{\delta^3_{mn}}{\tilde{\delta}^3_{mn}}\|D^3 f\|. \qquad (3.25)
\end{aligned}$$

As a result, it follows

Theorem 3.6. *For $f(x,y)\in C^3$, $(\tilde{x},\tilde{y})\in\Omega$,*

$$|D^{st}((f-V_{mn}(f))| \le \Big(\frac{2a_1}{3}+4b_1\Big)\frac{\delta^2_{mn}}{\tilde{\delta}_{mn}}\tilde{\omega}^2 f, \qquad s+t=1, \qquad (3.26a)$$

$$|D^{st}((f-V_{mn}(f))| \le \Big(\frac{2a_2}{3}+4b_2\Big)\frac{\delta^2_{mn}}{\tilde{\delta}^2_{mn}}\tilde{\omega}^2 f, \qquad s+t=2, \qquad (3.26b)$$

Table 4: The values of $D^{30}B_{ij}^2(x,y)$ and $D^{21}B_{ij}^2(x,y)$ on triangular cells in the support (Part I).

Triangular cells	$D^{30}B_{ij}^2(x,y)$	$D^{21}B_{ij}^2(x,y)$
Δ_1^2	$-\frac{6\lambda}{h_i^3}A_iB_{j+1}^{'}$	$\frac{6\lambda}{h_i^2k_{j+2}}A_iB_{j+1}^{'}$
Δ_2^2	$-\frac{6\lambda}{h_{i+1}^3}A_i^{'}B_{j+1}^{'}$	$-\frac{6\lambda}{h_{i+1}^2k_{j+2}}A_i^{'}B_{j+1}^{'}$
Δ_3^2	$-\frac{6\lambda}{h_i^3}A_i$	0
Δ_4^2	$-\frac{6\lambda}{h_i^3}A_iB_{j+1}^{'}$	$\frac{6\lambda}{h_i^2k_{j+1}}A_iB_{j+1}$
Δ_5^2	$-\frac{6\lambda}{h_i^3}A_iB_j$	$-\frac{6\lambda}{h_i^2k_j}A_iB_j$
Δ_6^2	$-\frac{6\lambda}{h_i^3}A_iB_j$	$-\frac{6\lambda}{h_i^2k_{j+1}}A_iB_j^{'}$
Δ_7^2	0	0
Δ_8^2	$-\frac{6\lambda}{h_{i+1}^3}A_i^{'}B_j$	$\frac{6\lambda}{h_{i+1}^2k_j}A_i^{'}B_j$
Δ_9^2	$\frac{6\lambda}{h_{i+2}^3}A_{i+1}^{'}B_j$	$-\frac{6\lambda}{h_{i+2}^2k_j}A_{i+1}^{'}B_j$
Δ_{10}^2	$\frac{6\lambda}{h_{i+1}^3}A_{i+1}B_j$	$\frac{6\lambda}{h_{i+1}^2k_j}A_{i+1}B_j$
Δ_{11}^2	$\frac{6\lambda}{h_{i+2}^3}A_{i+1}^{'}$	0
Δ_{12}^2	$\frac{6\lambda}{h_{i+2}^3}A_{i+1}^{'}B_j$	$-\frac{6\lambda}{h_{i+2}^2k_{j+1}}A_{i+1}^{'}B_j^{'}$
Δ_{13}^2	$\frac{6\lambda}{h_{i+2}^3}A_{i+1}^{'}B_{j+1}^{'}$	$\frac{6\lambda}{h_{i+2}^2k_{j+2}}A_{i+1}^{'}B_{j+1}^{'}$
Δ_{14}^2	$\frac{6\lambda}{h_{i+2}^3}A_{i+1}^{'}B_{j+1}^{'}$	$\frac{6\lambda}{h_{i+2}^2k_{j+1}}A_{i+1}^{'}B_{j+1}$
Δ_{15}^2	0	0
Δ_{16}^2	$\frac{6\lambda}{h_{i+1}^3}A_{i+1}B_{j+1}^{'}$	$-\frac{6\lambda}{h_{i+1}^2k_{j+2}}A_{i+1}B_{j+1}^{'}$
Δ_{17}^2	$\frac{6\lambda}{h_i^3}A_i(B_{j+1}-B_j)$	$\frac{6\lambda}{h_i^2k_{j+1}}A_i(B_{j+1}-B_j^{'})$
Δ_{18}^2	$\frac{6\lambda}{h_{i+1}^3}[A_i^{'}(B_{j+1}-B_j)+A_{i+1}]$	$\frac{6\lambda}{h_{i+1}^2k_{j+1}}A_i^{'}(B_j^{'}-B_{j+1})$

