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ON THE RESTRICTED PARAMETER IDENTIFICATION PROBLEMS FOR GENERAL VARIATIONAL INEQUALITIES *

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Abstract

A restricted parameter identification problem for general variational inequalities is studied using the so-called asymptotic regularization method. Some applications, including the evolution dam problem, are also briefly discussed.

1. Introduction

In [4], Hoffmann and Sprekels discussed an identification problem of general variational inequalities under a functional analytic framework. The problem they treated is the following:

Let H, X be two Hilbert spaces, V be a separable and reflexive Banach space with dual V^* and X_0 be another Banach space, such that the embeddings $V \subset H \subset V^*$ are dense and continuous and X_0 is a dense subspace of X. The dual pairing between elements of V^* and V is denoted by $\langle \cdot , \cdot \rangle$. The inner product in X is denoted by $[\cdot , \cdot]$. Let $C \subset V$ be nonempty, closed and convex. Then, the identification problem is as follows:

Problem $(\hat{P})$. Given $u^* \in D(S) \cap C$ and $f^* \in V^*$, find $a^* \in X$, with $(a^*, u^*) \in D(A_2)$, such that there exists a $w^* \in S(u^*)$ which satisfies the variational inequality

$$\langle w^* + A_1(a^*) + A_2(a^*, u^*) - f^*, v - u^* \rangle + \Psi(v) - \Psi(u^*) \geq 0, \quad \forall v \in C \tag{1.1}$$

where, S, A_1, A_2, Ψ are some given operators which will be defined in the next section (or see [4]).

The main purpose of this paper is to discuss a similar problem with the restriction $a^* \in X$ replaced by $a^* \in K$ for some nonempty, closed and convex subset K of X_0. In many physical problems, the parameters we want to find should belong to some specific convex and closed set. The method we use is a combination of those used in [1], [4] and [6].

2. Solution to Finite Dimensional Problems

Let H, X, V, V^*, X_0 be the same as in section 1. The norms of these spaces are denoted by $\| \cdot \|_H$, $\| \cdot \|_X$, $\| \cdot \|_V$, $\| \cdot \|_{V^*}$ and $\| \cdot \|_{X_0}$, respectively. Let $C \subset V$ be nonempty, convex and closed. Let $K \subset X_0$ be nonempty, convex and closed in X. we consider the following restricted parameter identification problem:

Problem (P). Given $u^* \in D(S) \cap C, f^* \in V^*$ find $a^* \in K$, such that there

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$$\{T_n(a, u), (\hat{a}, \hat{u})\} = \langle \frac{\varepsilon}{h}u + A_1(a) + A_2(a, u), \hat{u}\rangle + \left[\frac{a}{h}, \hat{a}\right]$$
$$- \langle A_1(\hat{a}), u\rangle - \langle A_2(\hat{a}, u), u - u^*\rangle \quad (2 \cdot 10)$$

$$g_n(a, u) = -\langle \frac{\varepsilon}{h}u_n + f^*, u\rangle + \Psi(u) - \left[\frac{a_n}{h}, a\right]$$
$$+ \langle A_1(a), u^*\rangle \quad (2 \cdot 11)$$

For $w \in V^* \subset V_N^*$ we set

$$\{(0, w), (a, u)\} = \langle w, u\rangle, \ \forall\ (a, u) \in B \quad (2 \cdot 12)$$

Thus, the $(n+1)$ -th step of Problem (P_h) is equivalent to solve the following problem:

Problem (T_n). Find $(a,u) \in G$, with $u \in D(S)$, such that there exists $w \in S(u)$, satisfying

$$\{(0, w) + T_n(a, u), (\hat{a}, \hat{u}) - (a, u)\} + g_n(\hat{a}, \hat{u}) - g_n(a, u) \geq 0,$$
$$\forall\ (\hat{a}, \hat{u}) \in G \quad (2 \cdot 13)$$

Theorem 2.1. *Suppose that* (A1) — (A7) *hold and* $\varepsilon > 0$ *is given. Then, there exists* $h_0 > 0$, *such that for any* $h \in (0, h_0]$, *Problem* (P_h) *has a solution* $\{(a_n, u_n)\}_{n=0}^{\infty}$.

Proof. It suffices to prove that for any $n \geq 1$, Problem (T_n) has a solution. To this end, like the proof of Theorem 2.1 of [4], we need to check several things.

