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GLOBAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS AND CAUCHY PROBLEM FOR TWO TYPES OF NONLINEAR PSEUDO-PARABOLIC SYSTEMS *

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1. Introduction

Various types of nonlinear evolutional systems of partial differential equations can be derived from a great amount of modern research in physics, chemical reactions, mechanics etc. such as the linear or nonlinear parabolic or pseudo-parabolic systems. So that it is meaningful and interesting to discuss the well-posedness in global of some basic problems as periodic boundary problems, initial-boundary value problems and Cauchy problems for the above mentioned systems. There appeared some papers concerned about the problems of some types of the nonlinear pseudo-parabolic equations and systems[1-12]. In this paper we are going to consider some problems for two new types of the nonlinear pseudo-parabolic systems, which are, in particular, different from the systems discussed in [1, 2].

Now let us give some conventions of notations for the following use.

Denote by (u, v) the inner product of two vectors u and v, $(u, v) = \int_{-X}^{X} u \cdot v dx$, also $|u(\cdot, t)|^2_{L_2(\Omega)} = (u, u)$, $\Omega \equiv (-X, X)$.

Denote by $[u, v]$ the integration from 0 to t of the inner product of two vectors u and v,

$$[u, v] = \int_0^t (u, v)\, dt = \iint_{Q_t} u \cdot v dx dt$$

also $\| u \|^2_{L_2(Q_t)} = [u, u]$, $Q_t = \{(x, \tau) \mid x \in \Omega \equiv (-X, X),\ \tau \in (0, t)\}$. Denote by $L_2(0, T; H^m(\Omega))$ the collection of functions (or vectors) $u(x, t)$, which when regarded as functions (or vectors) of variable x belong to space $H^m(\Omega)$ and when whose norms $|u(\cdot, t)|_{H^m(\Omega)}$ are regarded as functions (or vectors) of variable t belong to the space $L_2(0, T)$.

The following two lemmas will be used repeatedly in this paper.

Lemma 1 *(Nirenberg' s lemma)*[13]. *If* $\int_{-X}^{X} v dx = 0$ *or* $v|_{x=-X} = v|_{x=X} = 0$, *then we have*

$$|v(\cdot, t)|_{L_p(\Omega)} \le C |v_x(\cdot, t)|^{\alpha}_{L_q(\Omega)} |v(\cdot, t)|^{1-\alpha}_{L_r(\Omega)}, \quad \forall t \in [0, T] \tag{1}$$

where $\frac{1}{p} = \alpha(\frac{1}{q} - 1) + (1-\alpha)\frac{1}{r}$, $\alpha \in [0, 1]$, $r \ge 1$, $1 < p, q \le \infty$.

In particular, we have

$$|v(\cdot, t)|_{L_\infty(\Omega)} \le C |v_x(\cdot, t)|^{\frac{1}{2}}_{L_2(\Omega)} |v(\cdot, t)|^{\frac{1}{2}}_{L_2(\Omega)}, \quad \forall t \in [0, T] \tag{2}$$

Lemma 2[14]. *Let* $Q_T = \{(x, t) \mid x \in \Omega, t \in (0, T)\}$. *Suppose that* $G(Z_1, Z_2, \dots, Z_s)$

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is a function of g vectors $Z_1, Z_2, \ldots, Z_g$, $k\ (\geq 1)$ — times continuously differentiable, and $Z_i(x,t) \in L_\infty(Q_T) \cap L_2(0,T;H^k(\Omega))$. Let $\overline{M} = \max_{1\leq i\leq g} \sup_{(x,t)\in Q_T} |Z_i(x,t)|$. Then, we have the inequality

$$\left|\frac{\partial^k}{\partial x^k} G(Z_1(\cdot,t), \ldots, Z_g(\cdot,t))\right|^2_{L_2(\Omega)} \leq C(\overline{M}, k, g) \sum_{i=1}^{g} |Z_i(\cdot,t)|^2_{H^k(\Omega)}, \quad \forall t \in [0, t] \tag{3}$$

2. Periodic Boundary Problem. Cauchy Problem

In a rectangular domain $Q_T = \{(x,t) \mid x \in \Omega \equiv (-X, X), t \in (0,T)\}$ we consider the nonlinear pseudo-parabolic system

$$Lu \equiv u_t + (-1)^M A(t) u_{x^{2M}} + (-1)^M B u_{x^{2M}t} + \sum_{j=0}^{M} (-1)^j \frac{\partial^j}{\partial x^j}(F_j(u, \ldots, u_{x^{M-1}}) u_{x^j}) = f(x,t) \tag{4}$$

and the periodic boundary problem

$$\begin{cases} u(x,t) = u(x+2X, t) \\ u(x,0) = \varphi(x) \end{cases} \tag{5}$$

where $M \geq 1$ is an integer, $u = (u_1, \ldots, u_N)^T$ and $f = (f_1, \ldots, f_N)^T$ are N-dimensional vector valued functions, $A(t)$ and B are $N \times N$ symmetric matrices, $F_j(u, \ldots, u_{x^{M-1}})$ $(j = 0, 1, \ldots, M)$ are arbitrary nonlinear functions of N-dimensional vector variables $p_m = (p_{1m}, \ldots, p_{Nm})$, $p_{km} = u_{kx^m}$ $(m = 0, 1, \ldots, M-1; k = 1, \ldots, N)$.

