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ON PROPERTIES OF SOME OPERATORS IN DOUGLIS ALGEBRA AND THEIR APPLICATION TO PDE

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1. Introduction

Let e and i be two elements generation Douglis algebra[1], which are subject to the following multiplication rules:

$$i^2=-1,\ ie=ei,\ e^{r+1}=0,\ e^0=1$$

where r is a positive integer.

Definition 1. *We call a hypercomplex value if* $a=\sum_{k=0}^{r}a_ke^k$, *where* a_k $(k=0,\cdots,r)$ *are complex numbers,* a_0 *is called the complex part of* a. *Set* $\bar{a}=\sum_{k=0}^{r}\bar{a}_ke^k$, $|a|=\sum_{k=0}^{r}|a_k|$.

It is easy to know that $|ab|\le|a|\ |b|$, $a\bar{a}$ *is a real hypercomplex value and* $a\bar{a}\ne|a|^2$.

Let $D=\partial_{\bar{z}}+q(z)\partial_z$ be a differential operator, here $q(z)$ is known nilpotent function.

Definition 2. *A hypercomplex function* $w\in C^1(G)$ *is called hyperanalytic if it is a solution of* $Dw=0$.

A hypercomplex function $w\in C^1(G)$ *is called generalized hyperanalytic function if it is a solution of* $Dw+Aw+B\bar{w}=0$.

A. Douglis[1], R. P. Gilbert[2],[3], G. Hile[4], H. Begehr[5],[6] and Hou Zongyi[7],[8] have discussed properties of hyperanalytic and generalized hyperanalytic function and their boundary value problem.

Definition 3. *A hypercomplex function* $t(z)$ *is called a generating solution of the operator* D *if*

1) $t(z)$ *has the form* $t(z)=z+\sum_{k=1}^{r}t_k(z)e^k\triangleq z+T(z)$,

2) $T\in B^1(C)$ *and*

3) $Dt(z)=0$ *in* C.

By

$$\frac{1}{t(\zeta)-t(z)}=\sum_{k=0}^{r}(-1)^k\frac{\Delta(\zeta,z)^k}{(\zeta-z)^{k+1}} \tag{1・1}$$

where $\Delta(\zeta,z)=T(\zeta)-T(z)$, *we can get*

$$\left|\frac{1}{t(\zeta)-t(z)}\right|\le\frac{M}{|\zeta-z|},\ \zeta\ne z \tag{1・2}$$

where M *is a constant.*

In this paper we deal with some operators in a Douglis algebra and their application to PDE.

R. P. Gilbert[3] introduced Pompieu operator $J_Gf=-\frac{1}{\pi}\iint_G\frac{t_\zeta f(\zeta)d\sigma_\zeta}{t(\zeta)-t(z)}$ and discussed differential property of J_G, he obtained

$$DJ_G f = f \tag{1·3}$$

and then he investigated a series of properties of J_G, but he could not study operator Π, because the definition of operator J_G is not reasonable.

Now we introduce the differential operators

$$\partial = \alpha(z)\partial_{\bar z} + \beta(z)\partial_z,\ \bar\partial = \overline{\beta(z)}\partial_{\bar z} + \overline{\alpha(z)}\partial z \tag{1·4}$$

where

$$\alpha(z) = -\frac{\overline{t_{\bar z}}}{t_z\overline{t_z} - t_{\bar z}\overline{t_{\bar z}}},\ \beta(z) = \frac{\overline{t_z}}{t_z\overline{t_z} - t_{\bar z}\overline{t_{\bar z}}} \tag{1·5}$$

obviously we have

$$\partial t(z) = 1, \qquad \partial\overline{t(z)} = 0 \tag{1·6}$$

and we also introduce the integral operators.

$$\begin{cases} Tf = -\dfrac{1}{\pi}\displaystyle\iint_G \dfrac{t_\zeta \overline{Dt(\zeta)} f(\zeta) d\sigma_\zeta}{t(\zeta) - t(z)}, & \Pi^* f = -\dfrac{1}{\pi}\displaystyle\iint_G \dfrac{t_\zeta \overline{Dt(\zeta)} f(\zeta) d\sigma_\zeta}{(t(\zeta) - t(z))^2} \\ \Pi f = (\Pi^* - \sigma) f \end{cases} \tag{1·7}$$

where $\sigma = \frac{\overline{t_{\bar z}}}{t_z}$. Operator T is different from operator J, since the integrand has weight $\overline{Dt(\zeta)}$ and operator Π is new.

