J. Partial Differential Equations
Vol. 1 № 1 (1988), 43—52

ON A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS OF MIXED TYPE *

M. Y. Chi
Dept. of Math. Wuhan Univ.
Received November 24, 1986

Partial differential equations of mixed type has been a very active topic since F. Tricomi' s pioneering work on the equation

$$yu_{xx}+u_{yy}=0 \tag{1}$$

which bears his name. This is mainly due to the significant role it plays in the theory of transonic flow. It also appears in various fields, for instance in the theory of plasticity and the theory of deformation of surfaces, just to name a few of them. There is another type of partial differential equation of mixed type. M. Cibrario [1] considered the general second order equation

$$\begin{aligned} &A(x,\ y)u_{xx}+2B(x,\ y)u_{xy}+C(x,\ y)u_{yy}\\ &+D(x,\ y)u_x+E(x,\ y)u_y+F(x,\ y)u=0 \end{aligned} \tag{2}$$

where the coefficients are real analytic functions of real variables (x, y) and the discriminant $\Delta(x, y)=B^2-AC$ may change sign across the type-changing curve $\Gamma: \Delta=0$ and is of mixed type there. She proved that equation (1) can always be reduced to either of the following forms

$$y^{2m+1}u_{xx}+u_{yy}+a(x,\ y)u_x+b(x,\ y)u_y+c(x,\ y)u=0 \tag{3$_1$}$$

$$u_{xx}+y^{2m+1}u_{yy}+a(x,\ y)u_x+b(x,\ y)u_y+c(x,\ y)u=0 \tag{3$_2$}$$

Thus, Tricomi's equation is only the simplest model of (3_1) where Γ is not characteristic. Equations of the form (3_2) is also of considerable interest. The earliest example is

$$u_{xx}+yu_{yy}+\alpha u_y=0 \qquad \alpha=\text{const.} \tag{4}$$

which has been studied by I. P. Carol' [2]. Let Ω be a domain in (x, y) plane such that $\Omega\cap\{y\neq 0\}\neq\varnothing$. $\partial\Omega=\Gamma_1\cup\Gamma_2\cup\Gamma_3$, Γ_1 is an arc lying in $y\geq 0$ with end point A and B on $y=0$, Γ_2 and Γ_3 are characteristics of (4) in $y\leq 0$ through A and B respectively. When $\alpha<0$, Carol' proved that the Dirichlet problem (problem M) for (4) is well-posed, while for $\alpha>0$, boundary value can be assigned on Γ_1 (problem E).

Equations of the type (3_2) also appear in gas dynamics (for instance, conic flow) [3] where we are required to solve the Busemann equation

$$(1-x^2)u_{xx}-2xyu_{xy}+(1-y^2)u_{yy}+2\alpha xu_x+2\alpha yu_y-\alpha(\alpha+1)u=0 \tag{5}$$

The unit circle $x^2+y^2=1$ is the type-changing curve and also a characteristic curve for the equation (5), which in polar coordinates can be written as

$$(1-r^2)\frac{\partial^2u}{\partial r^2}+\frac{1}{r^2}\frac{\partial u}{\partial\theta^2}+\left(\frac{1}{r}+2\alpha r\right)\frac{\partial u}{\partial r}-\alpha(\alpha+1)u=0 \tag{5$_1$}$$

Near the type-changing curve $r=1$, (5_1) becomes asymptotically

$$\rho\frac{\partial^2u}{\partial\rho^2}+\frac{\partial^2u}{\partial\theta^2}-\left(\frac{1}{2}+\alpha\right)\frac{\partial u}{\partial\rho}-\alpha(\alpha+1)u=0, \qquad \rho=1-r \tag{5$_2$}$$

Gu Chao-hao proved in [4] that the Dirichlet problem is well-posed when $\alpha>\frac{1}{2}$, and when $\alpha<\frac{1}{2}$ boundary value can be assigned only on that part of the boundary inside the elliptic domain $x^2+y^2<1$.

---

* Project supported by the National Science Foundation of China

Equation of the type (3_2) also appears in magneto-hydrodynamics, see Seebass [5].

In this paper, we consider the second order linear equation

$$u_{xx}+yu_{yy}+a(x,\ y)u_x+b(x,\ y)u_y+c(x,\ y)u=0, \qquad c\leq 0 \tag{6}$$

(i. e. (3_2) with $m=0$) in a region $\Omega=\Omega_+\bigcup\Omega_-$, $\Omega_\pm=\Omega\bigcap\{y\geq,\leq 0\}$. We assume a, b, c are analytic functions of the real variables x and y. We also assume

$$b(x,\ 0)=b_0=\text{const.} \tag{7}$$

Our main idea is that, any solution of (6) is "glued up" from 2 solutions each in Ω_+ and Ω_-. The smoothness of the solution is determined by the constant b_0 in (7). Different requirements on smoothness lead to corresponding boundary value problem. More precisely, our main results are: first, we prove that all solutions of (6) admit an asymptotic expansion

$$u(x,\ y)\sim\sum_{n=0}^{\infty}a_n(x)y^n+y^\lambda\sum_{n=0}^{\infty}b_n(x)y^n \qquad \lambda=1-b_0 \tag{8}$$

near $y=0$ (but $y\neq 0$). Next, we prove that all solutions in Ω_+ or Ω_- can be extended "analytically" into Ω_- or Ω_+ respectively. From these results we can give well-posed boundary value problems.