First of all, since g_n is convex and continuous on B, we have

$$D(\partial g_n) = B \quad (2 \cdot 14)$$

Let us define $Q: B \to 2^{B^*}$ by $Q(a,u) = \{0\} \times S(u)$. Then, Q is maximal monotone. By (A1) and (A6) and the definition of G, we have

$$(P_{W_M}(a^*), u^*) \in \mathrm{int}(D(Q)) \cap G \quad (2 \cdot 15)$$

By (2.14), we have

$$\mathrm{int}(D(Q)) \cap D(\partial g_n) \neq \varnothing \quad (2 \cdot 16)$$

and

$$G \cap \mathrm{int} D(Q + \partial g_n) = G \cap \mathrm{int} D(Q) \neq \varnothing \quad (2 \cdot 17)$$

Also, the same argument as in [4], we have

$$\partial(g_n + X_G) = \partial g_n + \partial X_G \quad (2 \cdot 18)$$

where

$$X_G(x) = \begin{cases} 0 & x \in G \\ +\infty & x \in G^C \end{cases} \quad (2 \cdot 19)$$

Now, it remains to check T_n is continuous, bounded, coercive and pseudomonotone. Since $\dim W_M$, $\dim V_N < \infty$, we can assume that there exist $\alpha, \beta > 0$, such that

$$\| a \|_{X_0} \leq \alpha \| a \|_X, \quad \forall\ a \in W_M \quad (2 \cdot 20)$$
$$\| u \|_V \leq \beta \| u \|_H, \quad \forall\ u \in V_N \quad (2 \cdot 21)$$

It is easy to show that for any $(a,u), (\hat{a},\hat{u}) \in B$

$$\| T_n(a, u) - T_n(\hat{a}, \hat{u}) \|_{B^*}^2 \leq \Big\{\Big(\frac{\varepsilon}{h} + \alpha\beta^2 \| A_2 \| \ \| \hat{a} \|_X\Big) \| u - \hat{u} \|_H$$
$$+ (\beta \| A_1 \| + \alpha\beta^2 \| A_2 \| \ \| u \|_H) \| a - \hat{a} \|_X\}^2$$
$$+ \Big\{ [\beta \| A_1 \| + \alpha\beta^2 \| A_2 \| (\| u - u^* \|_H + \| u \|_H)]$$
$$\cdot \| u - \hat{u} \|_H + \frac{1}{h} \| a - \hat{a} \|_X \Big\}^2 \quad (2 \cdot 22)$$

which gives the continuity and boundedness of T_n. For the coercivity, let us take $(a,u) \in G$,

$$\{T_n(a, u), (a, u)\} = \langle \frac{\varepsilon}{h}u + A_1(a) + A_2(a, u), u\rangle + \frac{1}{h} \| a \|_X^2$$
$$- \langle A_1(a), u\rangle - \langle A_2(a, u), u - u^*\rangle$$
$$= \frac{\varepsilon}{h} \| u \|_H^2 + \frac{1}{h} \| a \|_X^2 + \langle A_2(a, u), u^*\rangle$$

exists a $w^* \in S(u^*)$, satisfying

$$\langle w^* + A_1(a^*) + A_2(a^*, u^*) - f^*, v - u^* \rangle + \Psi(v) - \Psi(u^*) \geq 0, \quad \forall v \in C \tag{2·1}$$

Here similar to [4], we assume:

(A1) $S: D(S) \subset V \to 2^{V^*}$ is a maximal monotone graph with $u^* \in \text{int } D(S)$.

(A2) $A_1: X \to V^*$ is linear and bounded, with

$$\| A_1(a) \|_{V^*} \leq \| A_1 \| \, \| a \|_X, \; \forall \; a \in X \tag{2·2}$$

(A3) $A_2: D(A_2) \subset X \times V \to V^*$ is bilinear with

$$X_0 \times V \subset D(A_2) \tag{2·3}$$

$$|\langle A_2(a, u), v \rangle| \leq \| A_2 \| \, \| a \|_{X_0} \| u \|_V \| v \|_V, \quad \forall (a, u, v) \in X_0 \times V \times V \tag{2·4}$$

$$\langle A_2(a, u - v), u - v \rangle \geq \delta \| u - v \|_V^2, \quad \forall (a, u, v) \in K \times C \times C \tag{2·5}$$

for some positive number δ.

(A4) $\Psi: V \to \mathbf{R}$ is continuous and convex.