Assume that the system (4) satisfies the following conditions:

$$\begin{cases} i)\ A(t) \text{ and } A'(t) \text{ are bounded matrices,} \\ ii)\ B \text{ is a positively definite constant matrix:} \\ \quad (B\xi, \xi) \geq b_0(\xi, \xi),\ \forall \xi \in R^N,\ b_0 > 0 \\ iii)\ F_j\ (j = 0, 1, \ldots, M) \text{ are nonnegative and } m+1\text{-times} \\ \quad \text{continuously differentiable, } m \geq M, \\ iv)\ f_{x^{m-M}t} \in L_2(Q_T),\ \varphi(x) \in H^{M+m}(\Omega),\ m \geq M, \\ \quad f_k \text{ and } \varphi_k \text{ are periodic functions of } x \text{ with period} \\ \quad 2X,\ k = 1, \ldots, N,\ f = (f_1, \ldots, f_N),\ \varphi = (\varphi_1, \ldots, \varphi_N) \end{cases} \tag{6}$$

Lemma 3. *The solutions of problem (4) (5) satisfy the following estimation*

$$|u(\cdot,t)|_{H^M(\Omega)} \leq C\{\|f\|^2_{L_2(Q_T)} + |\varphi|^2_{H^M(\Omega)}\},\ \forall t \in [0, T] \tag{7}$$

Proof. Taking the scalar product of the vector u and the system (4) and then making the integration $[Lu, u]$ in the domain $Q_t\ (0 < t \leq T)$, we get

$$\frac{1}{2}(u,u) + \frac{1}{2}(Bu_{x^M}, u_{x^M}) + \sum_{j=0}^{M} [F_j(u, \ldots, u_{x^{M-1}}) u_{x^j}, u_{x^j}]$$
$$= [f, u] + \frac{1}{2}(\varphi, \varphi) + \frac{1}{2}(B\varphi^{(M)}, \varphi^{(M)}) - [A(t) u_{x^M}, u_{x^M}] \tag{8}$$

Since $F_j \geq 0$ and the matrix $A(t)$ is bounded, then by applying the Gronwall's lemma we obtain the estimation (7)

Lemma 4. *The solutions of problem (4) (5) satisfy the estimate*

$$|u(\cdot,t)|_{H^{M+m}(\Omega)} \leq C(b_0, \|D_x^{m-M} f\|_{L_2(Q_T)}, |\varphi|_{H^{M+m}(\Omega)}),\ \forall t \in [0, T],\ m \geq M \tag{9}$$

Proof. By making the integration $[Lu, (-1)^m u_{x^{2m}}]\ (m \geq 1)$, we have

$$\frac{1}{2}(u_{x^m}, u_{x^m}) + \frac{1}{2}(Bu_{x^{M+m}}, u_{x^{M+m}}) + \sum_{j=0}^{M} \left[\frac{\partial^m}{\partial x^m}(F_j(u, \ldots, u_{x^{M-1}}) u_{x^j}), u_{x^{m+j}}\right]$$
$$= [f, (-1)^m u_{x^{2m}}] + \frac{1}{2}(\varphi^{(m)}, \varphi^{(m)})$$

$$+\frac{1}{2}(B\varphi^{(M+m)},\ \varphi^{(M+m)})-[A(t)u_{x^{M+m}},\ u_{x^{M+m}}] \tag{10}$$

The first and fourth terms on the right side in (10) may be estimated by

$$|[A(t)u_{x^{M+m}},\ u_{x^{M+m}}]|+|[f,\ (-1)^m u_{x^{2m}}]|$$
$$\leq \begin{cases} C(\|f\|^2_{L_2(Q_T)}+\|u_{x^{M+m}}\|^2_{L_2(Q_t)}), & \text{if } m\leq M, \\ C(\|D_x^{m-M}f\|^2_{L_2(Q_T)}+\|u_{x^{M+m}}\|^2_{L_2(Q_t)}), & \text{if } m\geq M \end{cases} \tag{11}$$

We estimate the last term on the left side in (10) as follows:

$$\sum_{j=0}^{M}\left[\frac{\partial^m}{\partial x^m}(F_j(u,\ \dots,\ u_{x^{M-1}})u_{x^j}),\ u_{x^{m+j}}\right]$$
$$=\sum_{j=0}^{M}[F_j(u,\ \dots,\ u_{x^{M-1}})u_{x^{m+j}},\ u_{x^{m+j}}]+\sum_{j=0}^{M}\left[\sum_{k=1}^{m}a_k\frac{\partial^k F_j}{\partial x^k}u_{x^{m+j-k}},\ u_{x^{m+j}}\right] \tag{12}$$

The first term is nonnegative, hence we with main strength estimate the second term. From Lemma 3 we have $u_{x^i}\in L_\infty(Q_T)$, $0\leq i\leq M-1$, and from Lemmas 1 and 2 we have

$$\sup_{-X\leq x\leq X}|u_{x^{m+j-k}}(\cdot,\ t)|$$
$$\leq C|u_{x^{m+j-k}}(\cdot,\ t)|^{\frac{1}{2}}_{L_2(\Omega)}|u_{x^{m+j-k+1}}(\cdot,\ t)|^{\frac{1}{2}}_{L_2(\Omega)},\ \forall t\in[0,\ T] \tag{13}$$

$$\left|\frac{\partial^k}{\partial x^k}F_j(u,\ \dots,\ u_{x^{M-1}})\right|^2_{L_2(\Omega)}\leq C\sum_{i=0}^{M-1}|u_{x^{k+i}}(\cdot,\ t)|^2_{L_2(\Omega)},\ 1\leq k\leq m,\ t\in[0,\ T] \tag{14}$$