Equation (6) is a Fuchsian type partial differential equation. From (8) it is seen that $y=0$ is a singularity of the solution, while for equation (3_1) there is no solution with remarkable singularity on $y=0$. This would help to explain the difference between (3_1) and (3_2).

From the condition (7), we have

$$b(x,\ y)=b_0+yb_1(x,\ y)$$

Introducing a new unknown function $v(x,\ y)$

$$v(x,\ y)=u(x,\ y)\exp\left[\frac{1}{2}\int a(x,\ 0)\,dx+\frac{1}{2}yb_1(x,\ 0)\right]$$

equation (6) becomes an equation in $v(x,\ y)$:

$$\begin{aligned}v_{xx}+yv_{yy}+[a(x,\ y)-a(x,\ 0)-yb_1'(x,\ 0)]v_x\\+[b(x,\ y)-yb_1(x,\ 0)]v_y+\tilde{c}v=0\end{aligned}$$

where $\tilde{c}(x,\ y)$ is an analytic function in $(x,\ y)$ near $y=0$. Since

$$a(x,\ y)-a(x,\ 0)-yb_1'(x,\ 0)=ya_1(x,\ y)$$

$$b(x,\ y)-yb_1(x,\ 0)=b_0+y[b_1(x,\ y)-b_1(x,\ 0)]=b_0+y^2b_2(x,\ y)$$

hence, without losing generality, we may assume that the coefficients of (6) $a(x,\ y)$, $b(x,\ y)$ are of the form

$$\begin{aligned}a(x,\ y)&=ya_1(x,\ y)\\ b(x,\ y)&=b_0+y^2b_1(x,\ y)\end{aligned} \tag{9}$$

Using characteristic variables

$$\xi=x+2(-y)^{\frac{1}{2}}$$

$$\eta=x-2(-y)^{\frac{1}{2}}$$

(6) can be written as

$$\begin{aligned}u_{\xi\eta}-\left[\frac{\beta'}{\xi-\eta}+(\xi-\eta)^2A(\xi,\ \eta)\right]u_\xi+\\+\left[\frac{\beta}{\xi-\eta}-(\xi-\eta)^2B(\xi,\ \eta)\right]u_\eta+C(\xi,\ \eta)u=0\end{aligned} \tag{10}$$

with $\beta=\beta'=-\frac{1}{2}+b_0=\text{const}$, $A(\xi,\ \eta)$, $B(\xi,\ \eta)$ and $C(\xi,\ \eta)$ analytic in ξ, η near $\xi=\eta$.

In hyperbolic region Ω_- where $y\leq 0$, ξ, η are real variables, while in elliptic region Ω_+ where $y\geq 0$, ξ, η are complex conjugate. But in the following we would treat ξ and η as independent complex variables.

Now we give the following

Definition. *If $(\xi-\eta)^{-\rho}f(\xi,\eta)$ is an analytic function of ξ and η near $\xi=\eta$, we say*

$f(\xi, \eta)$ belongs to the class $A(\rho)$.

If $f = f_1 + f_2$, where $f_1 \in A(0)$, $f_2 \in A(1-\beta-\beta')$, we say $f \in A$.

We are going to prove the following main theorem.

Theorem 1. *All solutions of the equation (6) which are analytic when $|\xi-\eta|>0$ is sufficiently small belong to the class A. Here we assume*

$$2\beta \notin \mathbf{Z}$$

As mentioned above, we assume that ξ and η are complex.

We need some lemmas for the proof of Theorem 1.

Lemma 1. *If $f_k(\xi, \eta) \in A(\rho)$, $\rho \geq 0$, converge uniformly near $\xi = \eta$ to a limit $f(\xi, \eta) = O(1) \cdot (\xi-\eta)^{\rho}$, then $f(\xi, \eta) \in A(\rho)$.*

When $\rho < 0$, the lemma is valid when assuming $(\xi-\eta)^{-\rho} f_k(\xi, \eta)$ converging uniformly.

Proof. We restrict ourselves to the case $\rho \geq 0$. It is easy to see $f(\xi, \eta) = (\xi-\eta)^{\rho} g(\xi, \eta)$, where $g(\xi, \eta)$ is analytic near $\xi-\eta=0$ (possibly with singularity at $\xi=\eta$) and is bounded. From the theorem of removable singularity for functions of several complex variables follows the lemma.

Now consider the Euler-Poisson equation

$$E(\beta, \beta'): u_{\xi\eta} - \frac{\beta'}{\xi-\eta} u_\xi + \frac{\beta}{\xi-\eta} u_\eta = 0 \tag{11}$$

with complex ξ and η. Its Riemann function is known [6] to be

$$R(\xi, \eta; \xi_0, \eta_0) = \frac{(\xi-\eta)^{\beta+\beta'}}{(\xi-\eta_0)^{\beta'} (\xi_0-\eta)^{\beta}} F(\beta, \beta', 1, \sigma) \tag{12}$$

$$\sigma = \frac{(\xi_0-\xi)(\eta-\eta_0)}{(\xi-\eta_0)(\xi_0-\eta)}$$

and all of its solution analytic near $\xi-\eta=0$ can be written as

$$u(\xi, \eta) = \int_\eta^\xi \varphi(\alpha)(\xi-\alpha)^{-\beta}(\alpha-\eta)^{-\beta'} d\alpha + (\xi-\eta)^{1-\beta-\beta'} \int_\eta^\xi \psi(\alpha)(\xi-\alpha)^{\beta'-1}(\alpha-\eta)^{\beta-1} d\alpha \tag{13}$$

where $\varphi(\alpha)$ and $\psi(\alpha)$ are two arbitrary analytic functions. Actually, we can repeat the proof given in [7] almost word by word. When $\mathrm{Re}\beta$ and $\mathrm{Re}\beta'$ don't lie in between 0 and 1, these integral should be understood in distributional sense, i. e. as Riemann-Liouville integrals. It is easy to see, $u(\xi, \eta)$ in (13) belongs to the class A.