Table 5: The values of $D^{30}B_{ij}^2(x,y)$ and $D^{21}B_{ij}^2(x,y)$ on triangular cells in the support (Part II).

Triangular cells	$D^{30}B_{ij}^2(x,y)$	$D^{21}B_{ij}^2(x,y)$
Δ_{19}^2	$\frac{6\lambda}{h_{i+1}^3}(A_{i+1}-A_i^{'})B_{j+1}^{'}$	$-\frac{6\lambda}{h_{i+1}^2k_{j+1}}(A_i^{'}+A_{i+1})B_{j+1}$
Δ_{20}^2	$\frac{6\lambda}{h_{i+1}^3}(A_{i+1}-A_i^{'})B_{j+1}^{'}$	$-\frac{6\lambda}{h_{i+1}^2k_{j+2}}(A_i^{'}+A_{i+1})B_{j+1}^{'}$
Δ_{21}^2	$\frac{6\lambda}{h_{i+1}^3}(A_{i+1}-A_i^{'})B_j$	$\frac{6\lambda}{h_{i+1}^2k_j}(A_i^{'}+A_{i+1})B_j$
Δ_{22}^2	$\frac{6\lambda}{h_{i+1}^3}(A_{i+1}-A_i^{'})B_j$	$\frac{6\lambda}{h_{i+1}^2k_{j+1}}(A_i^{'}+A_{i+1})B_j^{'}$
Δ_{23}^2	$\frac{6\lambda}{h_{i+2}^3}A_{i+1}^{'}(B_{j+1}^{'}-B_{j+1}^{'})$	$\frac{6\lambda}{h_{i+2}^2k_{j+1}}A_{i+1}^{'}(B_{j+1}-B_j^{'})$
Δ_{24}^2	$\frac{6\lambda}{h_{i+1}^3}[A_{i+1}(B_j-B_{j+1})-A_i^{'}]$	$\frac{6\lambda}{h_{i+1}^2k_{j+1}}A_{i+1}(B_j^{'}-B_{j+1})$

$$|D^{st}(V_{mn}(f))| \le \Big(\frac{4a_3}{9}+\frac{8b_3}{3}\Big)\frac{\delta_{mn}^3}{\tilde{\delta}_{mn}^3}\|D^3 f\|, \qquad s+t=3. \tag{3.26c}$$

By investigating the theorems in above, one can conclude that the item $\delta_{mn}/\tilde{\delta}_{mn}$ plays a role in the uniform approximation of the derivatives. Thus we shall recall the following definition introduced in [2]. Similarly,

Definition 3.1. *A sequence of type-2 triangulation $\Delta_{mn}^{(2)}$ of Ω is quasi-uniform if there exists a positive constant ρ such that*

$$\frac{h_{mn}}{\tilde{h}_{mn}} \le \rho, \quad \frac{h_{mn}}{\tilde{k}_{mn}} \le \rho, \quad \frac{k_{mn}}{\tilde{h}_{mn}} \le \rho, \quad \frac{k_{mn}}{\tilde{h}_{mn}} \le \rho. \tag{3.27}$$

Moreover a sequence of cubic spline space $S_3^{1,2}(\Delta_{mn}^{(2)})$ is quasi-uniform if they are based on a sequence of quasi-uniform type-2 triangulation.

Therefore, we can conclude

Remark 3.1. $D^{st}V_{mn}(f)$ approximates $D^{st}f$ uniformly as δ_{mn} approaches to 0 for $f \in C^{r-1}(\Omega)$, $1 < r \le s+t \le 3$ over quasi-uniform type-2 uniform. In fact, by Definition 3.1, it follows that $(\delta_{mn}/\tilde{\delta}_{mn})^{s+t}$ are bounded.