(A5) Problem (P) has at least one solution $a^* \in K$.

Now we turn to study a finite dimensional analogue of Problem (P). Tet $V_N \subset V, W_M \subset X_0$ be two finite dimensional subspaces, satisfying the following assumptions:

(A6) $u^* \in V_N, P_{W_M}(K) \subset K$, where, $P_{W_M}: X \to W_M$ is the orthogonal projection.

(A7) For any $a \in (I - P_{W_M})(K) \subset W_M^{\perp}, u, v \in C \cap V_N$,

$$\langle A_1(a) + A_2(a, u), v - u^* \rangle = 0 \tag{2·6}$$

Our finite dimensional identification problem is the following:

Problem (P^N). Given $u^* \in D(S) \cap C \cap V_N, f^* \in V^*$, find $a^N \in K \cap W_M$, such that there exists a $w^* \in S(u^*)$, satisfying

$$\langle w^* + A_1(a^N) + A_2(a^N, u^*) - f^*, v - u^* \rangle + \Psi(v) - \Psi(u^*) \geq 0 \quad \forall v \in C \cap V_N \tag{2·7}$$

We now introduce a sequence of systems of variational inequalities, the solutions of which will converge to a solution of Problem (P^N). Let $\varepsilon > 0, h > 0$ be given, we consider the following problem:

Problem (P_h). Given $(a_0, u_0) \in (K \cap W_M) \times (D(S) \cap C \cap V_N)$, find $(a_n, u_n) \in (K \cap W_M) \times (D(S) \cap C \cap V_N)$, such that there exists a $w_n \in S(u_n)$, satisfying

$$\begin{cases} \langle \varepsilon \dfrac{u_{n+1} - u_n}{h} + w_{n+1} + A_1(a_{n+1}) + A_2(a_{n+1}, u_{n+1}) - f^*, \hat{u} - u_{n+1} \rangle \\ \qquad + \Psi(\hat{u}) - \Psi(u_{n+1}) \geq 0 \\ \left[\dfrac{a_{n+1} - a_n}{h}, \hat{a} - a_{n+1} \right] - \langle A_1(\hat{a} - a_{n+1}) \\ \qquad + A_2(\hat{a} - a_{n+1}, u_{n+1}), u_{n+1} - u^* \rangle \geq 0 \end{cases} \tag{2·8}$$

$\forall (\hat{a}, \hat{u}) \in G \triangleq (K \cap W_M) \times (C \cap V_N), n = 1, 2, \cdots$

It is clear that at $(n+1)$- th step of Problem (P_h), we want to solve the following equivalent problem:

For $(a_n, u_n) \in (K \cap W_M) \times (D(S) \cap C \cap V_N)$, find $(a, u) \in (K \cap W_M) \times (D(S) \cap C \cap V_N)$, such that there exists a $w \in S(u)$, with

$$\langle \varepsilon \frac{u - u_n}{h} + w + A_1(a) + A_2(a, u) - f^*, \hat{u} - u \rangle + \Psi(\hat{u}) - \Psi(u) + \left[\frac{a - a_n}{h}, \hat{a} - a \right] - \langle A_1(\hat{a} - a) + A_2(\hat{a} - a, u), u - u^* \rangle \geq 0 \tag{2·9}$$

$\forall (\hat{a}, \hat{u}) \in G$

Now, let us set $B = W_M \times V_N$. The dual pairing between elements of B^* and B is denoted by $\{ \cdot, \cdot \}$. It is clear that dim $B < \infty$. We define $T_n: B \to B^*$ and $g_n: B \to R$ as follows:

$$\geq \frac{\varepsilon}{h}\| u \|_H^2 + \frac{1}{h}\| a \|_X^2 - \alpha\beta \| A_2 \| \, \| a \|_X \| u \|_H \| u^* \|_V$$
$$\geq \left(\frac{\varepsilon}{h} - \frac{\alpha\beta}{2}\| A_2 \| \, \| u^* \|_V\right)\| u \|_H^2 + \left(\frac{1}{h} - \frac{\alpha\beta}{2}\| A_2 \| \, \| u^* \|_V\right)\| a \|_X^2 \tag{2·23}$$

Hence, for given $\varepsilon > 0$, we can choose $h_0 > 0$, such that for any $h \in (0, h_0]$, T_* is coercive with respect to $0 \in B^*$.