Lemma 2. *If $\beta+\beta' \notin \mathbf{Z}$, let $f(\xi, \eta) \in A$ and write*

$$u(\xi_0, \eta_0) = \int_{\xi_1}^{\xi_0} \int_{\eta_1}^{\eta_0} R(\xi, \eta; \xi_0, \eta_0) f(\xi, \eta) d\xi d\eta \tag{14}$$

then $u(\xi, \eta)$, $(\xi-\eta)\frac{\partial u}{\partial \xi}$, $(\xi-\eta)\frac{\partial u}{\partial \eta}$ all belong to A.

Proof. Function u defined by (14) is the solution of the following problem

$$\begin{aligned} &u_{\xi\eta} - \frac{\beta'}{\xi-\eta} u_\xi + \frac{\beta}{\xi-\eta} u_\eta = f(\xi, \eta) \\ &u(\xi_1, \eta) = 0, \qquad u(\xi, \eta_1) = 0 \end{aligned} \tag{15}$$

We can construct a solution of the inhomogeneous equation in the form

$$V(\xi, \eta) = \sum_{n=0}^{N} v_n^{(1)}(\xi)(\xi-\eta)^n + V^{(1)}(\xi, \eta) + (\xi-\eta)^{1-\beta-\beta'} \left[\sum_{n=0}^{N} v_n^{(2)}(\xi)(\xi-\eta)^n + V^{(2)}(\xi, \eta)\right]$$

where $V^{(i)}(\xi, \eta) = O(1)(\xi-\eta)^{N+1}$. Substituting it into (15) and noticing that

$$f(\xi, \eta) = \sum_{n=0}^{N} f_n^{(1)}(\xi)(\xi-\eta)^n + F^{(1)}(\xi, \eta)$$

$$+ (\xi-\eta)^{1-\beta-\beta'}\left[\sum_{n=0}^{N} f_n^{(2)}(\xi)(\xi-\eta)^n + F^{(2)}(\xi,\eta)\right]$$

$F^{(i)}(\xi,\eta) = O(1)(\xi-\eta)^{N+1}$, it yields, for example

$$-(\beta+\beta')v_1^{(1)}(\xi) - \beta' v_0^{(1)\prime}(\xi) = 0$$

$$2(1+\beta+\beta')v_2^{(1)}(\xi) + (1+\beta')v_1^{(1)\prime}(\xi) = -f_0^{(1)}(\xi)$$

....

Since $\beta+\beta' \notin \mathbf{Z}$, we can solve for $v_n^{(1)}$ and $v_n^{(2)}$. $V^{(i)}(\xi,\eta)$ can be constructed by successive approximation as in [8], the method used there is also valid in complex case.

Then let $u = W + V$, we have

$$W_{\xi\eta} - \frac{\beta'}{\xi-\eta}W_\xi + \frac{\beta}{\xi-\eta}W_\eta = 0$$

$$w(\xi_1,\eta) = -v(\xi_1,\eta)$$

$$w(\xi,\eta_1) = -v(\xi,\eta_1)$$

and W can be solved in the form (13) and it is easy to prove that $W \in A$. Hence $u(\xi,\eta) \in A$. That $(\xi-\eta)^2\frac{\partial u}{\partial \xi} \in A$ and $(\xi-\eta)^2\frac{\partial u}{\partial \eta} \in A$ are evident.

We can now give the

Proof of Theorem 1

1° Any solution of (6) can be thought of as a solution of analytic Goursat problem

$$u(\xi_1,\eta) = \varphi(\eta),\ u(\xi,\eta_1) = \psi(\xi),\ \varphi(\eta_1) = \psi(\xi_1) \tag{16}$$

We first construct a solution $V(\xi,\eta)$ for the Euler-Poisson equation (11) $E(\beta,\beta')$ satisfying (16). Evidently, $V(\xi,\eta) \in A$. Then let $u = W + V$, we have

$$L(W) = (\xi-\eta)^2 A(\xi,\eta)\frac{\partial V}{\partial \xi} + (\xi-\eta)^2 B(\xi,\eta)\frac{\partial V}{\partial \eta} + C(\xi,\eta)V \tag{17}$$

$$W(\xi_1,\eta) = W(\xi,\eta_1) = 0$$

where L denotes the differential operator on the left hand side of (10). The right hand side of (17) still belongs to the class A, hence we may consider a more general problem

$$L(u) = f(\xi,\eta) \in A$$

$$u(\xi_1,\eta) = u(\xi,\eta_1) = 0$$

or

$$u_{\xi\eta} - \frac{\beta'}{\xi-\eta}u_\xi + \frac{\beta}{\xi-\eta}u_\eta$$

$$= f(\xi,\eta) + (\xi-\eta)^2 A(\xi,\eta)u_\xi + (\xi-\eta)^2 B(\xi,\eta)u_\eta + C(\xi,\eta)u$$

which can be reduced to the integro-differential equation

$$u(\xi_0,\eta_0) = f_1(\xi_0,\eta_0) + \int_{\xi_1}^{\xi_0}\int_{\eta_1}^{\eta_0} R(\xi,\eta;\xi_0,\eta_0)[(\xi-\eta)^2 A(\xi,\eta)u_\xi$$

$$+ (\xi-\eta)^2 B(\xi,\eta)u_\eta + C(\xi,\eta)u]d\xi d\eta \tag{18}$$

$$f_1(\xi_0,\eta_0) = \int_{\xi_1}^{\xi_0}\int_{\eta_1}^{\eta_0} R(\xi,\eta;\xi_0,\eta_0)f(\xi,\eta)d\xi d\eta$$

$R(\xi,\eta;\xi_0,\eta_0)$ is the Riemann function (12) of the Euler-Poisson equation.