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Appendix

The representation of the splines $B_{ij}^1(x,y)$, $B_{ij}^2(x,y)$ on each triangular cell in the supports is established as follows where $\lambda \in (-1,0)$.

$$B_{ij}^1(\Delta_1^1) = (1+\lambda)\Big[-\frac{1}{h_i^3}(x-x_{i-1}) - \frac{3}{h_i^2 k_{j+1}}(y-y_{j+1})\Big](x-x_{i-1})^2,$$
$$B_{ij}^1(\Delta_2^1) = (1+\lambda)\Big[-\frac{1}{h_i^3}(x-x_{i-1}) + \frac{3}{h_i^2 k_j}(y-y_{j-1})\Big](x-x_{i-1})^2,$$

$$B_{ij}^{1}(\Delta_3^1)=(1+\lambda)\Big[-\frac{1}{h_i^3}(x-x_{i-1})+\frac{3}{h_i^2k_j}(y-y_{j-1})\Big](x-x_{i-1})^2 -\frac{1+\lambda}{k_j^3}\Big[y-y_j+\frac{k_j}{h_i}(x-x_{i-1})\Big]^3,$$

$$B_{ij}^{1}(\Delta_4^1)=(1+\lambda)\Big[-\frac{1}{h_i^3}(x-x_{i-1})-\frac{3}{h_i^2k_{j+1}}(y-y_{j+1})\Big](x-x_{i-1})^2 +\frac{1+\lambda}{k_{j+1}^3}\Big[y-y_j-\frac{k_{j+1}}{h_i}(x-x_{i-1})\Big]^3,$$

$$B_{ij}^{1}(\Delta_5^1)=(1+\lambda)\Big[\frac{3}{h_ik_j^2}(x-x_{i-1})-\frac{1}{k_j^3}(y-y_{j-1})\Big](y-y_{j-1})^2,$$

$$B_{ij}^{1}(\Delta_6^1)=(1+\lambda)\Big[-\frac{3}{h_{i+1}k_j^2}(x-x_{i+1})-\frac{1}{k_j^3}(y-y_{j-1})\Big](y-y_{j-1})^2,$$

$$B_{ij}^{1}(\Delta_7^1)=(1+\lambda)\Big[-\frac{3}{h_{i+1}k_j^2}(x-x_{i+1})-\frac{1}{k_j^3}(y-y_{j-1})\Big](y-y_{j-1})^2 -\frac{1+\lambda}{k_j^3}\Big[y-y_{j-1}-\frac{k_j}{h_{i+1}}(x-x_i)\Big]^3,$$

$$B_{ij}^{1}(\Delta_8^1)=(1+\lambda)\Big[\frac{3}{h_ik_j^2}(x-x_{i-1})-\frac{1}{k_j^3}(y-y_{j-1})\Big](y-y_{j-1})^2 -\frac{1+\lambda}{k_j^3}\Big[y-y_{j-1}+\frac{k_j}{h_i}(x-x_i)\Big]^3,$$

$$B_{ij}^{1}(\Delta_9^1)=(1+\lambda)\Big[\frac{1}{h_{i+1}^3}(x-x_{i+1})+\frac{3}{h_{i+1}^2k_j}(y-y_{j-1})\Big](x-x_{i+1})^2,$$

$$B_{ij}^{1}(\Delta_{10}^1)=(1+\lambda)\Big[\frac{1}{h_{i+1}^3}(x-x_{i+1})-\frac{3}{h_{i+1}^2k_{j+1}}(y-y_{j+1})\Big](x-x_{i+1})^2,$$

$$B_{ij}^{1}(\Delta_{11}^1)=(1+\lambda)\Big[\frac{1}{h_{i+1}^3}(x-x_{i+1})-\frac{3}{h_{i+1}^2k_{j+1}}(y-y_{j+1})\Big](x-x_{i+1})^2 +\frac{1+\lambda}{k_{j+1}^3}\Big[y-y_j+\frac{k_{j+1}}{h_{i+1}}(x-x_{i+1})\Big]^3,$$