Finally, since T_* is continuous and $\dim B < \infty$ we have

$$\lim_{m\to\infty}\{T_*(a_m, u_m), (a_m, u_m) - (a, u)\} = \{T_*(\hat{a}, \hat{u}), (\hat{a}, \hat{u}) - (a, u)\} \quad \forall (a, u) \in B \tag{2·24}$$

whenever $(a_m, u_m) \to (\hat{a}, \hat{u})$ weakly. That means T_* is pseudomonotone. Then, our theorem follows a general theorem on variational inequalities (see [4] or [8]).

Theorem 2. 2. *Suppose (A1) — (A7) hold and $\{(a_n, u_n)\}_{n=1}^{\infty}$ is a solution to Problem (P_h). Then, there exists an $a_\infty \in K \cap W_M$, such that*

$$\| u_n - u^* \|_V \to 0 \quad as \quad n \to \infty \tag{2·25}$$
$$\| a_n - a_\infty \|_X \to 0 \quad as \quad n \to \infty \tag{2·26}$$

and a_∞ is a solution of Problem (P^N).

Proof. Set $q_n = u_n - u^*$, $\gamma_n = a_n - a^*$. Similar to [4], we can get the following a priori estimates.

$$\varepsilon \| q_{n+1} \|_H^2 + \| \gamma_{n+1} \|_X^2 + \delta \| q_{n+1} \|_V^2 \leq \varepsilon \| q_n \|_H^2 + \| \gamma_n \|_X^2 \tag{2·27}$$

$$\sup_{n\in N}\{\varepsilon \| q_n \|_H^2 + \| \gamma_n \|_X^2\} + 2\delta h \sum_{k=1}^{\infty} \| q_k \|_V^2 \leq \varepsilon \| q_0 \|_H^2 + \| \gamma_0 \|_X^2 < \infty \tag{2·28}$$

Thus, (2. 25) follows (2. 28). On the other hand, (2. 28) implies $\| a_n \|_X$ is bounded. Therefore there is a subsequence a_{n_k} such that

$$a_{n_k} \to a_\infty \in K \cap W_M \quad in \quad X \tag{2·29}$$

It is clear that a_∞ is a solution of Problem (P^N). The derivation of (2. 27) and (2. 28) shows that the same estimates also hold if we replace γ_n by $a_n - a_\infty$. Thus, we have, in particular,

$$\varepsilon \| q_{n+1} \|_H^2 + \| a_{n+1} - a_\infty \|_X^2 \leq \varepsilon \| q_n \|_H^2 + \| a_n - a_\infty \|_X^2 \tag{2·30}$$

This implies $\lim_{n\to\infty}(\varepsilon \| q_n \|_H^2 + \| a_n - a_\infty \|_X^2)$ exists. Together with the proved (2. 25), we have

$$\lim_{n\to\infty} \| a_n - a_\infty \|_X = d \tag{2·31}$$

Then, (2. 26) follows (2. 29) and (2. 31).

3. Stability and Solution of Problem (P)

In this section we first show that the limit point a_∞ of (P_h) is stable against perturbations in the initial data.

Theorem 3. 1. *Let (A1) — (A7) hold. Assume that initial data $(a_0^\lambda, u_0^\lambda)$, $|\lambda| \leq \lambda_0$, are given such that*

$$a_0^\lambda \to a_0^0, \; u_0^\lambda \to u_0^0, \; as \quad \lambda \to 0 \tag{3·1}$$

Then there is an $\hat{h} \in (0, h_0]$ such that for $0 < h \leq \hat{h}$ the following assertion holds: If $\{a_n^\lambda, u_n^\lambda\}_{n\geq 1}$ solves (P_h) with initial data $(a_0^\lambda, u_0^\lambda)$, $|\lambda| \leq \lambda_0$, and a_∞^λ is the limit point of $\{a_n^\lambda\}$, then