2° We can solve this equation by successive approximation, i. e, let

$$u_0(\xi_0,\eta_0) = f_1(\xi_0,\eta_0)$$

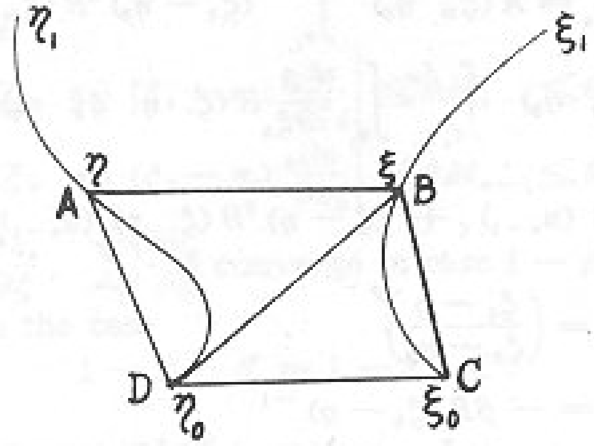
$$u_n(\xi_0,\eta_0) = \int_{\xi_1}^{\xi_0}\int_{\eta_1}^{\eta_0} R(\xi,\eta;\xi_0,\eta_0)[(\xi-\eta)^2 A(\xi,\eta)(u_{n-1})_\xi$$

$$+ (\xi-\eta)^2 B(\xi,\eta)(u_{n-1})_\eta + C(\xi,\eta)u_{n-1}]d\xi d\eta$$

From lemma 2, we have $u_n \in A$. We are going to prove that

$$\sum u_n,\quad \sum\frac{\partial u_n}{\partial \xi}\quad \text{and}\quad \sum\frac{\partial u_n}{\partial \eta}$$

all converge uniformly, and the sum $\sum u_n$ gives the solution of (18).



For the estimates of u_n and its derivatives, we note that ξ and η are complex variables, hence the paths of integration can be altered quite arbitrarily. With fixed and unequal ξ_0 and η_0, we choose the paths of integration such that at least near ξ_0 and η_0 we have

$$\left|\frac{\xi_0-\eta_0}{\xi-\eta_0}\right| = \frac{CD}{BD} = \frac{\operatorname{Sin} CBD}{\operatorname{Sin} BCD} \leq K$$
$$\left|\frac{\xi-\eta_0}{\xi-\eta}\right| = \frac{BD}{AB} = \frac{\operatorname{Sin} BAD}{\operatorname{Sin} ADB} \leq K$$
$$\left|\frac{\xi_0-\eta_0}{\xi-\eta}\right| = \left|\frac{\xi_0-\eta_0}{\xi-\eta_0}\right| \cdot \left|\frac{\xi-\eta_0}{\xi-\eta}\right| \leq K^2 \qquad (19)_1$$
$$\left|\frac{\eta-\eta_0}{\xi-\eta}\right| = \frac{AD}{AB} = \frac{\operatorname{Sin} ABD}{\operatorname{Sin} ADB} \leq K$$

Let $M = \max(K, K^2)$, we have analogously

$$\left|\frac{\xi_0-\eta_0}{\xi_0-\eta}\right| \leq M, \quad \left|\frac{\xi_0-\eta}{\xi-\eta}\right| \leq M, \quad \left|\frac{\xi-\xi_0}{\xi-\eta}\right| \leq M \qquad (19_2)$$

We also assume

$$\left|\frac{\xi-\xi_0}{\xi-\eta_0}\right| \leq 1, \quad \left|\frac{\eta-\eta_0}{\eta-\xi_0}\right| \leq 1$$

3° When estimating u_n, we should distinguish two cases. One is

$$1-\beta-\beta' = 1-2\beta > 0$$

In this case, all functions in A are bounded near $\xi-\eta=0$, and

$$|R(\xi, \eta; \xi_0, \eta_0)| = |\xi-\eta|^{2\beta}|\xi-\eta_0|^{-\beta}|\xi_0-\eta|^{-\beta}|F(\beta, \beta, 1, \sigma)|$$
$$\leq C|\xi-\eta|^{2\beta}|\xi-\eta_0|^{-\beta}|\xi_0-\eta|^{-\beta}$$

Here we note $\beta=\beta'$. If u_{n-1} can be estimated by

$$|(\xi-\eta)^2 A(\xi, \eta)(u_{n-1})_\xi + (\xi-\eta)^2 B(\xi, \eta)(u_{n-1})_\eta + C(\xi, \eta)u_{n-1}| \leq M_{n-1}$$

(which is valid for $n=1$), we have immediately

$$|u_n(\xi_0, \eta_0)| \leq CM_{n-1}\int_{\xi_1}^{\xi_0}\int_{\eta_1}^{\eta_0}\left|\frac{\xi-\eta}{\xi-\eta_0}\right|^\beta \cdot \left|\frac{\xi-\eta}{\xi_0-\eta}\right|^\beta |d\xi d\eta|$$