$$B_{ij}^{1}(\Delta_{12}^1)=(1+\lambda)\Big[\frac{1}{h_{i+1}^3}(x-x_{i+1})+\frac{3}{h_{i+1}^2k_j}(y-y_{j-1})\Big](x-x_{i+1})^2 -\frac{1+\lambda}{k_j^3}\Big[y-y_j-\frac{k_j}{h_{i+1}}(x-x_{i+1})\Big]^3,$$

$$B_{ij}^{1}(\Delta_{13}^1)=(1+\lambda)\Big[-\frac{3}{h_{i+1}k_{j+1}^2}(x-x_{i+1})+\frac{1}{k_{j+1}^3}(y-y_{j+1})\Big](y-y_{j+1})^2,$$

$$B_{ij}^1(\Delta_{14}^1) = (1+\lambda)\Big[\frac{3}{h_i k_{j+1}^2}(x-x_{i-1}) + \frac{1}{k_{j+1}^3}(y-y_{j+1})\Big](y-y_{j+1})^2,$$

$$B_{ij}^1(\Delta_{15}^1) = (1+\lambda)\Big[\frac{3}{h_i k_{j+1}^2}(x-x_{i-1}) + \frac{1}{k_{j+1}^3}(y-y_{j+1})\Big](y-y_{j+1})^2 + \frac{1+\lambda}{k_{j+1}^3}\Big[y-y_{j+1}-\frac{k_{j+1}}{h_i}(x-x_i)\Big]^3,$$

$$B_{ij}^1(\Delta_{16}^1) = (1+\lambda)\Big[-\frac{3}{h_{i+1} k_{j+1}^2}(x-x_{i+1}) + \frac{1}{k_{j+1}^3}(y-y_{j+1})\Big](y-y_{j+1})^2 + \frac{1+\lambda}{k_{j+1}^3}\Big[y-y_{j+1}+\frac{k_{j+1}}{h_{i+1}}(x-x_i)\Big]^3,$$

$$B_{ij}^2(\Delta_1^2) = \frac{\lambda h_i}{(h_i+h_{i+1})k_{j+2}^2(k_{j+1}+k_{j+2})}\Big[y-y_{j+2}-\frac{k_{j+2}}{h_i}(x-x_i)\Big]^3,$$

$$B_{ij}^2(\Delta_2^2) = \frac{\lambda}{k_{j+2}^2(k_{j+1}+k_{j+2})}(y-y_{j+2})^3 - \frac{\lambda h_{i+1}}{(h_i+h_{i+1})k_{j+2}^2(k_{j+1}+k_{j+2})}\Big[y-y_{j+2}+\frac{k_{j+2}}{h_{i+1}}(x-x_i)\Big]^3,$$

$$B_{ij}^2(\Delta_3^2) = -\frac{\lambda}{h_i^2(h_i+h_{i+1})}(x-x_{i-1})^3,$$

$$B_{ij}^2(\Delta_4^2) = -\frac{\lambda}{h_i^2(h_i+h_{i+1})}(x-x_{i-1})^3 + \frac{\lambda h_i}{(h_i+h_{i+1})k_{j+1}^2(k_{j+1}+k_{j+2})}\Big[y-y_{j+1}+\frac{k_{j+1}}{h_i}(x-x_{i-1})\Big]^3,$$

$$B_{ij}^2(\Delta_5^2) = -\frac{\lambda h_i}{(h_i+h_{i+1})k_j^2(k_j+k_{j+1})}\Big[y-y_j+\frac{k_j}{h_i}(x-x_{i-1})\Big]^3,$$

$$B_{ij}^2(\Delta_6^2) = -\frac{\lambda}{h_i^2(h_i+h_{i+1})}(x-x_{i-1})^3 - \frac{\lambda h_i}{(h_i+h_{i+1})k_{j+1}^2(k_j+k_{j+1})}\Big[y-y_j-\frac{k_{j+1}}{h_i}(x-x_{i-1})\Big]^3,$$

