J. Partial Differential Equations
Vol. 1 № 1 (1988), 85—95

HYPOELLIPTICITY OF NONLINEAR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

Xu Chaojiang
Dept. of Math., Wuhan univ.
Received May 21, 1987

1. Introduction

In order to study nonlinear PDE, the theory of paradifferential operator was introduced by J. M. Bony in [2]. Using this theory, the propagation and interaction of singularities of the solution of non-linear hyperbolic equations were studied by [1], [2] and [3]. This paper will use Bony's theory on the hypoellipticity of non-linear partial differential equations, an abstract of main results of this paper have been published in [8].

Consider the following nonlinear second order partial differential equations:

$$F(x,\ u,\ \nabla u,\ \nabla^2 u)=0 \tag{1.1}$$

where $x\in\Omega$, $\Omega\subset R^n$ is an open set; F is a real valued C^∞ function of real variables. Given a real function $u\in C^\rho_{loc}(\Omega)$, $\rho\geq 4$; we define

$$L=\sum_{j,k=1}^{n} a_{kj}(x)\,\partial_j\partial_k+\sum_{j=1}^{n} b_j(x)\,\partial_j+c(x) \tag{1.2}$$

which is the linearized operator associated with the equation (1.1) for u; where $a_{jk}=a_{kj}\ (j,\ k=1,\ 2,\ \ldots,\ n)$

$$\begin{cases} a_{jk}(x)=\dfrac{\partial F}{\partial u_{jk}}(x,\ u(x),\ \nabla u(x),\ \nabla^2 u(x)) \\ b_j(x)=\dfrac{\partial F}{\partial u_j}(x,\ u(x),\ \nabla u(x),\ \nabla^2 u(x)) \qquad j,\ k=1,\ 2,\ \ldots,\ n \\ c(x)=\dfrac{\partial F}{\partial u}(x,\ u(x),\ \nabla u(x),\ \nabla^2 u(x)) \end{cases} \tag{1.3}$$

are all real functions in $C^{\rho-2}_{loc}$. Let us first give the following definition:

Definition 1.1. *The linear operator (1.2) is said to be subelliptic, if $(a_{jk}(x))\geq 0$ for any $x\in\Omega$; and for every compact subset $K\subset\Omega$, there exist constants $\varepsilon>0$, $C>0$, such that for all $\varphi\in C_0^\infty(K)$, the subelliptic estimate:*

$$\|\varphi\|^2_\varepsilon\leq C\{|\langle L\varphi,\ \varphi\rangle|+\|\varphi\|^2_0\} \tag{1.4}$$

holds.

Our main theorem is as follows:

Theorem 1.2. *Let $u\in C^\rho_{loc}(\Omega)$, $\rho\geq 4$ be a real solution of equation (1.1). If the linearized operator defined by (1.2) is subelliptic, then the solution $u\in C^\infty(\Omega)$.*

If L is a self-adjoint operator, and the subelliptic index ε is independent of K. Then the consequence of theorem 1.2 is still true if we only suppose $\rho>4\text{-}2\varepsilon$. Now it remains to find the sufficient conditions for operator L to be subelliptic. First, if L is elliptic, it is also subelliptic, and $\varepsilon=1$ in this case; this is a classical result. Secondly if operator is degenerate, of course we will consider the so-called Hörmander conditions and Oleinik-Radkevic conditions (see[5], [7]) respectively.

For general operator (1.2), Let

$$\begin{cases} g_j(x, \xi) = \sum_{k=1}^{n} a_{jk}(x)(i\xi_k). & j=1, \ldots, n \\ g_{n+l}(x, \xi) = |\xi|^{-1} \sum_{j,k=1}^{n} \frac{\partial a_{jk}(x)}{\partial x_l} \xi_j \xi_k. & l=1, \ldots, n \\ g_0(x, \xi) = \sum_{j=1}^{n} (b_j(x) - \sum_{k=1}^{n} \frac{\partial a_{jk}}{\partial x_k}(x))(i\xi_j) & \end{cases} \tag{1.5}$$

Then functions $g_j(x, \xi)$ $(0 \leq j \leq 2n)$ are homogeneous of degree 1 and C^∞ in variable $\xi \neq 0$, and $C_{loc}^{\rho-3}$ in variable x. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$, $0 \leq \alpha_j \leq 2n$ be a multi-index, denote $|\alpha| = k$. If $\rho - 3 \geq |\alpha|$, we define

$$g_\alpha(x, \xi) = (-1)^{k-1}\{g_{\alpha_1}, \ldots, \{g_{\alpha_{k-1}}, g_{\alpha_k}\}, \ldots\} \tag{1.6}$$

to be the Poisson multi-bracket. Then function g_α is a homogeneous degree 1 and C^∞ in $\xi \neq 0$, and $C_{loc}^{\rho-k-2}$ in x. We have:

Theorem 1.3. *Let $u \in C_{loc}^{\rho}(\Omega)$ be a real solution of equation (1.1); if there exists positive integer p, such that $\rho \geq p+3$, and the linearized operator (1.2) satisfies:*
(i) $(a_{jk}(x)) \geq 0$ for all $x \in \Omega$.
(ii) For any compact subset $K \subset \Omega$, there exists $C > 0$, such that:

$$\sum_{|\alpha| \leq p} |g_\alpha(x, \xi)|^2 \geq C|\xi|^2 \quad \textit{for all } (x, \xi) \in K \times R^n, \ |\xi| \geq R > 0 \tag{1.7}$$

Then the linearized operator L is subelliptic, i. e. $u \in C^\infty$.