$$a_\infty^\lambda \to a_\infty^0, \; as \quad \lambda \to 0 \tag{3·2}$$

Proof. Let $0 < h \leq h_0$. From (2. 8), we have

$$\langle \varepsilon\frac{u_{n+1}^{\lambda}-u_n^{\lambda}}{h}+w_{n+1}^{\lambda}+A_1(a_{n+1}^{\lambda})+A_2(a_{n+1}^{\lambda},u_{n+1}^{\lambda})-f^*,\ v-u_{n+1}^{\lambda}\rangle$$
$$+\Psi(v)-\Psi(u_{n+1}^{\lambda})\geq 0,\quad \forall\ v\in C\cap V_N \tag{3·3}$$
with some $w_{n+1}^{\lambda}\in S(u_{n+1}^{\lambda})$,
$$\left[\frac{a_{n+1}^{\lambda}-a_n^{\lambda}}{h},\ \eta-a_{n+1}^{\lambda}\right]-\langle A_1(\eta-a_{n+1}^{\lambda})+A_2(\eta-a_{n+1}^{\lambda},u_{n+1}^{\lambda}),\ u_{n+1}^{\lambda}-u^*\rangle\geq 0$$
$$\forall\ \eta\in K\cap W_M \tag{3·4}$$
Let $y_n^{\lambda}=u_n^{\lambda}-u_n^0,\gamma_n^{\lambda}=a_n^{\lambda}-a_n^0,q_n^{\lambda}=u_n^{\lambda}-u^*$. Then it follows from (3. 3), (3. 4):
$$\langle \varepsilon\frac{u_{n+1}^{\lambda}-u_n^{\lambda}}{h}+w_{n+1}^{\lambda}+A_1(a_{n+1}^{\lambda})+A_2(a_{n+1}^{\lambda},u_{n+1}^{\lambda})-f^*,\ u_{n+1}^{\lambda}-u_{n+1}^0\rangle$$
$$+\Psi(u_{n+1}^{\lambda})-\Psi(u_{n+1}^0)\leq 0\leq\langle\varepsilon\frac{u_{n+1}^0-u_n^0}{h}+w_{n+1}^0+A_1(a_{n+1}^0)$$
$$+A_2(a_{n+1}^0,u_{n+1}^0)-f^*,\ u_{n+1}^{\lambda}-u_{n+1}^0\rangle+\Psi(u_{n+1}^{\lambda})-\Psi(u_{n+1}^0)$$
hence
$$\langle\varepsilon\frac{y_{n+1}^{\lambda}-y_n^{\lambda}}{h},\ y_{n+1}^{\lambda}\rangle+\langle A_2(a_{n+1}^0,y_{n+1}^{\lambda}),\ y_{n+1}^{\lambda}\rangle$$
$$+\langle A_1(\gamma_{n+1}^{\lambda})+A_2(\gamma_{n+1}^{\lambda},u_{n+1}^{\lambda}),\ y_{n+1}^{\lambda}\rangle\leq 0 \tag{3·5}$$
Putting $\eta=a_{n+1}^0$ in (3. 4) yields:
$$\left[\frac{\gamma_{n+1}^{\lambda}-\gamma_n^{\lambda}}{h},\ \gamma_{n+1}^{\lambda}\right]+\left[\frac{a_{n+1}^0-a_n^0}{h},\ \gamma_{n+1}^{\lambda}\right]-\langle A_1(\gamma_{n+1}^{\lambda})+A_2(\gamma_{n+1}^{\lambda},u_{n+1}^{\lambda}),\ y_{n+1}^{\lambda}\rangle$$
$$-\langle A_1(\gamma_{n+1}^{\lambda})+A_2(\gamma_{n+1}^{\lambda},u_{n+1}^{\lambda}),\ q_{n+1}^0\rangle\leq 0$$
and putting $\lambda=0,\eta=a_{n+1}^{\lambda}$ in (3. 4) yields:
$$\left[\frac{a_{n+1}^0-a_n^0}{h},\ \gamma_{n+1}^{\lambda}\right]\geq\langle A_1(\gamma_{n+1}^{\lambda})+A_2(\gamma_{n+1}^{\lambda},u_{n+1}^0),\ q_{n+1}^0\rangle$$
So we have
$$\langle A_1(\gamma_{n+1}^{\lambda})+A_2(\gamma_{n+1}^{\lambda},u_{n+1}^{\lambda}),\ y_{n+1}^{\lambda}\rangle$$
$$\geq\left[\frac{\gamma_{n+1}^{\lambda}-\gamma_n^{\lambda}}{h},\ \gamma_{n+1}^{\lambda}\right]-\langle A_2(\gamma_{n+1}^{\lambda},y_{n+1}^{\lambda}),\ q_{n+1}^0\rangle \tag{3·6}$$
Combining (3. 5) and (3. 6), we get
$$\langle\varepsilon\frac{y_{n+1}^{\lambda}-y_n^{\lambda}}{h},\ y_{n+1}^{\lambda}\rangle+\left[\frac{\gamma_{n+1}^{\lambda}-\gamma_n^{\lambda}}{h},\ \gamma_{n+1}^{\lambda}\right]$$
$$+\langle A_2(a_{n+1}^0,y_{n+1}^{\lambda}),\ y_{n+1}^{\lambda}\rangle-\langle A_2(\gamma_{n+1}^{\lambda},y_{n+1}^{\lambda}),\ q_{n+1}^0\rangle\leq 0 \tag{3·7}$$
This is the same inequality as (2. 25) in [4]. Starting from this inequality and using the same arguments as in [4], we can obtain the assertion of Theorem 3. 1.