$$B_{ij}^2(\Delta_7^2) = -\frac{\lambda}{k_j^2(k_j+k_{j+1})}(y-y_{j-1})^3,$$

$$B_{ij}^2(\Delta_8^2) = -\frac{\lambda}{k_j^2(k_j+k_{j+1})}(y-y_{j-1})^3 + \frac{\lambda h_{i+1}}{(h_i+h_{i+1})k_j^2(k_j+k_{j+1})}\Big[y-y_{j-1}-\frac{k_j}{h_{i+1}}(x-x_i)\Big]^3,$$

$$B_{ij}^2(\Delta_9^2) = -\frac{\lambda h_{i+2}}{(h_{i+1}+h_{i+2})k_j^2(k_j+k_{j+1})}\left[y - y_{j-1} - \frac{k_j}{h_{i+2}}(x - x_{i+1})\right]^3,$$

$$B_{ij}^2(\Delta_{10}^2) = -\frac{\lambda}{k_j^2(k_j+k_{j+1})}(y - y_{j-1})^3 + \frac{\lambda h_{i+1}}{(h_{i+1}+h_{i+2})k_j^2(k_j+k_{j+1})}\left[y - y_{j-1} + \frac{k_j}{h_{i+1}}(x - x_{i+1})\right]^3,$$

$$B_{ij}^2(\Delta_{11}^2) = \frac{\lambda}{h_{i+2}^2(h_{i+1}+h_{i+2})}(x - x_{i+2})^3,$$

$$B_{ij}^2(\Delta_{12}^2) = \frac{\lambda}{h_{i+2}^2(h_{i+1}+h_{i+2})}(x - x_{i+2})^3 - \frac{\lambda h_{i+2}}{(h_{i+1}+h_{i+2})k_{j+1}^2(k_j+k_{j+1})}\left[y - y_j + \frac{k_{j+1}}{h_{i+2}}(x - x_{i+2})\right]^3,$$

$$B_{ij}^2(\Delta_{13}^2) = \frac{\lambda h_{i+2}}{(h_{i+1}+h_{i+2})k_{j+2}^2(k_{j+1}+k_{j+2})}\left[y - y_{j+2} + \frac{k_{j+2}}{h_{i+2}}(x - x_{i+1})\right]^3,$$

$$B_{ij}^2(\Delta_{14}^2) = \frac{\lambda}{h_{i+2}^2(h_{i+1}+h_{i+2})}(x - x_{i+2})^3 + \frac{\lambda h_{i+2}}{(h_{i+1}+h_{i+2})k_{j+1}^2(k_{j+1}+k_{j+2})}\left[y - y_{j+1} - \frac{k_{j+1}}{h_{i+2}}(x - x_{i+2})\right]^3,$$

$$B_{ij}^2(\Delta_{15}^2) = \frac{\lambda}{k_{j+2}^2(k_{j+1}+k_{j+2})}(y - y_{j+2})^3,$$

$$B_{ij}^2(\Delta_{16}^2) = \frac{\lambda}{k_{j+2}^2(k_{j+1}+k_{j+2})}(y - y_{j+2})^3 - \frac{\lambda h_{i+1}}{(h_{i+1}+h_{i+2})k_{j+2}^2(k_{j+1}+k_{j+2})}\left[y - y_{j+2} - \frac{k_{j+2}}{h_{i+1}}(x - x_{i+1})\right]^3,$$

$$B_{ij}^2(\Delta_{17}^2) = -\frac{\lambda}{h_i^2(h_i+h_{i+1})}(x - x_{i-1})^3 + \frac{\lambda h_i}{(h_i+h_{i+1})k_{j+1}^2(k_{j+1}+k_{j+2})}\left[y - y_{j+1} + \frac{k_{j+1}}{h_i}(x - x_{i-1})\right]^3 - \frac{\lambda h_i}{(h_i+h_{i+1})k_{j+1}^2(k_j+k_{j+1})}\left[y - y_{j+1} - \frac{k_{j+1}}{h_i}(x - x_i)\right]^3,$$

$$B_{ij}^2(\Delta_{18}^2) = -\frac{\lambda}{h_i^2(h_i+h_{i+1})}(x - x_{i-1})^3 + \frac{\lambda h_i}{(h_i+h_{i+1})k_{j+1}^2(k_{j+1}+k_{j+2})}\left[y - y_{j+1} + \frac{k_{j+1}}{h_i}(x - x_{i-1})\right]^3$$