2. Differential Properties of Operator T

In this section we discuss differential properties of operator T in $C_\alpha^m(\bar G)$ and $L_p(\bar G)$.

Theorem 2.1. *Let $G \in C_\alpha^{m+1}$, $f(z) \in C_\alpha^m(\bar G)$, $q \in B^{0,\alpha}(C)$, $0<\alpha<1, m\geq 0$, then*

1) $T_G f \in C_\alpha^{m+1}(\bar G)$, T_G is a totally continuous operator in $C_\alpha^m(\bar G)$,

2) $\bar\partial T_G f = f, \partial T_G f = \Pi f$,

the integral of operator Π is in the Cauchy principle value sense and $\Pi f \in C_\alpha^m(\bar G)$.

Lemma 2.1. *Let G be a bounded domain and ∂G a piecewise smoothly closed curve, $w \in C^1(G)$, it turns out*

$$\iint_G t_\zeta \overline{Dt(\zeta)} \partial w d\sigma_\zeta = -\frac{1}{2i}\int_{\partial G} w d\overline{t(\zeta)} \tag{2·1}$$

$$\frac{1}{\pi}\iint_G \frac{t_\zeta \overline{Dt(\zeta)} d\sigma_\zeta}{(t(\zeta) - t(z))^2} = \frac{1}{2\pi i}\int_{\partial G} \frac{\overline{t(\zeta)} dt(\zeta)}{(t(\zeta) - t(z))^2} - \sigma(z) \tag{2·2}$$

where the integral in the left of (2,2) is in the Cauchy principle value sense.

Proof. By applying Green formula, Pompieu formula[3] and properties of $t(z)$, this lemma holds obviously.

Now we return to the proof of theorem 2.1.

Proof. we assume that $m=0$ at first.

$$\Pi^* f = -\frac{1}{\pi}\iint_G \frac{t_\zeta \overline{Dt(\zeta)}(f(\zeta) - f(z))}{(t(\zeta) - t(z))^2} d\sigma_\zeta - \frac{f(z)}{\pi}\iint_G \frac{t_\zeta \overline{Dt(\zeta)} d\sigma_\zeta}{(t(\zeta) - t(z))^2} \tag{2·3}$$

when $f \in C_\alpha(\bar G)$, the first integral is a weak singular integral. By use of lemma 2.1 the second integral is in the Cauchy principle value sense.

We set

$$g(z) = \Pi f,\ \Delta_1 = t(\zeta) - t(z_1),\ \Delta_2 = t(\zeta) - t(z),\ \Delta_0 = t(z_1) - t(z)$$

thus

$$g(z_1) - g(z) = -\frac{1}{\pi}\iint_G t_\zeta \overline{Dt(\zeta)} f(\zeta)\left[\frac{1}{\Delta_1^2} - \frac{1}{\Delta_2^2}\right] d\sigma_\zeta - f(z_1)\sigma(z_1) + f(z)\sigma(z)$$

$$= -\frac{\Delta_0}{\pi}\iint_G t_\zeta \overline{Dt(\zeta)}(f(\zeta) - f(z_1))\frac{d\sigma_\zeta}{\Delta_1^2\Delta_2} - \frac{\Delta_0}{\pi}\iint_G t_\zeta \overline{Dt(\zeta)}(f(\zeta) - f(z))\frac{d\sigma_\zeta}{\Delta_1\Delta_2^2}$$

$$-\left[f(z_1)\sigma(z_1)+\frac{\Delta_0}{\pi}f(z_1)\iint_G\frac{t_\zeta\overline{Dt(\zeta)}d\sigma_\zeta}{\Delta_1^2\Delta_2}\right]-\left[\frac{\Delta_0 f(z)}{\pi}\iint_G\frac{t_\zeta\overline{Dt(\zeta)}d\sigma_\zeta}{\Delta_1\Delta_2^2}-f(z)\sigma(z)\right]$$