Now we discuss the situation as dim $V_N\to\infty$. Let $\{V_N\}_{N\geq 2}$ be a sequence of subspaces of V such that (A6) holds and $u_0\in D(S)\cap C\cap V_N,\forall\ N\geq 2$. Assume to each N that there is a subspace $W_{M(N)}$ of X_0 such that (A7) holds and $a_0\in W_{M(N)}$.

We assume further that:

(A8) $(V,\|\cdot\|)$ is a Hilbert space, i. e., $\|\cdot\|_V$ is induced by an inner product $(\cdot,\cdot)$ on V.

(A9) $V_N\subset V_{N+1},\forall\ N\geq 2$; $\bigcup_{N=2}^{\infty}V_N$ is dense in V. $P_{V_N}(C)\subset C,\forall\ N\geq 2$, where P_{V_N} denotes the $(\cdot,\ \cdot)$-orthogonal projection operator onto V_N.

(A10) $a_n\in K,a_n\to a$ weakly in X implies $A_2(a_n,u^*)\to A_2(a,u^*)$ weakly in V^*.

Theorem 3. 3. *Let (A1) — (A10) hold. If $\{a_\infty^N\}$ denotes a sequence of limit points of the solution of $(P_{h(N)})$, then each subsequence has a weak cluster point in K which solves (P).*

Proof. By Theorem 2. 2 for any $N\geq 2$, there exists an $a_\infty^N\in K\cap W_{M(N)}$ such that with some $w_\infty^N\in S(u^*)$
$$\langle w_\infty^N+A_1(a_\infty^N)+A_2(a_\infty^N,u^*)-f^*,\ v-u^*\rangle+\Psi(v)-\Psi(u^*)\geq 0, \tag{3·8}$$

$$\forall v \in C \cap V_N$$

The a priori estimate (2.28) gives that $\{a_\infty^N\}$ is bounded in X. Thus a subsequence, again denoted by a^N, converges weakly in X to some $a \in K$. By the boundedness of $S(u^*)$ we may assume that $w^N \to w^*$ weakly in V^*, where $w^* \in S(u^*)$ by the maximal monotonicity.

Now we show that a_∞ is a solution of (P) with $w^* \in S(u^*)$.

Let $v \in C$ be arbitrary. By (A9), $P_{V_N}v \in C \cap V_N$ and $\| P_{V_N}v - v \|_V \to 0$, as $N \to \infty$. We have

$$\begin{aligned}
&\langle w^* + A_1(a_\infty) + A_2(a_\infty, u^*) - f^*, v - u^* \rangle + \Psi(v) - \Psi(u^*) \\
&= [\langle w_\infty^N + A_1(a_\infty^N) + A_2(a_\infty^N, u^*) - f^*, P_{V_N}v - u^* \rangle + \Psi(P_{V_N}v) - \Psi(u^*)] \\
&\quad + [\langle w^* - w_\infty^N + A_1(a_\infty - a_\infty^N) + A_2(a_\infty - a_\infty^N, u^*), P_{V_N}v - u^* \rangle \\
&\quad + \Psi(v) - \Psi(P_{V_N}v)] + [\langle w^* + A_1(a_\infty) + A_2(a_\infty, u^*) - f^*, v - P_{V_N}v \rangle]
\end{aligned}$$

The first bracket is nonnegative by (3.8), and the second and third brackets approach 0 as $N \to \infty$. Hence passing to the limit as $N \to \infty$ yields

$$\langle w^* + A_1(a_\infty) + A_2(a_\infty, u^*) - f^*, v - u^* \rangle + \Psi(v) - \Psi(u^*) \geq 0, \ \forall v \in C$$

i. e. a_∞ solves Problem (P).