$$
\begin{aligned}
&- \frac{\lambda h_i}{(h_i+h_{i+1})k_{j+1}^2(k_j+k_{j+1})}\Big[y-y_{j+1}-\frac{k_{j+1}}{h_i}(x-x_i)\Big]^3\\
&+ \frac{3\lambda}{h_ih_{i+1}}\Big(\frac{1}{k_j+k_{j+1}}-\frac{1}{k_{j+1}+k_{j+2}}\Big)(x-x_i)^2(y-y_{j+1})\\
&+ \frac{\lambda(h_{i+1}-h_i)}{h_i^2h_{i+1}^2}\Big(\frac{k_j}{k_j+k_{j+1}}-\frac{k_{j+1}}{k_{j+1}+k_{j+2}}\Big)(x-x_i)^3\\
&+ \frac{\lambda}{h_{i+1}^2(h_{i+1}+h_{i+2})}(x-x_i)^3+\frac{3\lambda k_{j+2}}{h_ih_{i+1}(k_{j+1}+k_{j+2})}(x-x_i)^2,
\end{aligned}
$$

$$
\begin{aligned}
B_{ij}^2(\Delta_{19}^2) = &\frac{\lambda}{k_{j+2}^2(k_{j+1}+k_{j+2})}(y-y_{j+2})^3\\
&- \frac{\lambda h_{i+1}}{(h_i+h_{i+1})k_{j+2}^2(k_{j+1}+k_{j+2})}\Big[y-y_{j+2}+\frac{k_{j+2}}{h_{i+1}}(x-x_i)\Big]^3\\
&- \frac{\lambda h_{i+1}}{(h_{i+1}+h_{i+2})k_{j+2}^2(k_{j+1}+k_{j+2})}\Big[y-y_{j+1}-\frac{k_{j+2}}{h_{i+1}}(x-x_i)\Big]^3\\
&+ \frac{3\lambda}{k_{j+1}k_{j+2}}\Big(\frac{1}{h_i+h_{i+1}}-\frac{1}{h_{i+1}+h_{i+2}}\Big)(x-x_i)(y-y_{j+1})^2\\
&+ \frac{\lambda(k_{j+1}-k_{j+2})}{k_{j+1}^2k_{j+2}^2}\Big(\frac{h_{i+1}}{h_{i+1}+h_{i+2}}-\frac{h_i}{h_i+h_{i+1}}\Big)(y-y_{j+1})^3\\
&- \frac{\lambda}{k_{j+1}^2(k_j+k_{j+1})}(y-y_{j+1})^3+\frac{3\lambda h_i}{(h_i+h_{i+1})k_{j+1}k_{j+2}}(y-y_{j+1})^2,
\end{aligned}
$$

$$
\begin{aligned}
B_{ij}^2(\Delta_{20}^2) = &\frac{\lambda}{k_{j+2}^2(k_{j+1}+k_{j+2})}(y-y_{j+2})^3\\
&- \frac{\lambda h_{i+1}}{(h_i+h_{i+1})k_{j+2}^2(k_{j+1}+k_{j+2})}\Big[y-y_{j+2}+\frac{k_{j+2}}{h_{i+1}}(x-x_i)\Big]^3\\
&- \frac{\lambda h_{i+1}}{(h_{i+1}+h_{i+2})k_{j+2}^2(k_{j+1}+k_{j+2})}\Big[y-y_{j+1}-\frac{k_{j+2}}{h_{i+1}}(x-x_i)\Big]^3,
\end{aligned}
$$

$$
\begin{aligned}
B_{ij}^2(\Delta_{21}^2) = &-\frac{\lambda}{k_j^2(k_j+k_{j+1})}(y-y_{j-1})^3\\
&+ \frac{\lambda h_{i+1}}{(h_{i+1}+h_{i+2})k_j^2(k_j+k_{j+1})}\Big[y-y_{j-1}+\frac{k_j}{h_{i+1}}(x-x_{i+1})\Big]^3\\
&+ \frac{\lambda h_{i+1}}{(h_i+h_{i+1})k_j^2(k_j+k_{j+1})}\Big[y-y_j-\frac{k_j}{h_{i+1}}(x-x_{i+1})\Big]^3,
\end{aligned}
$$