Then, the second term on the right side in (12) may be estimated by

$$\left|\sum_{j=0}^{M}\left[\sum_{k=1}^{m}a_k\frac{\partial^k F_j}{\partial x^k}u_{x^{m+j-k}},\ u_{x^{m+j}}\right]\right|$$
$$\leq C\sum_{j=0}^{M}\sum_{k=1}^{m}\int_0^t\sup_{-X\leq x\leq X}|u_{x^{m+j-k}}(\cdot,\ t)|\left|\frac{\partial^k F_j}{\partial x^k}\right|_{L_2(\Omega)}|u_{x^{m+j}}(\cdot,\ t)|_{L_2(\Omega)}dt$$
$$\leq\tilde{C}\sum_{j=0}^{M}\sum_{k=1}^{m}\int_0^t|u_{x^{m+j-k}}(\cdot,\ t)|^{\frac{1}{2}}_{L_2(\Omega)}|u_{x^{m+j-k+1}}(\cdot,\ t)|^{\frac{1}{2}}_{L_2(\Omega)}$$
$$\cdot\left(\sum_{i=0}^{M-1}|u_{x^{k+i}}(\cdot,\ t)|^2_{L_2(\Omega)}\right)^{\frac{1}{2}}|u_{x^{m+j}}(\cdot,\ t)|_{L_2(\Omega)}dt \tag{15}$$

Hence, from (10) we derive the inequality:

$$(u_{x^m},\ u_{x^m})+(u_{x^{M+m}},\ u_{x^{M+m}})+\sum_{j=0}^{M}[F_j(u,\ \dots,\ u_{x^{M-1}})u_{x^{m+j}},\ u_{x^{m+j}}]$$
$$\leq C_1\{\|D_x^h f\|^2_{L_2(Q_T)}+|\varphi|^2_{H^{M+m}(\Omega)}+\|u_{x^{M+m}}\|^2_{L_2(Q_t)}\}$$
$$+C_2\sum_{j=0}^{M}\sum_{k=1}^{m}\int_0^t|u_{x^{m+j-k}}(\cdot,\ t)|^{\frac{1}{2}}_{L_2(\Omega)}|u_{x^{m+j-k+1}}(\cdot,\ t)|^{\frac{1}{2}}_{L_2(\Omega)}|u_{x^{m+j}}(\cdot,\ t)|_{L_2(\Omega)}$$
$$\cdot\left(\sum_{i=0}^{M-1}|u_{x^{k+i}}(\cdot,\ t)|^2_{L_2(\Omega)}\right)^{\frac{1}{2}}dt \tag{16}$$

where $h=0$ if $m\leq M$, $h=m-M$ if $m>M$. We are going to prove the boundedness of the terms on the left side of the inequality (16) by the method of mathematical induction. For the case when $m=1$, since $k=m=1$ and then $k+i\leq M$, $m+j-k\leq M$, $\forall\ 0\leq i\leq M-1$, $0\leq j\leq M$, hence, the second term on the right side of (16) may be estimated by

$$I\equiv C_2\sum_{j=0}^{M}\sum_{k=1}^{m}\int_0^t|u_{x^{m+j-k}}(\cdot,\ t)|^{\frac{1}{2}}_{L_2(\Omega)}|u_{x^{m+j-k+1}}(\cdot,\ t)|^{\frac{1}{2}}_{L_2(\Omega)}$$

The first term on the right side of (25) may be estimated by

$$|[f, (-1)^m u_{x^{2m}}]| \leq \frac{b_0}{4}[u_{x^{M+m}}, u_{x^{M+m}}] + C(b_0) \| D_x^h f \|^2_{L_2(Q_T)} \tag{27}$$

where $h = 0$ when $m \leq M$, $h = m - M$ when $m > M$.

Then third term on the right side of (25) may be estimated by

$$\left| \sum_{j=0}^{M} \left[\frac{\partial^j}{\partial x^j}(F_j(u, \ldots, u_{x^{M-1}}) u_{x^j}), (-1)^{m+j+1} u_{x^{2m}} \right] \right|$$

$$= \left| \sum_{j=0}^{M} \left[\frac{\partial^{m+j-M}}{\partial x^{m+j-M}}(F_j(u, \ldots, u_{x^{M-1}}) u_{x^j}), (-1)^{M+j+1} u_{x^{M+m}} \right] \right|$$

$$\leq \frac{b_0}{4}[u_{x^{M+m}}, u_{x^{M+m}}] + C\{ \| u_{x^{M+m}} \|^2_{L_2(Q_t)} + \| u \|^2_{L_2(Q_t)} \} \tag{28}$$

Then, by Lemma 4 from (25) we derive the inequality (24).