If (1.1) is a quasi-linear equation, i. e.

$$\sum_{j=1}^{n} X_j^2 u + X_0 u + f(x, u) = 0 \tag{1.8}$$

where $X_j = \sum_{k=1}^{n} a_{kj}(x, u)\partial_k$, $j = 0, 1, \ldots, m$; a_{kj} and f are all C^∞ real valued functions of real variables. Then replacing $g_j(x, \xi)$ above by

$$X_j(x, \xi) = \sum_{k=1}^{n} a_{kj}(x, u(x))(i\xi_k), \qquad j = 0, 1, \ldots, m$$

theorem 1.3 still holds, under the condition $\rho \geq \max\{2, p\}$.

In theorem 1.2, we need the solution u to be at least C^4. This condition can hardly be improved when (1.1) is a general non-linear and genuinely degenerate equation. C. Zuily[12] proved that there is a solution for a class of degenerate Monge-Ampère equations, which belongs to $C^{2+\varepsilon}$, but not to C^3. In theorem 1.3, more smoothness for the solution u is required. This is because that we need the coefficients of operator to be smooth enough under our assumption, for the Poisson brackets to be definable. In order to improve the condition in theorem 1.3, we introduced the so-called Fefferman-Phong condition in [9], which is kind of geometric subelliptic condition. In [9], we need only $u \in C^4$. Because of the subelliptic conditions in the preceding theorems are all given on a linearized operator of solution u, which must be dependent on u. In [12], for Monge-Ampère equation $\det(u_{ij})(x) = \psi(x)$, C. Zuily gave a condition on function $\psi(x)$, such that its linearized operator satisfies the condition in theorem 1.3, this means the condition in theorem 1.3 may be independent of solution u under some cases. On the other hand, we studied the boundary value problem for a class of non-linear equation (1.1) in [11]; and in [10], higher order equation was discussed.

The plan of this paper is as follows: In Section 2, we will prove the so-called paralinearization theorem of equation (1.1). Section 3 will give the proofs of theorem 1.2 and 1.3. Finally some degenerate cases of theorem 1.3 will be discussed in Section 4.

This paper is a part of my doctorate thesis at the université de Paris-Sud (Orsay). I want to express my gratitude to my advisor Jean-Michel Bony for his enthusiastic

help and guidance.

2. Paralinearization Theorem

First we recall briefly Bony's theory on paradifferential operator. For the details in this field, see [1], [2], [3] and [4].

Let $l(x, \xi)$ be a C^∞ function homogeneous of degree m ($m \in R^1$) in $\xi \neq 0$, C^ρ ($\rho > 0$) in x, and $l(x, \xi)$ has compact support in x; we define operator T_l as follows:

$$T_l \hat{u}(\xi) = (2\pi)^{-n} \int \chi(\xi - \eta, \eta) \hat{l}(\xi - \eta, \eta) S(\eta) \hat{u}(\eta) d\eta \qquad (2.1)$$

where $\chi \in C^\infty(R^n \times R^n \backslash \{0\})$ is homogeneous in ξ of degree zero; and for sufficiently small $0 < \varepsilon_1 < \varepsilon_2$, satisfies:

$$\begin{cases} \chi(\theta, \eta) = 1 & \text{if} \quad |\theta| \leq \varepsilon_1 |\eta| \\ \chi(\theta, \eta) = 0 & \text{if} \quad |\theta| \geq \varepsilon_2 |\eta| \end{cases}$$

$S \in C^\infty(R^n)$ satisfies the condition that $S(x) = 0$ in a neighborhood of zero; $S(x) = 1$ outside a compact set. $\hat{l}(\xi, \eta)$ is the partial Fourier transformation in x of function $l(x, \eta)$.

We know for every $s \in R^1$, $T_l: H^s \to H^{s-m}$ is a continuous mapping. It is a class of completely new operators; comparing operator (2.1) with pseudo-differential operators, we see the symbol $l(x, \xi)$ here is non—smooth.

Hence as pseudo-differential operators $l(x, D)$, we just have $l(x, D): C^\infty \to C^\rho$. This is the reason that we cannot deal with $l(x, D)$ simply as an operator of $S^m_{1,0}$ class. But for operator T_l defined by (2.1), the case is different; in fact, T_l belongs to a subset of operator $S^m_{1,1}$ (see [4]). On the other hand, the definition (2.1) is dependent on choice of cut-off functions χ and S; but it is easy to prove that if χ and S in the definition of the operator T_l changes the operator remains unchanged modulo a $(\rho - m)$ — regularizing operator (i. e. an operator of order $(m - \rho)$).

Definition 2. 1. *Let Ω be an open set of R^n, for $m \in R^1$ and positive number $\rho > 0$ ($\rho \overline{\in} N$), we denote by $\sum^m_\rho(\Omega)$ the set of such functions $l(x, \xi)$, which are defined on $\Omega \times R^n \backslash \{0\}$, and have the following form:*

$$l(x, \xi) = l_m(x, \xi) + \cdots + l_{m-[\rho]}(x, \xi) \qquad (2.2)$$

where $l_{m-k}(x, \xi)$ is C^∞ and homogeneous of degree $(m - k)$ in $\xi \neq 0$, and $C^{\rho-k}_{loc}(\Omega)$ in x.