If $\beta>0$, then from $1-\beta-\beta'=1-2\beta>0$ we have $0<\beta<\frac{1}{2}$. Hence the above integral converges, so we have

$$|u_n(\xi_0, \eta_0)| \leq \theta M_{n-1} \leq \theta^n M_0, \quad 0<\theta<1 \text{ small enough} \qquad (20)$$

when $|\xi_0-\xi_1|$ and $|\eta_0-\eta_1|$ are sufficiently small. If $\beta<0$, from (19) we know that the integrand is bounded:

$$\left|\frac{\xi-\eta}{\xi-\eta_0}\right|^\beta \cdot \left|\frac{\xi-\eta}{\xi_0-\eta}\right|^\beta \leq M^{-2\beta}$$

hence (20) still holds.

For the derivatives of u_n we have

$$(\xi_0-\eta_0)^2A(\xi_0,\eta_0)(u_n)_{\xi_0}=A(\xi_0,\eta_0)\int_{\eta_1}^{\eta_0}(\xi_0-\eta_0)^2R(\xi_0,\eta;\xi_0,\eta_0)[\quad]_{\xi=\xi_0}d\eta+$$

$$+(\xi_0-\eta_0)^2A(\xi_0,\eta_0)\int_{\xi_1}^{\xi_0}\int_{\eta_1}^{\eta_0}\frac{\partial}{\partial\xi_0}R(\xi,\eta;\xi_0,\eta_0)[\quad]d\xi d\eta \tag{21}$$

where [ ] stands for

$$(\xi-\eta)^2A(\xi,\eta)(u_{n-1})_\xi+(\xi-\eta)^2B(\xi,\eta)(u_{n-1})_\eta+C(\xi,\eta)u_{n-1}$$

Note that

$$R(\xi_0,\eta;\xi_0,\eta_0)=\left(\frac{\xi_0-\eta}{\xi_0-\eta_0}\right)^\beta$$

$$\frac{\partial}{\partial\xi_0}R(\xi,\eta;\xi_0,\eta_0)=-\beta R(\xi_0-\eta)^{-1}$$

$$+\frac{(\xi-\eta)^{2\beta}}{(\xi-\eta_0)^\beta(\xi_0-\eta)^\beta}\beta^2F(\beta+1,\beta+1,2,\sigma)\frac{\partial\sigma}{\partial\xi_0}$$

We now estimate the two terms in (21) separately.

First for the simple integral. When $\beta>0$, we have $0<\beta<\frac{1}{2}$, hence

$$|A(\xi_0,\eta_0)(\xi_0-\eta_0)^2R(\xi_0,\eta;\xi_0,\eta_0)|\le\max|A|\,|\xi_0-\eta_0|^{2-\beta}|\xi_0-\eta|^\beta\le\mu$$

when $\beta\le0$, we have

$$|A(\xi_0,\eta_0)(\xi_0-\eta_0)^2R(\xi_0,\eta;\xi_0,\eta_0)|\le\max|A|\,|\xi-\eta_0|^2\left|\frac{\xi_0-\eta_0}{\xi_0-\eta}\right|^{-\beta}\le\mu$$

Here we also made use of (19). Summing up, we know the norm of the simple integral is less than $\frac{\theta}{2}M_{n-1}$, $0<\theta<1$.

Next consider the double integral. When $\beta>0$, $2-(\beta+1)-(\beta+1)<0$, hence

$$F(\beta+1,\beta+1,2,\sigma)=O(1)(1-\sigma)^{-2\beta}=O(1)\left[\frac{(\xi_0-\eta)(\xi-\eta_0)}{(\xi_0-\eta_0)(\xi-\eta)}\right]^{2\beta}$$

$$\frac{\partial\sigma}{\partial\xi_0}=\frac{\eta-\eta_0}{\xi-\eta_0}\cdot\frac{\xi_0-\eta-\xi_0+\xi}{(\xi_0-\eta)^2}=\frac{(\eta-\eta_0)(\xi-\eta)}{(\xi_0-\eta_0)(\xi_0-\eta)^2}$$

hence

$$\left|A(\xi_0,\eta_0)(\xi_0-\eta_0)^2\frac{\partial R}{\partial\xi_0}\right|\le\max|A|\left\{\beta\cdot\frac{|\xi_0-\eta_0|}{|\xi_0-\eta|}C\cdot\frac{|\xi-\eta|^2}{|\xi-\eta_0|^\beta|\xi_0-\eta|^\beta}\right.$$

$$\left.+\beta^2C_1|\xi_0-\eta_0|^{2-2\beta}|\xi_0-\eta|^\beta|\xi-\eta_0|^\beta\left|\frac{(\eta-\eta_0)(\xi-\eta)}{(\xi-\eta_0)(\xi_0-\eta)^2}\right|\right\}$$

From (19), $\left|\frac{\eta-\eta_0}{\xi-\eta_0}\right|\le M$, $\left|\frac{\xi_0-\eta_0}{\xi_0-\eta}\right|\le M$, hence the last term in the above expression will not be bigger than

$$C_2|\xi_0-\eta_0|^{1-2\beta}|\xi_0-\eta|^{\beta-1}|\xi-\eta|\cdot|\xi-\eta_0|^\beta$$