$$\triangleq J_1+J_2+J_3+J_4 \tag{2·4}$$

By use of Pompieu formula, we obtain

$$\overline{t(z)}=\frac{1}{2\pi i}\int_{\partial G}\frac{t_\zeta dt(\zeta)}{t(\zeta)-t(z)}-\frac{1}{\pi}\iint_G\frac{t_\zeta\overline{Dt(\zeta)}d\sigma_\zeta}{t(\zeta)-t(z)} \tag{2·5}$$

thus

$$J_3=-\partial\Phi(z_1)+\left(-\frac{\overline{\Delta}_0}{\Delta_0}+\frac{\Phi(z_1)-\Phi(z)}{\Delta_0}\right) \tag{2·6}$$

So we get

$$g(z_1)-g(z)=J_1+J_2+(f(z_1)-f(z))\left[-\frac{\overline{\Delta}_0}{\Delta_0}+\frac{\Phi(z_1)-\Phi(z)}{\Delta_0}-\partial\Phi(z)\right]$$
$$-f(z_1)[\partial\Phi(z_1)-\partial\Phi(z)] \tag{2·7}$$

where $\Phi(z)=\frac{1}{2\pi i}\int_{\partial G}\frac{t_\zeta dt(\zeta)}{t(\zeta)-t(z)}$. Since $G\in C_\alpha^{m+1}, 0<\alpha<1$, similar to the discussion on properties of analytic function, we have $\Phi(z)\in C_\alpha^{m+1}(\overline{G}), \partial\Phi(z)\in C_\alpha^m(\overline{G})$.

On the other hand for $f\in C_\alpha(\overline{G})$, we have

$$|f(z_1)-f(z)|\leq H(G)|z_1-z|^\alpha \tag{2·8}$$

$$|J_1|\leq M_0H(G)|z_1-z|^{\alpha-1},\ |J_2|\leq M_0H(G)|z_1-z|^{\alpha-1} \tag{2·9}$$

where M_0 is a constant independent of G.

Therefore

$$|g(z_1)-g(z)|\leq M_\alpha(G)C_\alpha(f,\overline{G})|z_1-z|^\alpha \tag{2·10}$$

where $M_\alpha(G)=1+2M_0+C_\alpha(\partial\Phi,\overline{G})+H(\Phi,\overline{G})$ and

$$|\Pi f|\leq M_0H(G)+C(f,\overline{G})C(\partial\Phi,\overline{G})\leq M_\alpha(G)C_\alpha(f,\overline{G}) \tag{2·11}$$

Moreover, by use of (2.10), (2.11), we can obtain

$$C_\alpha(\Pi f,\overline{G})\leq 2M_\alpha(G)C_\alpha(f,\overline{G}) \tag{2·12}$$

that is, when $f\in C_\alpha(\overline{G}), \Pi_G f\in C_\alpha(\overline{G})$ and $\Pi_G f$ is a linear bounded operator from $C_\alpha(\overline{G})$ to itself.

To study differential properties of $T_G f$, set $h(z)=T_G f$

$$\frac{h(z_1)-h(z)}{t(z_1)-t(z)}-\Pi f=-\frac{\Delta_0}{\pi}\iint_G\frac{t_\zeta\overline{Dt(\zeta)}(f(\zeta)-f(z))}{\Delta_1\Delta_2^2}d\sigma_\zeta$$
$$-f(z)\left[-\frac{\overline{\Delta}_0}{\Delta_0}+\frac{\Phi(z_1)-\Phi(z)}{\Delta_0}-\partial\Phi(z)\right]$$

then we can get estimate

$$\left|\frac{h(z_1)-h(z)}{t(z_1)-t(z)}-\Pi f-\frac{\overline{\Delta}_0}{\Delta_0}f(z)\right|$$
$$\leq M_\alpha H(G)|z_1-z|^\alpha+\left|\partial\Phi(z)-\frac{\Phi(z_1)-\Phi(z)}{\Delta_0}\right|C(f,\overline{G}) \tag{2·13}$$

thus

$$\frac{h_x}{t_x}-\Pi f-\frac{\overline{t_x}}{t_x}f(z)=0,\quad \frac{h_y}{t_y}-\Pi f-\frac{\overline{t_y}}{t_y}f(z)=0 \tag{2·14}$$

we have

$$\begin{cases}h_z=\frac{1}{2}(h_x-ih_y)=\Pi f(t_z)+f(z)(\bar{t})_z\\ h_{\bar{z}}=\Pi f(t_{\bar{z}})+f(z)(\bar{t})_{\bar{z}}\end{cases} \tag{2·15}$$