It is obvious that if $l(x, \xi)$ has compact support in x, we can define operator $T_l = \sum_{j=m}^{m-[\rho]} T_{l_j}$, which is the same (2.1). We know T_l is still a continuous mapping from H^s to H^{s-m}

Definition 2. 2. *Let Ω be an open set of R^n, $L: \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$, is a properly supported operator. If there exists $l \in \sum^m_\rho(\Omega)$, for every compact subset K of Ω and function $\chi \in C^\infty_0(\Omega)$, which is equal to l in a neighborhood of K, operator $L - \chi T_{\chi l}$ is a continuous mapping from $H^s_{comp}(K)$ to $H^{s+m-\rho}_{comp}(\Omega)$ ($\forall s \in R$), we call the operator L a C^ρ paradifferential operator of order m and denote $L \in OP(\sum^m_\rho(\Omega))$.*

Let $L \in OP(\sum^m_\rho(\Omega))$, it maps $H^s_{loc}(\Omega)$ into $H^{s-m}_{loc}(\Omega)$. Let $\sigma(L) = l$ be the symbol of L; and $\sigma_m(L) = l_m$ be the principal symbol of L. On the other hand, for an operator $L \in OP(\sum^m_\rho(\Omega))$, its symbol is unique; and the following symbol mapping:

$$\sigma: OP(\textstyle\sum^m_\rho(\Omega)) \to \sum^m_\rho(\Omega)$$

is surjective; the kernel of the mapping above is a $(\rho - m)$ — regularizing operator. As we pointed out before, for paradifferential operator, we also have the symbolic calculus similar to that of classical pseudo-differential operators.

Lemma 2. 3

a) Let $L_j \in OP(\sum_\rho^{m_j}(\Omega))$, $j=1, 2$, then:

$$L_1 \circ L_2 \in OP(\sum_\rho^{m_1+m_2}(\Omega))$$

where $L_1 \circ L_2 = L + R$, L is a paradifferential operator of order (m_1+m_2), whose symbol is

$$l(x, \xi) = \sum_{k_1+k_2+|\alpha| \leq (\rho)} \frac{1}{\alpha!} \partial_\xi^\alpha l^1_{m_1-k_1} D_x^\alpha l^2_{m_2-k_2}$$

and R is a paradifferential operator of ordr $(m_1+m_2-\rho)$.

b) Let $L \in OP(\sum_\rho^m(\Omega))$, then $L^* \in OP(\sum_\rho^m(\Omega))$, and

$$\sigma(L^*) = \sum_{k+|\alpha| \leq (\rho)} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar{l}_{m-k}$$

c) Let $L_j \in OP(\sum_\rho^{m_j}(\Omega))$, $j=1, 2$, $\rho > 1$. then

$$[L_1, L_2] \in OP(\sum_{\rho-1}^{m_1+m_2-1}(\Omega))$$

and its principal symbol is

$$\sigma_{m_1+m_2-1}([L_1, L_2]) = \frac{1}{i}\{\sigma_{m_1}(L_1), \sigma_{m_2}(L_2)\}$$

Now, for a function $l \in \sum_\rho^m(\Omega)$, we have a paradifferential operator L. Of course we may define a non smooth pseudo-differential operator $l(x, D)$ by classical method. The relations between L and $l(x, D)$ are the following:

Lemma 2. 4

a) Let $l \in \sum_\rho^m(\Omega)$ has compact support in x, and $\rho \geq m$. L is the paradifferential operator corresponding to l. Then for all $\sigma > 0$, $L - l(x, D)$ is a continuous operator from H^σ to L^2.

b) Let $l \in S^m_{1,0}$ then $L - l(x, D)$ is a regularizing operator of order infinity. This implies for all s, $s' \in R$ the operator $L - l(x, D)$ is a continuous mapping from H^s to $H^{s'}$

Let us consider the following non-linear partial differential equation of order m:

$$F(x, u, \ldots, \partial^\beta u, \ldots)_{|\beta| \leq m} \equiv \sum_{k_0 < k \leq m} \sum_{|\alpha| = k} A_\alpha(x, u, \ldots, \partial^\beta u, \ldots)_{|\beta| \leq p(k)} \partial^\alpha u + A_{k_0}(x, u, \ldots, \partial^\beta u, \ldots)_{|\beta| \leq k_0} = 0 \qquad (2.3)$$

where A_α and A_{k_0} are C^∞ functions. We may assume $p(k) < k$ in the preceding equation, at the same time denote $p(k) = -\infty$ if A_k depends only on x. Let:

$$d = \max\left(k_0, \frac{k+p(k)}{2}\right)$$

Then we have $d = k_0 = m$ if (2. 3) is a fully nonlinear equation; and $d = m - \frac{1}{2}$ if (2. 3) is a quasi-linear equation; and $d = -\infty$ if (2. 3) is a linear equation.