Lemma 7. *There is the estimation of the solutions of the problem (4) (5)*

$$\| u_{x^m t^2} \|^2_{L_2(Q_t)} + \| u_{x^{M+m} t^2} \|^2_{L_2(Q_t)} \leq C\left(b_0, \ \| \frac{\partial^{m-M+1}}{\partial x^{m-M} \partial t} f \|_{L_2(Q_T)}, \ |\varphi|_{H^{M+m}(\Omega)} \right), \ m \geq M \tag{29}$$

Proof. Taking the differentiation of the system (4) with respect to t, we have

$$(Lu)_t \equiv u_{tt} + (-1)^M A(t) u_{x^{2M} t} + (-1)^M A'(t) u_{x^{2M}} + (-1)^M B u_{x^{2M} t^2} + \sum_{j=0}^{M} (-1)^j \frac{\partial^{j+1}}{\partial x^j \partial t}(F_j(u, \ldots, u_{x^{M-1}}) u_{x^j}) = f_t \tag{30}$$

Then, by making the integration $[(Lu)_t, (-1)^m u_{x^{2m} t^2}]$, we have

$$[u_{x^m t^2}, u_{x^m t^2}] + [B u_{x^{M+m} t^2}, u_{x^{M+m} t^2}]$$
$$= [f_t, (-1)^m u_{x^{2m} t^2}]$$
$$- [(A(t) u_{x^{M+m} t} + A'(t) u_{x^{M+m}}), u_{x^{M+m} t^2}] +$$
$$+ \left[\sum_{j=0}^{M} (-1)^{j+1} \frac{\partial^{j+1}}{\partial x^j \partial t}(F_j(u, \ldots, u_{x^{M-1}}) u_{x^j}), (-1)^m u_{x^{2m} t^2} \right] \tag{31}$$

The first term on the right side of (31) may be estimated by

$$|[f_t, (-1)^m u_{x^{2m} t^2}]| = |[D_x^h f_t, (-1)^M u_{x^{M+m} t^2}]|$$

$$\leq \frac{b_0}{4}[u_{x^{M+m} t^2}, u_{x^{M+m} t^2}] + C(b_0) \| D_x^h f_t \|^2_{L_2(Q_T)} \tag{32}$$

where $h = 0$ when $m = M$, $h = m - M$ when $m > M$.

The second and third terms on the right side of (31) may be estimated respectively by

$$|[(A(t) u_{x^{M+m} t} + A'(t) u_{x^{M+m}}), u_{x^{M+m} t^2}]|$$

$$\leq \frac{b_0}{4}[u_{x^{M+m} t^2}, u_{x^{M+m} t^2}] + C(b_0) \{ \| u_{x^{M+m} t} \|^2_{L_2(Q_t)} + \| u_{x^{M+m}} \|^2_{L_2(Q_t)} \} \tag{33}$$

$$\left| \left[\sum_{j=0}^{M} (-1)^{j+1} \frac{\partial^{j+1}}{\partial x^j \partial t}(F_j(u, \ldots, u_{x^{M-1}}) u_{x^j}), (-1)^m u_{x^{2m} t^2} \right] \right|$$

$$= \left| \left[\sum_{j=0}^{M} \frac{\partial x^{m+j-M+1}}{\partial x^{m+j-M} \partial t}(F_j(u, \ldots, u_{x^{M-1}}) u_{x^j}), (-1)^{M-j+1} u_{x^{M+m} t^2} \right] \right|$$

$$\leq \frac{b_0}{4}[u_{x^{M+m} t^2}, u_{x^{M+m} t^2}] + C(b_0) \{ \| u_{x^{M+m} t} \|^2_{L_2(Q_t)} + \| u_t \|^2_{L_2(Q_t)} \} \tag{34}$$

Then, by Lemmas 4 and 6, from (31) we may derive the inequality (29).

Corollary. *From (29) we have*

$$|u_{x^{M+m} t}(\cdot, t)|_{L_2(\Omega)} \leq \text{const.} \quad \forall t \in [0, T] \tag{35}$$

Having the estimates (9) and (29), we may prove the following main theorem.

Theorem 1. *Suppose that the conditions in (6) are satisfied. Then, when $m = M$ the periodic boundary problem (4) (5) has a global solution $u(x, t) \in Z \equiv H^2(0, T; H^{2M}(\Omega))$*

When $m \geq M + 1$ the periodic boundary problem (4) (5) has a global smooth solution,

$$\cdot \left| u_{x^{m+j}}(\cdot,\ t) \right|_{L_2(\Omega)} \Big(\sum_{i=0}^{M-1} \left| u_{x^{k+i}}(\cdot,\ t) \right|^2_{L_2(\Omega)}\Big)^{\frac{1}{2}} dt$$

$$\leq C' \sum_{j=0}^{M}\sum_{k=1}^{m}\sum_{i=0}^{M-1} \sup_{0\leq\tau\leq t} \left| u_{x^{k+i}}(\cdot,\ \tau) \right|_{L_2(\Omega)}$$

$$\cdot \left(\| u_{x^{m+j-k}} \|^2_{L_2(Q_t)} + \| u_{x^{m+j-k+1}} \|^2_{L_2(Q_t)} + \| u_{x^{m+j}} \|^2_{L_2(Q_t)} \right)$$

$$\leq C'' \{ \| u_{x^{M+m}} \|^2_{L_2(Q_t)} + \| u \|^2_{L_2(Q_t)} \},\ (\text{when } m=1),\ \forall\, t \in [0,\ T] \tag{17}$$

Then, by applying the Gronwall's lemma we have

$$\left| u_{x^{M+1}}(\cdot,\ t) \right|_{L_2(\Omega)} \leq C(b_0,\ \| f \|_{L_2(Q_T)},\ |\varphi|_{H^{M+1}(\Omega)}),\ \forall\, t\in[0,\ T] \tag{18}$$

It proves that the boundedness of the term $|u_{x^{M+m}}(\cdot,t)|_{L_2(\Omega)}$ holds for $m=1$.

Now we assume that the boundedness of $|u_{x^{M+m}}(\cdot,t)|_{L_2(\Omega)}$ is known for $m=d$. we have to derive the boundedness of it for $m=d+1$.