4. Applications

The method in this paper can be applied to many physical problems including all the examples in [4]. For the sake of comparison, we take Example 1 in [4] as our first example and discuss it once again using our method. Then we take nonstationary dam problem as our second example. In our examples below $\Omega \subset R^m$ is open and bounded with a sufficiently smooth boundary $\partial\Omega$.

Example 1. Consider the problem

Given $u^* \in H_0^1(\Omega) \cap H^{1,\infty}(\Omega)$ and $f^* \in H^{-1}(\Omega)$ find a matrix $a^* = (a_{ij}^*)$ with $a^* \in K$ such that $-\nabla \cdot (a^* \nabla u^*) = f^*$ (4.1)

If we assume that

$$K = \{a = (a_{ij}) : a_{ij} \in L^2(\Omega)\}$$

then problem (4.1) is just the same as Example 1 in [4]. Now let K be the set of physically admissible parameters which we assume to be of the form

$$K = \{a = (a_{ij}) : a_{ij} \in L^\infty(\Omega),\ a_{ij} = a_{ji},\ \| a_{ij} \|_{L^\infty(\Omega)} \leq \beta, \quad \xi^T a(x) \xi \geq \alpha |\xi|^2,\ \forall \xi \in R^m, \text{ a. e. in } \Omega\} \tag{4.2}$$

here $\alpha > 0, \beta > 0$ are fixed constants.

Problem (4.1) with K satisfying (4.2) is the restricted parameter identification problem. After making some specifications as in [4], one can see that the assumptions (A1) — (A10) might be satisfied. Therefore we can apply the method which is developed in previous sections to this problem.

Example 2. (Nonstationary dam problem)

We consider nonstationary fluid (say water) flows through a porous medium (say dam) $\Omega \subset R^m$. Let Γ_1 be the boundary to the imperious ground, $T \in R, 0 < T < \infty$, and set $Q = \Omega \times (0,T)$, $\sum_1 = \Gamma_1 \times (0,T)$, $\sum = \partial\Omega \times (0,T)$, $\sum_2 = \sum \setminus \sum_1$. Assume that the boundary data for pressure $u(x,t)$ are given by

$$g^* \in L^2(0, t; H^{1,\infty}(\Omega)),\ g^* \geq 0 \tag{4.3}$$

Now we can state the weak formulation for the nonstationary dam problem as follows

Given $a^* \in K$ (see (4.2)), find a pair

$(u^*, \gamma^*) \in L^2(0,T; H^1(\Omega)) \times L^\infty(Q)$ with $u^* \geq 0$ in Q, $u^* = g^*$ on $\sum_2$,

$\gamma^* \subset H(u^*), \gamma^*(x,0) = \gamma_0$ in Ω, such that

$$\iint_Q \nabla v \cdot a^* (\nabla u^* + \gamma^* \vec{e}) - \gamma^* v_t \leq 0, \quad \forall v \in H^1(Q), \tag{4.4}$$

$$v = 0 \quad \text{on} \quad \Omega \times \{0, T\},\ v = 0 \quad \text{on} \quad \textstyle\sum_2 \cap [g^* > 0],$$

$$v \geq 0 \quad \text{on} \quad \sum\nolimits_2 \cap [g^* = 0]$$

where $\vec{e} = \nabla x_m$. Set

$$\hat{H}^1(\Omega) = \{v \in H^1(\Omega); \quad v = 0 \quad \text{on} \quad \partial\Omega \backslash \Gamma_1\} \tag{4·5}$$

and let (u^*, λ^*) be a solution of (4. 4). Then for any $v \in \mathscr{D}(0,T;\hat{H}^1(\Omega))$, we have

$$\iint\limits_Q \nabla v \cdot a^*(\nabla u^* + \gamma^* \vec{e}) - \gamma^* v_t = 0$$

i. e.

$$\iint\limits_Q \gamma^* v_t = \iint\limits_Q \nabla v \cdot a^*(\nabla u^* + \gamma^* \vec{e})$$

Therefore

$$\left| \iint\limits_Q \gamma^* v_t \right| \leq \left\{ \iint |a^*(\nabla u^* + \gamma^* \vec{e})|^2 \right\}^{1/2} \cdot \left\{ \iint\limits_Q |\nabla v|^2 \right\}^{1/2}$$
$$\leq C \cdot \| v \|_{L^2(0,T;H^1(\Omega))}$$