Lemma 2. 5. *Let $u \in C^\rho_{loc}(\Omega)$ be a real function, $\rho > \max\{k_0, p(k)\}$*

Suppose:

$$p(x, \xi) = \sum_{|\beta| > 2d-\rho} \frac{\partial F}{\partial u_\beta}(x, u(x), \ldots)(i\xi)^\beta \qquad (2.4)$$

Then $p \in \sum^m_{\rho+m-2d}(\Omega)$.

The fundamental relation between paradifferential operator and non-linear partial differential equations is the following:

Theorem 2. 6. *Let $u \in C^\rho_{loc}(\Omega) \cap H^s_{loc}(\Omega)$, $\rho > \max\{k_0, p(k)\}$, $s > 0$, be a real solution of equation (2. 3). P is the paradifferential operator whose symbol is defined by (2. 4). Then there exists a function $f \in C^{2\rho-2d}_{loc}(\Omega) \cap H^{s+\rho-2d}_{loc}(\Omega)$, such that:*

$$Pu = f \qquad (2.5)$$

The proof of theorem 2. 6 is the modification of theorem 5. 3 in [2]. Here we only point out the difference in their proofs.

We first recall the "dyadic decomposition" of spaces H^s and C^ρ. Let $K>1$ be a constant, $p\in N$ Denote:

$$C_p=\{\xi\in R^n;\ K^{-1}2^p\leq|\xi|\leq K2^{p+1}\}$$

and denote $C_{-1}=B(0,R)=\{\xi\in R^n;|\xi|\leq R\}$, Then:

Proposition 2. 7. *Let $\alpha\in R_+\backslash N$, then the following conditions are equivalent:*

a) $u\in C^\alpha(R^n)$;

b) There exists a decomposition $u=u_{-1}+\sum_{p=0}^{+\infty}u_p$, satisfies *supp* $\widehat{u}_p\subset C_p$ *and* $\|u_p\|_{L^\infty}\leq C2^{-p\alpha}$;

c) There exists a decomposition $u=\sum_{p=0}^{+\infty}u_p$, satisfies *supp* $\widehat{u}\subset B(0,K'2^p)$, *and* $\|u_p\|_{L^\infty}\leq C2^{-p\alpha}$;

d) There exists a decomposition $u=\sum_{p=0}^{+\infty}u_p$, $u_p\in C^\infty$; *and for all* $\lambda\in N^n$, *there exists* $B_\lambda>0$, *such that* $\|D^\lambda u_p\|_{L^\infty}\leq B_\lambda 2^{-(\alpha-|\lambda|)p}$.

For Sobolev spaces, we have similar results as follows:

Proposition 2. 8. *Let $s>0$, then the following condition are equivalent:*

a) $u\in H^s(R^n)$;

b) $u=u_{-1}+\sum_{p=0}^{+\infty}u_p$, *satisfies supp* $\widehat{u}_p\subset C_p$; *and there exists* $(b_p)\in l^2$, *such that* $\|u_p\|_{L^2}\leq b_p2^{-ps}$

c) $u=\sum_{p=0}^{+\infty}u_p$ *satisfies supp* $\widehat{u}_p\subset B(0,K'2^p)$; *and there exists* $(b_p)\in l^2$, *such that* $\|u_p\|_{L^2}\leq b_p2^{-ps}$;

d) $u=\sum_{p=0}^{+\infty}u_p$, $u_p\in C^\infty$; *and for all* $\lambda\in N^n$, *there exists* $(b_{\lambda,p})\in l^2$, *such that* $\|D^\lambda u_p\|_{L^2}\leq b_{\lambda,p}2^{-p(s-|\lambda|)}$.

The proofs of the two preceding propositions are based on the so-called dyadic decomposition of u which is sub-ordinate to the phase space of $\{C_p\}$. This decomposition is the basic starting point for the theory of paradifferential operator. We will not prove them here. Similar to [2], the proof of theorem 2. 6 can be easily deduced from the following:

Proposition 2. 9. *Let* $u=(u_1,\ldots,u_m)\in C^\rho\cap H^s$, $\rho,s>0$; *and* $F\in C^\infty(\Omega\times R^m)$. *Then:*

$$F(x,u_1,\ldots,u_m)-\sum_{j=1}^m\widetilde{T}_{\frac{\partial F}{\partial u_j}}u_j\in C^{2\rho}\cap H^{s+\rho}\tag{2.6}$$

where

$$\widetilde{T}_a u=\sum_{p\geq N_0}\Big(\sum_{q=-1}^{q-N_0}a_q\Big)u_p\tag{2.7}$$

(a_q) and (u_p) are the decompositions satisfying condition b) of the proposition 2. 7 and 2. 8 respectively.