From all these, we see the norm of the double integral is less than $\frac{\theta}{2}M_{n-1}$. When $\beta<0$, we have

$$2-(\beta+1)-(\beta+1)>0,$$

hence

$$F(\beta+1,\beta+1,2,\sigma)=O(1)$$

and

$$\left|A(\xi_0,\eta_0)(\xi_0-\eta_0)^2\frac{\partial u}{\partial\xi_0}\right|\le\max|A|\left\{|\beta|\left|\frac{\xi_0-\eta_0}{\xi_0-\eta}\right|^2C\frac{|\xi-\eta|^{2\beta}}{|\xi-\eta_0|^\beta|\xi_0-\eta|^\beta}\right.$$

$$\left.+C_1|\xi_0-\eta_0|^2\left|\frac{\eta-\eta_0}{\xi-\eta_0}\right|\frac{|\xi-\eta|}{|\xi_0-\eta|^2}\cdot\frac{|\xi-\eta|^{2\beta}}{|\xi-\eta_0|^\beta|\xi_0-\eta|^\beta}\right\}$$

Hence we still have that the norm of the double integral is not bigger that $\frac{\theta}{2}M_{n-1}$.

Summing up, we have

$$\left|A(\xi_0,\ \eta_0)\ (\xi_0-\eta_0)\ ^{2}\frac{\partial u}{\partial \xi_0}\right| \leq \theta M_{n-1} \leq \theta^n M_0$$

$$\left|B(\xi_0,\ \eta_0)\ (\xi_0-\eta_0)\ ^{2}\frac{\partial u}{\partial \eta_0}\right| \leq \theta M_{n-1} \leq \theta^n M_0$$

and we know $\sum u_n$, $\sum\frac{\partial u_n}{\partial \xi}$, $\sum\frac{\partial u_n}{\partial \eta}$ converge in case $1-\beta-\beta'=1-2\beta>0$.

4° We are left with the case

$$1-\beta-\beta'=1-2\beta<0$$

Functions in A are $O(1)\ (\xi-\eta)^{1-\beta-\beta'}$ in this case. Let

$$u_n(\xi,\ \eta)=(\xi-\eta)^{1-2\beta}v_n(\xi,\ \eta)$$

we know that the v_n is bounded and

$$\left|A(\xi,\ \eta)\ (\xi-\eta)\ ^{2}\frac{\partial u_{n-1}}{\partial \xi}+B(\xi,\ \eta)\ (\xi-\eta)\ ^{2}\frac{\partial u_{n-1}}{\partial \eta}+C(\xi,\ \eta)\,u_{n-1}\right|$$

$$=|\xi-\eta|^{1-2\beta}\cdot\left|\bar{A}(\xi-\eta)\ ^{2}\frac{\partial v_{n-1}}{\partial \xi}+\bar{B}(\xi,\ \eta)\ (\xi-\eta)\ ^{2}\frac{\partial v_{n-1}}{\partial \eta}+\bar{C}(\xi,\ \eta)\,v_{n-1}\right|$$

If we can prove that

$$\left|\bar{A}(\xi-\eta)\ ^{2}\frac{\partial v_{n-1}}{\partial \xi}+\bar{B}(\xi-\eta)\ ^{2}\frac{\partial v_{n-1}}{\partial \eta}+\bar{C}(\xi,\ \eta)\,v_{n-1}\right|\leq M_{n-1}$$

which is valid for $n=1$, then v_n can be estimated as follows

$$(\xi_0-\eta_0)^{1-2\beta}v_n(\xi_0,\ \eta_0)=\int_{\xi_1}^{\xi_0}\int_{\eta_1}^{\eta_0}R(\xi,\ \eta;\ \xi_0,\ \eta_0)\ (\xi-\eta)^{1-\beta-\beta'}[\quad]d\xi d\eta$$

where $[\quad]$ stands for

$$\bar{A}(\xi-\eta)\ ^{2}\frac{\partial v_{n-1}}{\partial \xi}+\bar{B}(\xi-\eta)\ ^{2}\frac{\partial v_{n-1}}{\partial \eta}+\bar{C}v_{n-1}$$

and

$$\begin{aligned} R(\xi,\ \eta;\ \xi_0,\ \eta_0) &= \frac{(\xi-\eta)^{2\beta}}{(\xi_0-\eta)^{\beta}(\xi-\eta_0)^{\beta}}F(\beta,\ \beta,\ 1,\ \sigma) \\ &= (1-\sigma)^{1-2\beta}(\xi-\eta)^{2\beta}(\xi_0-\eta)^{-\beta}(\xi-\eta_0)^{-\beta} \\ &\quad\cdot F(1-\beta,\ 1-\beta,\ 1,\ \sigma) \\ &= O(1)\ (\xi_0-\eta_0)^{1-2\beta}(\xi-\eta)\ (\xi_0-\eta)^{\beta-1}(\xi-\eta_0)^{\beta-1} \end{aligned}$$

hence

$$v_n(\xi_0,\ \eta_0)=\int_{\xi_1}^{\xi_0}\int_{\eta_1}^{\eta_0}O(1)\ (\xi-\eta)^{2-2\beta}(\xi-\eta_0)^{\beta-1}(\xi_0-\eta)^{\beta-1}[\quad]d\xi d\eta$$

Denote $1-\beta=\beta_1$, we have $1-2\beta_1=2\beta-1>0$ and

$$v_n(\xi_0,\ \eta_0)=\int_{\xi_1}^{\xi_0}\int_{\eta_1}^{\eta_0}O(1)\ (\xi-\eta)^{2\beta_1}(\xi_0-\eta)^{-\beta_1}(\xi-\eta_0)^{-\beta_1}[\quad]d\xi d\eta$$

Hence we can repeat what has been done in 3°, and Theorem 1 is proved.