This shows that

$$\gamma_t^* \in L^2(0, T; \hat{H}^1(\Omega)^*) \tag{4·6}$$

and

$$\iint\limits_Q \nabla v \cdot a^*(\nabla u^* + \gamma^* \vec{e}) + \langle \gamma_t^* | v \rangle = 0, \ \forall\, v \in L^2(0, T; \hat{H}^1(\Omega)) \tag{4·7}$$

where $\langle \cdot | \cdot \rangle$ denotes the dual pairing between elements of $L^2(0,T;\hat{H}^1(\Omega)^*)$ and $L^2(0,T;\hat{H}^1(\Omega))$. (4. 7) can be rewritten as:

$$\iint\limits_Q \nabla v \cdot a^*(\nabla u^* + \nabla g^* + \gamma^* \vec{e}) + \langle \gamma_t^* | v \rangle = 0, \ \forall\, v \in L^2(0, T; \hat{H}^1(\Omega)) \tag{4·8}$$

where $u^* \in L^2(0,T;\hat{H}^1(\Omega)), u^* \geq -g^*$.

Now we can describe the corresponding restricted parameter identification problem as follows:

$$\begin{cases} \text{Given } u^* \in L^2(0, T; \hat{H}(\Omega)) \text{ with } u^* \geq -g^* \text{ and } \gamma^* \subset H(u^* + g^*), \\ \gamma_t^* \in L^2(0, T; \hat{H}^1(\Omega)^*), \text{ find } a^* \in K \text{ such that (4. 8)holds.} \end{cases} \tag{4·9}$$

We show that the theory of above sections applies.

Let

$H = L^2(Q)$

$V = L^2(0,T;\hat{H}^1(\Omega))$ where $\hat{H}^1(\Omega)$ is defined by (4. 5)

$C = V$

$X = \{a = (a_{ij}); a_{ij} \in L^2(\Omega), \forall\, i, j\}$

$X_0 = \{a = (a_{ij}); a_{ij} \in L^\infty(\Omega), \forall\, i, j\}$

K is defined by (4. 2)

and let

$S = \theta$

$$\langle A_1(a) | v \rangle = \iint\limits_Q \nabla v \cdot a(\nabla g^* + \gamma^* \vec{e})$$

$$\langle A_2(a, u) | v \rangle = \iint\limits_Q \nabla v \cdot a \nabla u$$

$\langle f | v \rangle = -\langle \gamma_t^* | v \rangle$

$\Psi = 0$

Using these notations, it is easy to see that problem (4.9) is a special case of Problem (P) which is described in section 2. Using the same arguments as in Example 3 (Dam problem) of [4], one can esaily see that all the assumptions (A1) — (A10) might be satisfied if we additionally have that $u^* \in L^2(0,T;H^{1,\infty}(\Omega))$.

References

[1] H. W. Alt, K. -H Hoffmann, J. Sprekels, A Numerical Procedure to Solve Certain Identification Problems, *Intern. Ser. Numer. Math.* **68**(1984),11-43.

[2] V. Barbu, Th. Precupanu, Convexity and Optimization in Banach Spaces, Sijthoff & Noordhoff,1978.

[3] G. Gilardi, A New Approach to Evolution Free Boundary Problems. *Comm. P. D. E.*, **4**(1979),1099-1122.

[4] K. -H. Hoffmann, J. Sprekels, On the Identification of Parameters in General Variational Inequalities by Asymptotic Regularization. To appear.

[5] K. -H. Hoffmann, J. Sprekels, On the Identification of Elliptic Problems by Asymptotic Regularization. To appear.

[6] K. -H. Hoffmann, J. Sprekels, The Method of Asymptotic Regularization and Restricted Parameter Identification Problems in Variational Inequalities, Free Boundary Problems-Theory and Applications, (A. Bossavit et al. eds.), Vol. IV. 508-513, Boston, Pitman, 1985.

[7] K. -H. Hoffmann, J. Sprekels, On the Identification of Heat Conductivity and Latent Heat in a One-phase Stefan Problem, *Control and Cybernetics*, **14** (1985), No. 1-3, pp. 37-51.

[8] R. Kluze, Nichtlineare Variationsungleichungen und Extremalaufgaben, Berlin, VEB Deutscher Verlag der Wissenschaften, 1979.