When $m=d+1$, since $k+i\leq M+d$, $m+j-d\leq M+d$ $\forall$ $1\leq k\leq m$, $0\leq i\leq M-1$, $0\leq j\leq M$, hence the terms $|u_{x^{m+j-k}}(\cdot,t)|_{L_2(\Omega)}$ and $|u_{x^{k+i}}(\cdot,t)|_{L_2(\Omega)}$ $(0\leq i\leq M-1,\ 1\leq k\leq m,\ 0\leq j\leq M)$ in I are bounded, and we obtain the result

$$I \leq C'' \{ \| u_{x^{M+d+1}} \|^2_{L_2(Q_t)} + \| u \|^2_{L_2(Q_t)} \},\ \forall\, t\in[0,\ T] \tag{19}$$

Then, by applying the Gronwall's lemma we derive from (16) the boundedness of $|u_{x^{M+d+1}}(\cdot,t)|_{L_2(\Omega)}$. Hence, it is valid for all $m\,(\geq 1)$ the estimate

$$\left| u_{x^{M+m}}(\cdot,\ t) \right|_{L_2(\Omega)} \leq C(b_0,\ \| D^k_x f \|_{L_2(Q_T)},\ |\varphi|_{H^{M+m}(\Omega)}),\ \forall\, t\in[0,\ T] \tag{20}$$

In particular, when $m\geq M$ the estimation relation (9) follows.

Lemma 5. *The solutions of problem (4) (5) satisfy the estimate*

$$\| u_t \|^2_{L_2(Q_t)} + \| u_{x^M t} \|^2_{L_2(Q_t)} \leq C\{ \| f \|^2_{L_2(Q_T)} + |\varphi|^2_{H^M(\Omega)} \} \tag{21}$$

Proof. By making the integration $[Lu,\ u_t]$, we have

$$[u_t,\ u_t] + [Bu_{x^M t},\ u_{x^M t}]$$

$$= [f,\ u_t] - [A(t)u_{x^M},\ u_{x^M t}] + \sum_{j=0}^{M} [(-1)^{j+1}\frac{\partial^j}{\partial x^j}(F_j(u,\ \dots,\ u_{x^{M-1}})u_{x^j}),\ u_t] \tag{22}$$

Since $A(t)$ is bounded, hence by Lemma 3, we have

$$|[A(t)u_{x^M},\ u_{x^M t}]| \leq \frac{b_0}{2}[u_{x^M t},\ u_{x^M t}] + C$$

The first and third terms on the right side of (22) may be estimated by

$$|[f,\ u_t]| + \left| \sum_{j=0}^{M} [(-1)^{j+1}\frac{\partial^j}{\partial x^j}(F_j(u,\ \dots,\ u_{x^{M-1}})u_{x^j}),\ u_t] \right|$$

$$\leq \frac{1}{2}[u_t,\ u_t] + \| f \|^2_{L_2(Q_T)} + C\| u \|^2_{L_2(0,T;H^{2M}(\Omega))} \tag{23}$$

Then, by Lemma 4 we derive from (22) the inequality (21).

Lemma 6. *There is the estimation for the solutions of the problem (4) (5)*

$$\| u_{x^m t} \|^2_{L_2(Q_t)} + \| u_{x^{M+m} t} \|^2_{L_2(Q_t)} \leq C\{ \| D_x^{m-M} f \|^2_{L_2(Q_T)} + |\varphi|^2_{H^{M+m}(\Omega)} \},\ m\geq M \tag{24}$$

Proof. By making the integration $[Lu,\ (-1)^m u_{x^{2m} t}]$, we have

$$[u_{x^m t},\ u_{x^m t}] + [Bu_{x^{M+m} t},\ u_{x^{M+m} t}]$$

$$= [f,\ (-1)^m u_{x^{2m} t}] - [A(t)u_{x^{M+m}},\ u_{x^{M+m} t}]$$

$$+ \sum_{j=0}^{M} \left[\frac{\partial^j}{\partial x^j}(F_j(u,\ \dots,\ u_{x^{M-1}})u_{x^j},\ (-1)^{m+j+1} u_{x^{2m} t} \right] \tag{25}$$

Since $A(t)$ is bounded, then by Lemma 4 we have

$$|[A(t)u_{x^{M+m}},\ u_{x^{M+m} t}]| \leq \frac{b_0}{4}[u_{x^{M+m} t},\ u_{x^{M+m} t}] + C(b_0,\ f,\ \varphi) \tag{26}$$

and the continuous derivatives $u_{x^{M+m-1}}$, $u_{x^{M+m-1}t}$.

Proof. In order to prove the existence of the global solution for the problem (4) (5), we take the functional space $G \equiv H^1(0, T; W_\infty^{2M-1}(\Omega))$ as the base space for the fixed point theorem treatment.

For every $v(x, t) \in G$, We construct a N-dimensional vector valued function $u(x, t)$ defined as the solution of the periodic boundary problem (5) for the linear pseudo-parabolic system

$$u_t + (-1)^M A(t) u_{x^{2M}} + (-1)^M B u_{x^{2M}t} + \lambda \sum_{j=0}^{M} (-1)^j \frac{\partial^j}{\partial x^j} (F_j(v, \ldots, v_{x^{M-1}}) u_{x^j}) = f(x, t) \tag{36}$$

with a parameter $0 \leq \lambda \leq 1$. It can be easily seen that the solution of (36) (5) satisfy all the estimates (7), (9), (21), (24) and (29), so by means of the method of continuation of parameter or by the theorem 2.1 in [2], it follows that the periodic boundary problem (36) (5) has a unique solution $u(x, t)$ in the functional space Z.