Proof: Without losing generatily, we first assume that F has compact support in x, this implies $F\in C^\infty(R^{n+m})$. Secondly let $u_{m+j}=x_j$, $j=1,\ldots,n$, and $\widetilde{T}_{\frac{\partial F}{\partial x_j}}x_j\in H^{+\infty}$. Then the problem above can be reduced to the case when F is independent of x. On the other hand, we may assume $m=1$ simply. Because of $u \in C^{\rho}$, and $\rho > 0$; we have by b) of proposition 2.7:

$$\lim_{q\to+\infty} \| u - \sum_{p=-1}^{q} u_p \|_{L^\infty} = 0$$

Hence we suppose $S_q u = \sum_{p=-1}^{q} u_p$, then

$$F(u) = F(S_0 u) + \sum_{q=0}^{+\infty} \{F(S_{q+1}u) - F(S_q u)\}$$

For any N_0, we have $S_{N_0}u \in H^{+\infty}$; hence we also have $F(S_{N_0}u) \in H^{+\infty}$. Now

$$F(S_{q+1}u) - F(S_q u) = u_q \int_0^1 F'(S_q u + t u_q)\,dt$$

This implies

$$F(u) - \widetilde{T}_{F'(u)} u = F(S_{N_0}u) + \sum_{q\geq N_0} u_q \left\{ \int_0^1 F'(S_q u + t u_q)\,dt - S_{q-N_0}(F'(u)) \right\}$$

It remains to prove that:

$$g = \sum_{q\geq N_0} a_q = \sum_{q\geq N_0} u_q \left\{ \int_0^1 F'(S_q u + t u_q)\,dt - S_{q-N_0}(F'(u)) \right\}$$

belongs to $C^{2\rho} \cap H^{s+\rho}$. By d) of proposition 2.7 and 2.8, it needs only to prove that (a_q) satisfy the estimates in d). Thus by b), we have at once

$$\| u_q \|_{L^\infty} \leq C 2^{-q\rho}, \qquad q = 1, 2, \ldots$$
$$\| u_q \|_{L^2} \leq b_q 2^{-qs}, \qquad (b_q) \in l^2$$

Because $\operatorname{supp}\widehat{u}_q \subset C_q$, hence for all $\alpha \in N^n$, we have

$$\| D^\alpha u_q \|_{L^\infty} \leq C 2^{-q(\rho-|\alpha|)}$$
$$\| D^\alpha u_q \|_{L^2} \leq b_{\alpha q} 2^{-q(s-|\alpha|)}, \qquad (b_{\alpha q}) \in l^2$$

On the other hand, using the method in [4], since $u \in C^\rho$, $\rho > 0$, we can prove that the estimate:

$$\| D^\alpha \left\{ \int_0^1 F'(S_q u + t u_q)\,dt - s_{q-N_0}(F'(u)) \right\} \|_{L^\infty} \leq C_\alpha 2^{-q(\rho-|\alpha|)}$$

hold for all $\alpha \in N^n$. Summing up the preceding process, we have obtained for all $\alpha \in N^n$

$$\| D^\alpha a_q \|_{L^\infty} \leq C'_\alpha 2^{-q(2\rho-|\alpha|)}$$
$$\| D^\alpha a_q \|_{L^2} \leq C'_{\alpha q} 2^{-q(s+\rho-|\alpha|)}, \quad (C'_{\alpha q}) \in l^2$$

Using proposition 2.7 and 2.8, we have proved that $g \in C^{2\rho} \cap H^{s+\rho}$. This means that the proposition 2.9 is proved.

In order to prove the theorem 2.6, let us consider the equation (2.3). Set $u_j = \partial^\beta u$ in (2.6), where u is a real solution of equation (2.3) satisfying the conditions of theorem 2.6. Using proposition 2.9 and method in [2], we can get

$$\sum_{|\beta|>2d-\rho} \widetilde{T}_{\frac{\partial F}{\partial u_\beta}} \partial^\beta u \in C^{2\rho-2d}_{loc}(\Omega) \cap H^{s+\rho-2d}_{loc}(\Omega) \tag{2.8}$$

Denote now

$$P' = \sum_{|\beta|>2d-\rho} \widetilde{T}_{\frac{\partial F}{\partial u_\beta}} \partial^\beta$$

Then P' is a linear operator, whose symbol is denoted by

$$\sigma(P') = \sum_{|\beta|>2d-\rho} \frac{\partial F}{\partial u_\beta}(i\xi)^\beta = p(x,\xi)$$

then we may define a paradifferential operator P by symbol $p(x,\xi)$. Because $p \in \sum^m_{\rho+m-2d}$, and $m \geq d$, $\rho - d > 0$, we can get from the theorem 2.1 of [2] that $P -$

P' is a $(\rho - 2d)$ - regularizing operator. From $u \in C^{\rho}_{loc}(\Omega) \cap H^{s}_{loc}(\Omega)$ and (2. 8), we can obtain at once

$$Pu \in C^{2\rho-2d}_{loc}(\Omega) \cap H^{s+\rho-2d}_{loc}(\Omega)$$

Hence the theorem 2. 6 is proved.

The difference between theorem 2. 6 and Bony's theorem is that for $s>0$, $\rho>0$, the algebra studied by us is $C^{\rho}_{loc} \cap H^{s}_{loc}$. This means we can establish the energy inequality on C^{ρ}.

For simplicity, we will replace $C^{m}(\Omega)$ $(m \in N)$ by Zygmund class $C^{m}(\Omega)$, which are the sets of functions that satisfy the equivalent conditions in proposition 2. 7. Thus the assumption above, we may denote the Holder spaces in general by $C^{\rho}(\Omega)$, $\rho>0$.