$$
B_{ij}^2(\Delta_{22}^2) = -\frac{\lambda}{k_j^2(k_j+k_{j+1})}(y-y_{j-1})^3
$$

$$
\begin{aligned}
&+\frac{\lambda h_{i+1}}{(h_i+h_{i+1})k_j^2(k_j+k_{j+1})}\Big[y-y_{j-1}-\frac{k_j}{h_{i+1}}(x-x_i)\Big]^3\\
&+\frac{\lambda h_{i+1}}{(h_{i+1}+h_{i+2})k_j^2(k_j+k_{j+1})}\Big[y-y_j+\frac{k_j}{h_{i+1}}(x-x_i)\Big]^3\\
&+\frac{3\lambda}{k_jk_{j+1}}\Big(\frac{1}{h_i+h_{i+1}}-\frac{1}{h_{i+1}+h_{i+2}}\Big)(x-x_i)(y-y_j)^2\\
&+\frac{\lambda(k_{j+1}-k_j)}{k_j^2k_{j+1}^2}\Big(\frac{h_i}{h_i+h_{i+1}}-\frac{h_{i+1}}{h_{i+1}+h_{i+2}}\Big)(y-y_j)^3\\
&+\frac{\lambda}{k_{j+1}^2(k_{j+1}+k_{j+2})}(y-y_j)^3+\frac{3\lambda h_i}{(h_i+h_{i+1})k_jk_{j+1}}(y-y_j)^2,
\end{aligned}
$$

$$
\begin{aligned}
B_{ij}^2(\Delta_{23}^2)=&\frac{\lambda}{h_{i+2}^2(h_{i+1}+h_{i+2})}(x-x_{i+2})^3\\
&+\frac{\lambda h_{i+2}}{(h_{i+1}+h_{i+2})k_{j+1}^2(k_{j+1}+k_{j+2})}\Big[y-y_{j+1}-\frac{k_{j+1}}{h_{i+2}}(x-x_{i+2})\Big]^3\\
&-\frac{\lambda h_{i+2}}{(h_{i+1}+h_{i+2})k_{j+1}^2(k_j+k_{j+1})}\Big[y-y_{j+1}+\frac{k_{j+1}}{h_{i+2}}(x-x_{i+1})\Big]^3,
\end{aligned}
$$

$$
\begin{aligned}
B_{ij}^2(\Delta_{24}^2)=&\frac{\lambda}{h_{i+2}^2(h_{i+1}+h_{i+2})}(x-x_{i+2})^3\\
&+\frac{\lambda h_{i+2}}{(h_{i+1}+h_{i+2})k_{j+1}^2(k_{j+1}+k_{j+2})}\Big[y-y_{j+1}-\frac{k_{j+1}}{h_{i+2}}(x-x_{i+2})\Big]^3\\
&-\frac{\lambda h_{i+2}}{(h_{i+1}+h_{i+2})k_{j+1}^2(k_j+k_{j+1})}\Big[y-y_{j+1}+\frac{k_{j+1}}{h_{i+2}}(x-x_{i+1})\Big]^3\\
&+\frac{3\lambda}{h_{i+1}h_{i+2}}\Big(\frac{1}{k_j+k_{j+1}}-\frac{1}{k_{j+1}+k_{j+2}}\Big)(x-x_{i+1})^2(y-y_{j+1})\\
&+\frac{\lambda(h_{i+1}-h_{i+2})}{h_{i+1}^2h_{i+2}^2}\Big(\frac{k_{j+1}}{k_{j+1}+k_{j+2}}-\frac{k_j}{k_j+k_{j+1}}\Big)(x-x_{i+1})^3\\
&-\frac{\lambda}{h_{i+1}^2(h_i+h_{i+1})}(x-x_{i+1})^3+\frac{3\lambda k_{j+2}}{h_{i+1}h_{i+2}(k_{j+1}+k_{j+2})}(x-x_{i+1})^2.
\end{aligned}
$$

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