The correspondence of v to u defines a functional mapping $T_\lambda: G \to Z$, where $\lambda \in [0, 1]$ is a parameter. For every $v \in G$, the image $T_\lambda v = u$ belongs to $Z \subset G$. Since the imbedding mapping $Z \to G$ is compact, for every $0 \leq \lambda \leq 1$, the mapping $T_\lambda: G \to Z \to G$ is completely continuous.

Let S be a bounded set of G. For any $v \in S \subset G$ and any $0 \leq \lambda, \bar{\lambda} \leq 1$, there are $T_\lambda v = u_\lambda$ and $T_{\bar{\lambda}} v = u_{\bar{\lambda}}$. The difference vector $w = u_\lambda - u_{\bar{\lambda}}$ satisfies the linear pseudo-parabolic system

$$\begin{aligned} &w_t + (-1)^M A(t) w_{x^{2M}} + (-1)^M B w_{x^{2M}t} + \lambda \sum_{j=0}^{M} (-1)^j \frac{\partial^j}{\partial x^j} (F_j(v, \ldots, v_{x^{M-1}}) w_{x^j}) \\ &= (\lambda - \bar{\lambda}) \left\{ - \sum_{j=0}^{M} (-1)^j \frac{\partial^j}{\partial x^j} (F_j(v, \ldots, v_{x^{M-1}}) u_{\bar{\lambda} x^j} \right\} \end{aligned} \tag{37}$$

and the periodic boundary condition

$$\begin{cases} w(x, t) = w(x + 2X, t) \\ w(x, 0) = 0 \end{cases} \tag{38}$$

It follows immediately by Lemma 4 the estimate

$$\| w \|_G = \| u_\lambda - u_{\bar{\lambda}} \|_G \leq C |\lambda - \bar{\lambda}| \tag{39}$$

which means that for any bounded subset S of G, the mapping $T_\lambda: S \to G$ is uniformly continuous for $0 \leq \lambda \leq 1$.

When $\lambda = 0$, for any $v \in G$, $T_0 v = u_0$ is a fixed vector.

Now we turn to consider the a priori estimations of the solutions of the periodic boundary problem (5) for the nonlinear pseudo-parabolic system:

$$u_t + (-1)^M A(t) u_{x^{2M}} + (-1)^M B u_{x^{2M}t} + \lambda \sum_{j=0}^{M} (-1)^j \frac{\partial^j}{\partial x^j} (F_j(u, \ldots, u_{x^{M-1}}) u_{x^j}) = f(x, t) \tag{40}$$

with parameter $0 \leq \lambda \leq 1$. By Lemmas 3-6 it follows that all possible solutions of the problem (40) (5) are uniformly bounded for $0 \leq \lambda \leq 1$ in the base space G.

Therefore by the Leray-Schauder's fixed point principle the problem (4) (5) has at least one global solution $u(x, t) \in Z$.

When $m \geq M + 1$ the existence of the global smooth solution for the porblem (4) (5) is the direct consequence of the estimates (9) and (35) and the Sobolev′ s imbedding theorem.

Theorem 2. *The global solution of problem (4) (5) is unique.*

Proof. Suppose that there are two solutions u and v of the problem (4) (5). Then $w = u - v$ satisfies the following system and periodic boundary condition:

$$w_t + (-1)^M A(t) w_{x^{2M}} + (-1)^M B w_{x^{2M}t} + \sum_{j=0}^{M} (-1)^j \frac{\partial^j}{\partial x^j} (F_j(u, \ldots, u_{x^{M-1}}) w_{x^j}) +$$

$$+\sum_{j=0}^{M}(-1)^j\frac{\partial^j}{\partial x^j}\left(\left(\sum_{k=0}^{M-1}\frac{\partial \widetilde{F}_j}{\partial p_k}w_{x^k}\right)v_{x^j}\right)=0 \tag{41}$$

$$\begin{cases} w(x,\ t)=w(x+2X,\ t) \\ w(x,\ 0)=0 \end{cases} \tag{42}$$

where the symbol "$\sim$" denotes that the functions take the mean values. By the Lemmas 4 and 6 the vector functions u_{x^k}, v_{x^k} ($k=0,\ 1,\ \ldots,\ 2M$) are all bounded in Q_T, hence the vectors $\frac{\partial \widetilde{F}_j}{\partial p_k}$ ($p_k=u_{x^k}$, $k=0,\ 1,\ \ldots,\ M-1$; $j=0,\ 1,\ \ldots,\ M$) are also bounded.

Taking the scalar product of w with the system (41) and integrating in Q_t ($0<t\leq T$) by parts, we have

$$\frac{1}{2}(w,\ w)+\frac{1}{2}(Bw_{x^M},\ w_{x^M})+\sum_{j=0}^{M}[F_j(u,\ \ldots,\ u_{x^{M-1}})w_{x^j},\ w_{x^j}]$$

$$=-[A(t)w_{x^M},\ w_{x^M}]-\sum_{j=0}^{M}\left[\left(\left(\sum_{k=0}^{M-1}\frac{\partial \widetilde{F}_j}{\partial p_k}w_{x^k}\right)v_{x^j}\right),\ w_{x^j}\right] \tag{43}$$

By applying the Gronwall's lemma we have

$$|w(\cdot,\ t)|_{L_2(\Omega)}+|w(\cdot,\ t)|_{H^M(\Omega)}=0 \tag{44}$$

Then, it follows that $w\equiv 0$, i. e., $u=v$.