3. Proofs of Theorem 1. 2 and Theorem 1. 3

Let us prove theorem 1. 2 now. We know $d=2$ for equation (1. 1). Let $u \in C^{\rho}_{loc}$, $\rho \geq 4$ be a real solution of equation (1. 1). Similar to (1. 2), we have

$$p(x, \xi) = p_2(x, \xi) + p_1(x, \xi) + p_0(x) \in \sum\nolimits^{2}_{\rho-2}(\Omega)$$

and the corresponding paradifferential operator

$$P = \sum_{k=1}^{n} \partial_k G_k + G_0 + P_0 \tag{3. 1}$$

where $G_k \in OP(\sum^{1}_{\rho-2}(\Omega))(k=1, \ldots, n)$, $G_0 \in OP(\sum^{1}_{\rho-3}(\Omega))$, are defined by g_j in (1. 5). Since $u \in C^{\rho}_{loc}(\Omega) = C^{\rho}_{loc}(\Omega) \cap H^{4}_{loc}(\Omega)$, so we have from theorem 2. 6:

$$Pu = g \in C^{\rho}_{loc}(\Omega) \cap H^{\rho}_{loc}(\Omega) \tag{3. 2}$$

which is a linear equation. Now we can establish its energy estimates as handling linear equation.

Proposition 3. 1. *Let P be the operator (3. 1) and $(a_{jk}(x)) \geq 0$. Then for every compact subset $K \subset \Omega$ and $s \in \mathbf{R}$, there exists a constant $C>0$, such that for any $v \in C^{\infty}_{0}(K)$ and $\sigma>0$, we have*

$$\sum_{j=1}^{2n} \| G_j v \|^2_s + \| G_0 v \|^2_{s-\frac{1}{2}} \leq C\{ \| Pv \|^2_s + \| v \|^2_{s+\sigma} \}$$

Before proving the proposition, we first give a lemma in [7], which is a result about convex functions.

Lemma 3. 2. *If $(a_{jk}(x)) \geq 0$ for any $x \in \Omega$, and $a_{jk}(x) \in C^{\rho}_{loc}(\Omega)$, $\rho \geq 2$. Then for every compact subset $K \subset \Omega$, there exists a constant M, which is dependent on K and the upper bound of second order derivatives of a_{jk} on K only, such that for all $v \in C^{\infty}_{0}(K)$ and $1 \leq \alpha \leq n$, the following estimate:*

$$\left| \sum_{j,k=1}^{n} \frac{\partial a_{kj}(x)}{\partial x_\alpha} v_{kj} \right|^2 \leq M \sum_{k,j,s=1}^{n} a_{kj}(x) v_{ks} v_{js}$$

holds. On the other hand, for all $(x, \xi) \in \Omega \times R^n$, we have

$$\left| \sum_{k=1}^{n} a_{kj}(x) \xi_k \right|^2 \leq 2a_{jj}(x) \sum_{k,l=1}^{n} a_{kl}(x) \xi_k \xi_l$$

The proof of proposition 3. 1:

We will prove proposition 3. 1 in three steps:

a) We first estimate $\sum_{k=1}^{n} \| G_k v \|^2_0$ where $G_k = \sum_{j=1}^{n} \widetilde{T}_{a_{kj}} \partial_j$. By lemma 3. 2, for any $v \in C^{\infty}_{0}(K)$, we have:

$$\left| \sum_{j=1}^{n} a_{kj} \partial_j v \right|^2 \leq C \sum_{k,j=1}^{n} a_{kj} \partial_k v \overline{\partial_j v}$$

Hence

$$\sum_{k=1}^{n} \| g_k(x, D) v \|_0^2 \leq C \sum_{k, j=1}^{n} (a_{kj}\partial_k v, \partial_j v)$$

Because $a_{kj} \in C_{loc}^{\rho-2}(\Omega)$, $\rho \geq 4$; from lemma 2.4, for any $\sigma > 0$, the mappings $g_j(x, D) - G_j$: $H^\sigma \to L^2$ and $\sum_{k=1}^{n} \partial_k G_k - \sum_{k, j=1}^{n} \partial_k a_{kj} \partial_j$: $H^\sigma \to L^2$ are all continuous. This implies that:

$$\sum_{j=1}^{n} \| G_j v \|_0^2 \leq C \{ \sum_{j=1}^{n} (G_j v, \partial_j v) + \| v \|_\sigma^2 \}$$

On the other hand, $G_0 \in OP(\sum_{\rho-3}^{1}(\Omega))$ with real coefficients. Hence we can get at once

$$\sum_{j=1}^{n} (G_j v, \partial_j v) \leq C \{ | (Pv, v) | + \| v \|_\sigma^2 \} \tag{3.3}$$

Thus

$$\sum_{j=1}^{n} \| G_j v \|_0^2 \leq C \{ | (Pv, v) | + \| v \|_\sigma^2 \} \tag{3.4}$$

b) Let us estimate $\sum_{j=n+1}^{2n} \| G_j v \|_0^2$ and $\| G_0 v \|_{-\frac{1}{2}}$. Since $G_{n+l} = L_l E^{-1}$ and $T = E^{-\frac{1}{2}} E^{-\frac{1}{2}} G_0$; where $\sigma(E^s) = (1 + |\xi|)^s$, a pseudo-differential operator; T is a paradifferential operator of order zero; L_l is a second order paradifferential operator whose symbol is $\sum_{k, j=1}^{n} \frac{\partial a_{kj}}{\partial x_l} (i\xi_k)(i\xi_j)$. Hence we only need to estimate the commutator $[P, E^s]$ by using lemma 3.2. just as we do in a). We have the following