Remark.

1. If we return to the original variables, what we have proved is that, all solutions of equation (6) which are analytic when $y\neq 0$, can be expanded in the form

$$u(x, y) = \sum_{n=0}^{\infty} a_n(x) y^{n/2} + y^{\lambda} \sum_{n=0}^{\infty} b_n(x) y^{n/2}$$

2. Equation (6) belongs to the class of totally characteristic equations [9], [10]. In case when (6) is elliptic, it is proved in [9] that all its distribution solutions can be expanded as

$$u(x, y) = \sum_{n=0}^{\infty} a_n(x) y^n + y^{\lambda} \sum_{n=0}^{\infty} b_n(x) y^n$$

and can be extended across $y = 0$. But in our case of mixed type equation, we cannot make use of this result directly. That is why we made use of the Riemann function which is closely related to fundamental solution.

In the following, we shall follow the method of solving boundary value problem, i. e. first solve the problem in either the elliptic or hyperbolic region, then solve it in the other region, hence we need an extension theorem.

Theorem 2. *All solutions of class A of (6) in the elliptic (hyperbolic) region can be extended "analytically" into the other region.*

Here, analytic extension of $u = u_1 + y^{\lambda} u_2$, $u_1, u_2 \in A(0)$, *means analytic extension of* u_1 *and* u_2.

Proof. Let $u(x, y) = u(\xi, \eta)$ be a solution of class A of (6) in $y > 0$. We have proved in Theorem 1 that $u = u_1 + (\xi - \eta)^{1-2\beta} u_2$ where $u_1(\xi, \eta)$ and $u_2(\xi, \eta)$ are analytic near $\xi = \eta$.

First take $u_1(\xi, \eta)$. Expand it in Taylor series

$$u_1(\xi, \eta) = \sum_{n=0}^{\infty} u_n^{(1)}(\xi)(\xi - \eta)^n$$

It is evident that $u^{(1)}(\xi)$ is analytic in ξ. Returning to x, y coordinates yields

$$u_1(x, y) = \sum_{n=0}^{\infty} u_n^{(1)}(x + 2(-y)^{\frac{1}{2}}) 4^n (-y)^{\frac{n}{2}}$$
$$= \sum_{n=0}^{\infty} \tilde{u}_n(x) y^{\frac{n}{2}} \tag{22}$$

where $\tilde{u}_0(x)$ is analytic in x.

Substituting (22) into (6) gives recurrence formulas for $u_n(x)$ which can be divided into two groups, one for even n' s, the other for odd n' s. Since the coefficients $a(x, y)$, $b(x, y)$ and $c(x, y)$ of (6) are analytic in x, y and in their Taylor expansion in y there is no half power of y, we see all $u_n(x) = 0$ for n odd. Hence (22) can be put into the form

$$u_1(x, y) = \sum_{n=0}^{\infty} u_n^{*}(x) y^n \tag{23}$$

Where $u_0^{*}(x) = u_0^{(1)}(x)$ is analytic in x.

In [11], we proved that we can find an analytic solution $U(x, y)$ of the Fuchsian type equation (6) in the form

$$U(x, y) = \sum_{n=0}^{\infty} U_n(x) y^n \tag{24}$$

with $U_0(x)$ arbitrary. Using $u_0^{*}(x)$ as $U_0(x)$ we get a solution of (6). Since the recurrence formulas for $U_n(x)$ are the same as those for $u_n^{*}(x)$, we know $U(x, y) = u_1(x, y)$ for $y > 0$. But as a power series in y, (24) converges also in a region in $y < 0$, we know $U(x, y)$ is an analytic extension of $u_1(x, y)$.

Similarly, we can extend $u_2(x, y)$ across $y = 0$ into $y < 0$. Theorem 2 is proved.

We can now give the formulation of boundary value problems for (6) under various situations. Let the region Ω be bounded by a Jordan curve Γ_1 in $y \geq 0$ with extremities $A(0, 0)$ and $B(1, 0)$ and two characteristic curves Γ_2 and Γ_3 in $y \leq 0$ through A and B respectively. Γ_2 and Γ_3 intersect at a point C.

$$AC: x - 2(-y)^{\frac{1}{2}} = 0$$

$$BC: x + 2(-y)^{\frac{1}{2}} = 1$$

Formulation of the boundary value problems depends on the sign of

$$1 - \beta - \beta' = 1 - 2\beta = 2(1 - b_0)$$

1° $b_0 < 1$. In this case $1 - \beta - \beta' > 0$. We can pose the Dirichlet problem. i. e., to find a solution $u(x, y)$ of (6) in Ω continuous in $\bar{\Omega}$ and satisfies the following conditions:

$$u|_{AC} = \varphi_1(x)$$

$$u|_{BC} = \varphi_2(x)$$

$$u|_{\Gamma_1} = \varphi_0(s)$$

s: length of arc Γ_1

Here we assume φ_1 and φ_2 analytic, φ_0 continuous and

$$\varphi_1(C) = \varphi_2(C), \quad \varphi_1(A) = \varphi_0(A), \quad \varphi_2(B) = \varphi_0(B)$$

Theorem 3. *When $b_0 < 1$ and $-1 + 2b_0 \notin \mathbf{Z}$, the Dirichlet problem for (6) is well-posed, the solution is analytic in Ω when $y \neq 0$, and across $y = 0$, it is C^k continuous*

$$k = [1 - b_0] \geq 0$$

Proof. We first solve the Goursat problem for (6):

$$u|_{AC} = \varphi_1(x), \quad u|_{BC} = \varphi_2(x)$$

This problem is evidently well-posed. By theorems 1 and 2, its solution u can be written as

$$u(x, y) = \sum_{n=0}^{\infty} \tau_n(x) y^n + y^{1-b_0} \sum_{n=0}^{\infty} \nu_n(x) y^n, \quad y \leq 0 \tag{25}$$

$\tau(x)$ is analytic in x.