Since the estimations given in the above lemmas are all independent of the width $2X$ of the rectangular domain Q_T, by taking the limiting process for $X\to\infty$, we can obtain the solution of the Cauchy problem

$$u(x,\ 0)=\varphi(x),\ -\infty<x<+\infty \tag{45}$$

for the system (4).

Theorem 3. *Suppose that all conditions for $f(x,\ t)$ and $\varphi(x)$ in the Theorem 1 hold in $\widetilde{Q}_T\equiv\{-\infty<x<+\infty,\ 0<t<T\}$. Then the Cauchy problem (4) (45) has a unique global solution $u(x,\ t)\in H^2(0,\ T;H^{2M}(-\infty,\ \infty))$ when $m=M$, and has a unique global smooth solution with the continuous derivatives $u_{x^{M+m-1}}$ and $u_{x^{M+m-1}t}$ when $m\geq M+1$.*

3. Initial-Boundary Value Problem

(A) In Q_T we consider the nonlinear pseudo-parabolic system

$$Lu\equiv u_t+(-1)^M A(t)u_{x^{2M}}+(-1)^M Bu_{x^{2M}t}+$$

$$+\sum_{j=0}^{[\frac{M}{2}]}(-1)^j\frac{\partial^j}{\partial x^j}(F_j(u,\ \ldots,\ u_{x^{M-1}})u_{x^j})=f(x,\ t) \tag{46}$$

and the IBV problem

$$\begin{cases} u_{x^k}(-X,\ t)=u_{x^k}(X,\ t)=0, & k=0,\ 1,\ \ldots,\ M-1,\ t\in[0,\ T] \\ u(x,\ 0)=\varphi(x),\ x\in[-X,\ X] \end{cases} \tag{47}$$

Assume that

(48)
- *i)* $A(t)$ is a bounded matrix,
- *ii)* B is a positively definite constant matrix,
- *iii)* $F_j\left(j=0,\ 1,\ \ldots,\ \left[\frac{M}{2}\right]\right)$ are nonnegative and $\left[\frac{M}{2}\right]$-times continuously differentiable,
- *iv)* $f\in L_2(Q_T)$; $\varphi(x)\in H^{2M}(\Omega)$ and vanishes together with its derivatives of order up to $M-1$ at the ends of the interval $[-X,\ X]$.

By making the integrations $[Lu,\ u]$, $[Lu,\ (-1)^M u_{x^{2M}}]$, $[Lu,\ (-1)^M u_{x^{2M}t}]$ respectively and applying the Lemmas 1 and 2, we derive without difficulties the following estimates of the solutions of the IBV problem (46) (47) immediately:

$$|u(\cdot,t)|^2_{H^M(\Omega)} \leq C\{\|f\|^2_{L_2(Q_T)} + |\varphi|^2_{H^M(\Omega)}\},\ \forall t\in[0,T] \tag{49}$$

$$|u(\cdot,t)|^2_{H^{2M}(\Omega)} \leq C\{\|f\|^2_{L_2(Q_T)} + |\varphi|^2_{H^{2M}(\Omega)}\},\ \forall t\in[0,T] \tag{50}$$

$$\|u_{x^Mt}\|^2_{L_2(Q_t)} + \|u_{x^{2M}t}\|^2_{L_2(Q_t)} \leq C\{\|f\|^2_{L_2(Q_T)} + |\varphi|^2_{H^{2M}(\Omega)}\} \tag{51}$$

Theorem 4. *Suppose that the conditions in (48) are satisfied. Then the IBV problem (46) (47) has a unique global generalized solution* $u(x,t)\in H^1(0,T;H^{2M}(\Omega))$.

This theorem is proved by applying the fixed point technique as used in the proof of Theorem 1.

(B) In $Q_T=\{(x,t)\,|\,x\in\Omega\equiv(-X,X),\ t\in(0,T)\}$ we consider another nonlinear pseudo-parabolic system

$$\widetilde{L}u \equiv u_t + (-1)^M A(t)u_{x^{2M}} + (-1)^M Bu_{x^{2M}t} + \\ + \sum_{j=0}^{M-1}(-1)^j\frac{\partial^j}{\partial x^j}(F_j(u,\ \dots,\ u_{x^{M-1-j}})u_{x^jt}) = f(x,t) \tag{52}$$

Assume that

$$\begin{cases} i)\ A(t) \text{ and } B \text{ are } N\times N \text{ matrices, } A(t) \text{ is bounded, } B \text{ is a positively definite constant matrix, as defined in (6),} \\ ii)\ F_j(u,\ \dots,\ u_{x^{M-1-j}})\ (j=0,1,\dots,M-1) \text{ are nonnegative functions } \forall (u,\dots,u_{x^{M-1-j}})\in R^{N(M-j)} \text{ and } j\ (j=0,1,\dots,M-1)\text{-times} \\ \quad \text{continuously differentiable with respect to all their variables,} \\ iii)\ f\in L_2(Q_T),\ \varphi(x)\in H^{2M}(\Omega) \text{ and vanishes together with all its derivatives of order up to } M-1 \text{ at the ends of the interval } [-X,X]. \end{cases} \tag{53}$$