Proposition 3.3. *There exist pseudo-differential operators* $E_j^s \in S_{1,0}^s$ *and* $R_0 \in OP(\sum_{\rho-4}^{s}(\Omega))$, *such that*

$$[P, E^s] = \sum_{j=1}^{2n} E_j^s G_j + R_0 \tag{3.5}$$

Using this proposition, we can get immediately

$$\sum_{j=1}^{2n} \| G_j v \|_0^2 \leq C \{ | (Pv, v) | + | (E^0 Pv, E^0 v) | + \| v \|_\sigma^2 \} \tag{3.6}$$

At the same time similar estimate holds as well for $\| G_0 v \|_{-\frac{1}{2}}$.

c) Let E_s be a properly supported pseudo-differential operator. By proposition 3.3, we have

$$\begin{aligned}
&\sum_{j=1}^{2n} \| G_j v \|_s^2 + \| G_0 v \|_{s-\frac{1}{2}}^2 \\
&\leq C \{ \sum_{j=1}^{2n} \| G_j E_s v \|_0^2 + \| G_0 E_s v \|_{-\frac{1}{2}}^2 + \| v \|_{s+\sigma}^2 \} \\
&\leq C \{ | (PE_s v, E_s v) | + | (E^0 P E_s v, E^0 E_s v) | + \| v \|_{s+\sigma}^2 \} \\
&\leq C \{ \| Pv \|_s^2 + \mu \| [P, E^s] v \|_0^2 + C(\mu) \| v \|_{s+\sigma}^2 \} \\
&\leq C \{ \| Pv \|_s^2 + \mu \sum_{j=1}^{2n} \| G_j v \|_s^2 + C(\mu) \| v \|_{s+\sigma}^2 \}
\end{aligned}$$

Taking $\mu > 0$ sufficiently small, the above estimate will imply the proposition 3.1.

By proposition 3.1, proposition 3.3 and lemma 2.4, if operator P satisfies the condition of theorem 1.2, and $0 < \sigma < \varepsilon$; then for any $v \in C_0^\infty(K)$ and $t \in R^1$, the estimate

$$\| v \|^2_{t+\varepsilon} \leq C\{ \| Pv \|^2_t + \| v \|^2_t\} \tag{3.7}$$

holds (where constant C depends only on K and t). By density theorem we know that the estimate (3.7) is still hold for any $v \in H^{t+2}_{comp}(K)$.

Finally using the so-called localizing and regularizing technique, we can prove:

Proposition 3. 4. *Under the assumptions of theorem 1. 2, let P be the paradifferential operator (3. 1), and $u \in H^s_{loc}(\Omega)$ be a solution of equation (1. 1). For any compact subset K of Ω, suppose $\varphi \in C^\infty_0(\Omega)$, and $\varphi = 1$ on K. Then there exist φ_1, $\varphi_2 \in C^\infty_0(\Omega)$, $f \in H^s_{comp}(\Omega)$, and constants $\varepsilon > 0$. $C(K, s) > 0$, such that*

$$\| \varphi u \|^2_{s+\varepsilon} \leq C\{ \| \varphi_1 Pu \|^2_s + \| \varphi_2 u \|^2_s + \| f \|^2_s\} \tag{3.8}$$

where ε depends only on K.

From this proposition, we can at once deduce the theorem 1. 2 as follows: Since $u \in C^\rho_{loc}(\Omega) = C^\rho_{loc}(\Omega) \cap H^4_{loc}(\Omega)$, so we first have $u \in H^4_{loc}(\Omega)$. By (3. 2) we have $Pu \in H^\rho_{loc}(\Omega) \subset H^4_{loc}(\Omega)$. Thus from (3. 8) we can deduce $u \in H^{4+\varepsilon}(K)$. Where ε independs on s. Secondly let $s = 4 + \varepsilon$, we can also deduce $u \in H^{4+2\varepsilon}(K)$. Repeating the process above, we finally obtain $u \in H^{+\infty}(K) \subset C^\infty(K)$, where K is an arbitrary compact subset of Ω. The proof of theorem 1. 2 is completed.

For the proof of proposition 3. 4, we only point out the regularizing operator which we will use here is:

$$T_\delta = \varphi_1(x)\,(1 - \delta\Delta)^{-1}\varphi(x) \tag{3.9}$$

where $\varphi_1 \in C^\infty_0(\Omega)$, and $\varphi_1(x) = 1$ on suppφ. Since T_δ is still a paradifferential operator; and for general convolution regular, the commutator is loss convenient then T_δ.