Next, we solve a Dirichlet problem in $y \geq 0$

$$u|_{\Gamma} = \varphi_0(s), \quad u|_{AB} = \tau_0(x)$$

for degenerate elliptic equation (6). Keldysh's fundamental result yields (e. g. see [12]) that this problem is well-posed, and theorems 1 and 2 show us that

$$u(x, y) = \sum_{n=0}^{\infty} \tilde{\tau}_n(x) y^n + y^{1-b_0} \sum_{n=0}^{\infty} \tilde{\nu}_n(x) y^n, \quad y \geq 0 \tag{26}$$

and $\tilde{\tau}_0(x) = \tau_0(x)$. Since the recurrence formulas for $\tilde{\tau}_n(x)$, $n \geq 1$ and $\tau_n(x)$, $n \geq 1$ are the same, we have $\tilde{\tau}_n(x) = \tau_n(x)$, hence

$$u(x, 0_+) - u(x, 0_-) = y^{1-b_0}([\tilde{\nu}_0(x) - \nu_0(x)] + O(1))$$

The theorem is proved.

The proof shows that the solution is actually "glued up" from two solutions, hence it has a weak discontinuity along $y = 0$ and the smoothness of the solution depends on the magnitude of b_0. Using this principle, we can give other boundary value probems. Of course, we may demand that $\tilde{\nu}_0(x) = \nu_0(x)$ also and hence $\tilde{\nu}_n(x) = \nu_n(x)$ and call the solution, if it exists, "analytic solution". It is very probable that the Tricomi problem for (6) is well-posed in the class of analytic solutions when $b_0 < 1$.

2° Next, consider the case $b_0 > 1$. In this case $1 - \beta - \beta' < 0$ and the solutions (25) and (26) are in general unbounded. Physical consideration shows that we should look for bounded solution.

Theorem 4. *When $b_0 > 1$ and $-1 + 2b_0 \in \mathbf{Z}$, the following problem for (6) is well-posed and possesses a unique bounded analytic solution:*

$$u|_{\Gamma_1} = \varphi_0(s)$$

That is to say, in this case, no conditions should be imposed in hyperbolic regions.

Proof. By Keldysh's results the following 'problem E'

$$u_{\Gamma_1} = \varphi_0(s), \qquad u \text{ bounded}$$

is well-posed for the degenerate elliptic equations (6).

By theorems 1 and 2, the solution can be written as (26). Since u is bounded, $\gamma_0(x) = 0$ and hence $\gamma_n(x) = 0$ and the unique solution is

$$u(x, y) = \sum_{n=0}^{\infty} \tau_n(x) y^n, \qquad y \geq 0$$

Using $\tau_0(x)$ and $\gamma_0(x) = 0$ as initial data, we can solve the Cauchy problem for the Fuchsian type equation (6) in $y \leq 0$. This solution exists in Ω_-. Theorem 4 is proved.

The above theorems can explain the results of Carol' and Gu chao-hao.

References

〔1〕 M. Cibrabrio, Sulle reduzione a forma canonica delle equazioni lineari alle derivate parziali di secondo ordine di tipo misto, *Rend. Lombardo, Vol.* 65, (1932), 889-906.

〔2〕 I. P. Carol', On the theory of boundary value problems for the equations of mixed elliptic-hyperbolic equations, *Matem Sbornik,* Vol. 38, (1955), 261-282, (Russian).

〔3〕 A. Ya. Sagomonian, Space problems of non-steady motion of compressible fluids. Mock. Univ. Press, (1962), (Russian)

〔4〕 Gu Chao-hao, *Scientia Sinica,* (1965).

〔5〕 Seebass, *Quarterly of Apple. Math.*, Vol. 19, (1961), 231.

〔6〕 G. Darboux, Theorie des surfaces, t. II. Gauthier-Villars (1914), Paris.

〔7〕 M. Y. Chi, On the Cauchy problem for a class of hyperbolic equations with data on the parabolic-degenerating line, *Acta Math. Sinica,* Vol. 8 (1958), 521-530 (Chinese).

〔8〕 M. Y. Chi, On the Cauchy problem for second order hyperbolic equation in two variables with initial data given on the parabolic degenerating line, *Scientia Sinica,* Vol. 12, (1963) 1413—1424.

〔9〕 L. Hormander, *Analysis of linear partial differential operators,* Vol. III, Springer-Verlag, (1985), Berlin.

〔10〕 R. Melrose and G. Mendoza, Elliptic operators of totally characteristic type. Preprint, (1986).

〔11〕 M. Y. Chi, on Fuchsian type partial differential equations, *J. of Wuhan Univ.* (Nat. Sci. Edition), 1978, 1-6 (Chinese).

〔12〕 A. V. Bitsadze, Equation of mixed type Itogi Nauki, Acad. Nauk USSR, (1959) (Russian).