Lemma 8. *The solution of IBV problem (52) (47) has estimation*

$$\|u_t\|^2_{L_2(Q_t)} + |u(\cdot,t)|^2_{H^M(\Omega)} + \|u_{x^Mt}\|^2_{L_2(Q_t)} \\ \leq C\{\|f\|^2_{L_2(Q_T)} + |\varphi|^2_{H^M(\Omega)}\},\ \forall t\in[0,T] \tag{54}$$

Proof. By making the integration $[\widetilde{L}u, u_t]$, we have

$$[u_t,u_t] + [Bu_{x^Mt}, u_{x^Mt}] + \sum_{j=0}^{M-1}[F_ju_{x^jt}, u_{x^jt}] = [f,u_t] - [A(t)u_{x^M}, u_{x^Mt}] \tag{55}$$

The terms on the right side of (55) are estimated respectively by

$$|[f,u_t]| \leq \frac{1}{4}[u_t,u_t] + [f,f] \tag{56}$$

$$|[A(t)u_{x^M}, u_{x^Mt}]| \leq \frac{b_0}{4}[u_{x^Mt}, u_{x^Mt}] + C(b_0)[u_{x^M}, u_{x^M}] \tag{57}$$

The third term on the left side of (55) is nonnegative. We add a term $[u_{x^M}, u_{x^Mt}]$ simultaneously to the both sides of the equality (55), and on the left side we transform it into the form $\frac{1}{2}(u_{x^M},u_{x^M}) - \frac{1}{2}(\varphi^{(M)},\varphi^{(M)})$, on the right side we estimate it by

$$[u_{x^M}, u_{x^Mt}] \leq \frac{b_0}{4}[u_{x^Mt}, u_{x^Mt}] + C(b_0)[u_{x^M}, u_{x^M}] \tag{58}$$

Then, by applying Gronwall's lemma, we derive the inequality (54) immediately.

Lemma 9. *The solution of IBV problem (52) (47) satisfies the estimate*

$$\|u_{x^Mt}\|_{L_2(Q_t)} + |u(\cdot,t)|_{H^{2M}(\Omega)} + \|u_{x^{2M}t}\|_{L_2(Q_t)} \\ \leq C(a_0, b_0, \|f\|_{L_2(Q_T)}, |\varphi|_{H^{2M}(\Omega)}),\ \forall t\in[0,T] \tag{59}$$

Proof. Taking the scalar product of the vector $(-1)^M u_{x^{2M}t}$ with the system (52) and then making the integration $[Lu, (-1)^M u_{x^{2M}t}]$, we have

$$[u_{x^Mt}, u_{x^Mt}] + [Bu_{x^{2M}t}, u_{x^{2M}t}] = [f, (-1)^M u_{x^{2M}t}] - [A(t)u_{x^{2M}}, u_{x^{2M}t}] + \\ + \sum_{j=0}^{M-1}\left[(-1)^j\frac{\partial^j}{\partial x^j}(F_j(u,\ \dots,\ u_{x^{M-1-j}})u_{x^jt}),\ (-1)^{M-1}u_{x^{2M}t}\right] \tag{60}$$

By the Lemma 8 we have $u_{x^k}\in L_\infty(Q_T)$, $k=0,1,\dots,M-1$. Then by the Lemma 2, the first and third terms on the right side of (60) may be estimated by

$$
\begin{aligned}
&|[f,\ (-1)^M u_{x^{2M}t}]| + \left|\sum_{j=0}^{M-1}\left[(-1)^j \frac{\partial^j}{\partial x^j}(F_j(u,\ \ldots,\ u_{x^{M-1-j}})\, u_{x^j t}),\ (-1)^{M-1} u_{x^{2M}t}\right]\right| \\
&\quad \le \frac{b_0}{8}[u_{x^{2M}t},\ u_{x^{2M}t}] + C(b_0)\left\{\|f\|^2_{L_2(Q_T)} + \sum_{j=0}^{2M-2}\|u_{x^j t}\|^2_{L_2(Q_t)}\right\} \\
&\quad \le \frac{b_0}{4}[u_{x^{2M}t},\ u_{x^{2M}t}] + C'(b_0)\left\{\|f\|^2_{L_2(Q_T)} + \|u_t\|^2_{L_2(Q_t)}\right\} \qquad (61)
\end{aligned}
$$

The second term on the right side of (60) may be estimated by

$$
|[A(t)\, u_{x^{2M}},\ u_{x^{2M}t}]| \le \frac{b_0}{4}[u_{x^{2M}t},\ u_{x^{2M}t}] + C_1(b_0)\,[u_{x^{2M}},\ u_{x^{2M}}] \qquad (62)
$$

We add a term $[u_{x^{2M}},\ u_{x^{2M}t}]$ simultaneously to the both sides of the equality (60), and on the left side we transform it into the form $\frac{1}{2}(u_{x^{2M}},\ u_{x^{2M}}) - \frac{1}{2}(\varphi^{(2M)},\ \varphi^{(2M)})$, on the right side we estimate it by

$$
|[u_{x^{2M}},\ u_{x^{2M}t}]| \le \frac{b_0}{4}[u_{x^{2M}t},\ u_{x^{2M}t}] + C_2(b_0)\,[u_{x^{2M}},\ u_{x^{2M}}] \qquad (63)
$$

Then, by applying the Gronwall′s lemma, we derive the inequality (59) immediately.

Having the estimation (59), then applying the fixed point technique as used in the proof of Theorem 1, we have.

Theorem 5. *Suppose that the conditions in* (53) *are satisfied. The IBV problem* (52) (47) *has a unique global generalized solution* $u(x,\ t) \in H^1(0,\ T;\ H^{2M}(\Omega))$.

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