Following the proof of theorem 1. 2, for theroem 1. 3, we just need to prove the subelliptic estimate of its linearized operator. We have the following:

Proposition 3. 5. *Let L be the linearized operator (1. 2), satisfies $\sum_{k,j=1}^{n} a_{kj}(x)\,\xi_k\xi_j \geq 0$; and the assumptions of theorem 1. 3 hold. Then for every compact subset $K \subset \Omega$, there exist constants $C(K) > 0$, $\varepsilon(K) > 0$, such that for all $v \in C^\infty_0(K)$, we have:*

$$\| v \|^2_\varepsilon \leq C\{ \| Lv \|^2_0 + \| v \|^2_0\} \tag{3.10}$$

Proof: Since $\rho \geq p + 3$, so for $\alpha \in N^n$, $|\alpha| \leq p$, we have $g_\alpha \in \sum^1_1(\Omega)$. Similar to the estimates in [7], we can get the estimate for the commutator $g_\alpha(x, D)$:

$$\sum_{|\alpha| \leq p} \| g_\alpha(x, D) v \|^2_{\varepsilon-1} \leq C\{ \| Lv \|^2_0 + \| v \|^2_0\} \tag{3.11}$$

where $v \in C^\infty_0(K)$ and ε is at least equal to $1/4^p$. The proof of the estimate (3. 11) is similar to that of the estimate in [7], except for g_α we have only first order derivatives in x here. Thus we have to make several technical modifications for commutators and conjugate operations.

Under the assumptions of theorem 1. 3, $\sum_{|\alpha| \leq p} g^*_\alpha(x, D) \cdot g_\alpha(x, D)$ is an elliptic pseudo-differential operator, hence for $0 < \varepsilon \leq 1$, there exists $C > 0$, such that for all $v \in C^\infty_0(K)$, we have:

$$\| v \|^2_\varepsilon \leq C\{ \sum_{|\alpha| \leq p} \| g_\alpha(x, D) v \|^2_{\varepsilon-1} + \| v \|^2_0\} \tag{3.12}$$

This is the Gårding inequality. From the estimates (3. 11) and (3. 12), we obtain at once (3. 10).

Now let L_0 be the principal part of L, then L_0 is a nonnegative operator. From the estimate (3. 10), we can get:

$$\begin{aligned} \| v \|^2_{\varepsilon/2} &\leq C\{ \langle L_0 v, v\rangle + \| v \|^2_0\} \\ &\leq C\{ \langle Lv, v\rangle + \| v \|^2_0\} \end{aligned}$$

Theorem 1. 3 is proved.

4. Global Hypoellipticity

Actually, the conditions of theorem 1. 3 give the local hypoellipticity of linearized operators. In this section, we will consider the case where the conditions of theorem 1. 3 cannot be satisfied on some subset as in [7]. Thus we can obtain the so-called global hypoellipticity.

Let N be an $(n-1)$ -dimension submanifold in Ω. Thus N can be expressed locally as the zero set of a smooth function $\varphi(x)$ on Ω; i. e. $\{\varphi(x)=0\}$, satisfying $\mathrm{grad}\varphi(x)=0$. Let $M\subset N$ be a compact subset in Ω, then we have:

Theorem 4. 1. *Let $u\in C^{\rho}_{loc}(\Omega)$, $\rho>\max\{4, p+3\}$, be a real solution of equation (1. 1). If the conditions of theorem 1. 3 hold on $\Omega\backslash M$, and for $x\in M$, the linearized operator (1. 2) satisfies:*

$$\sum_{k,j=1}^{n} a_{kj}(x)\partial_k\varphi(x)\partial_j\varphi(x)+\Big|\sum_{k,j=1}^{n} a_{kj}(x)\partial^2_{kj}\varphi(x)+\sum_{j=1}^{n} b_j(x)\partial_j\varphi(x)\Big|>0 \tag{4. 1}$$

Thus $u\in C^{\infty}(\Omega)$.

In the following, we only sketch the proof of theorem 4. 1. without lossing generlity, we can assume by Heine-Borel theorem, $N=\{\varphi(x)=0\}$, grad $\varphi(x)\neq 0$. After a C^{∞} transformation of variable, we have $N=\{x_n=0, x\in\Omega\}$, $x=(x', x_n)$. This transformation will preserve the conditions of theorem 1. 3 which is invariant. Under the preceding transformation, (4. 1) becomes: $a_{nn}\neq 0$ or $b_n\neq 0$ in a neighborhood of N. Now let $N_{\beta}=\{x\in\Omega; |x_n|\leq\beta\}$, β is a constant to be determined. We have:

Propositon 4. 2. *Under the assumptions of theorem 4. 1, there exist constants $C>0$, $\mu>0$ such that for all $v\in C_0^{\infty}(N_{\beta})$, we have:*

$$\| v\|_0^2\leq\frac{C}{\mu}\| Lv\|_0^2+C\| v\|_{-r}^2, \tag{4. 2}$$

where r is sufficiently large, C and β are independent of ε; and μ will be infinity if $\beta\to 0$.

In general, we call the estimate (4. 2) the global hypoelliptic estimate. It is obvious that the estimate (4. 2) is weaker than (1. 4). On the other hand, from the estimates (4. 2) and (3. 10), where (3. 10) holds only on $\Omega\backslash M$; we can deduce the following estimate;

$$\| v\|_s^2\leq\frac{C}{\mu}\{\| Pv\|_s^2+\| v\|_{-r}^2\}$$

for all $v\in C_0^{\infty}(K)$. This implies that theroem 4. 1 holds.

On the other hand, if $M=\{x_0\}$ is an isolated point, then the condition (4. 1) becomes:

$$\sum_{j=1}^{n}(a_{jj}(x_0)+|b_j(x_0)|)>0$$

At the same time, in the proof of theorem 4. 1, we know that if $N=\{\varphi(x)=0\}$, then the condition that M is a compact subset in Ω can be dropped. Of course for equation (1. 8), we also have several similar results, which are omitted here.

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