That is

$$\partial Tf=\partial h=\alpha h_{\bar{z}}+\beta h_z=(\partial t)\Pi f+f(z)(\partial\bar{t})=\Pi f \tag{2·16}$$

$$\bar{\partial}Tf=\bar{\partial}h=\bar{\beta}h_{\bar{z}}+\bar{\alpha}h_z=(\bar{\partial}t)\Pi f+f(z)(\overline{\partial t})=f(z) \tag{2·17}$$

It implies that theorem 2. 1 holds for $m=0$.

For $m>1$, hence $f\in C^1_\alpha(\bar G)$ and then

$$\Pi f=-\frac{1}{\pi}\iint_G\frac{t_\zeta D\overline{t(\zeta)}f(\zeta)d\sigma_\zeta}{(t(\zeta)-t(z))^2}-f(z)\sigma(z)$$
$$=-\frac{1}{2\pi i}\int_{\partial G}\frac{f(\zeta)d\overline{t(\zeta)}}{t(\zeta)-t(z)}$$
$$+\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{|\zeta-z|=\varepsilon}\frac{f(\zeta)d\overline{t(\zeta)}}{t(\zeta)-t(z)}+T_G(\partial f)-f(z)\sigma(z)\qquad(2\cdot 18)$$

by simple calculation, it's easy to get the following formula of Πf

$$\Pi f=\Psi(z)+T_G(\partial f)\qquad(2\cdot 19)$$

where $\Psi(z)=-\frac{1}{2\pi i}\int_{\partial G}\frac{f(\zeta)d\overline{t(\zeta)}}{t(\zeta)-t(z)}$ is a hyperanalytic function, then we have

$$\bar\partial\Pi f=\bar\partial\Psi+\bar\partial(T_G(\partial f))=\partial f\qquad(2\cdot 20)$$
$$\partial\Pi f=\partial\Psi+\partial(T_G(\partial f))=\partial\Psi+\Pi(\partial f)\qquad(2\cdot 21)$$

By these formulas we get $\Pi f\in C^1_\alpha(\bar G)$ when $G\in C^2_\alpha, f\in C^1_\alpha(\bar G)$ and recursively we also get $\Pi f\in C^m_\alpha(\bar G)$, $T_G f\in C^{m+1}_\alpha(\bar G)$ when $G\in C^{m+1}_\alpha$ and $f\in C^m_\alpha(\bar G)$, finally we get

$$C^m_\alpha(\Pi f,\bar G)\le C^m_\alpha(\partial T_G f,\bar G)\le KC^{m+1}_\alpha(T_G f,\bar G)\le M(m,\alpha)C^m_\alpha(f,\bar G),\qquad(2\cdot 22)$$

where K is a constant independent of m,α.

It means that T_G is totally continuous operator from $C^m_\alpha(\bar G)$ to itself and Π is a linear bounded operator from $C^m_\alpha(\bar G)$ to itself.

Theorem 2. 2. *If G is a bounded domain in plane and $f\in L_p(\bar G)$, $(p>1)$, then*

1) $\Pi_G f\in L_p(\bar G)$,

2) $\Pi_G f$ is a linear bounded operator from $L_p(\bar G)$ to itself and we have the estimation:

$$L_p(\Pi f,\bar G)\le\Lambda_p L_p(f,\bar G)\qquad(2\cdot 23)$$

where

$$L_p(f,\bar G)=\left(\iint_G|f(\zeta)|^p d\sigma_\zeta\right)^{\frac{1}{p}}$$

Proof. By singularity estimation of $\frac{1}{t(\zeta)-t(z)}$ and the method of [10].

Theorem 2. 3. *If $f\in L_p(\bar G)$, $p>1$, then there exists the generalized derivatives of $T_G f$*

$$\bar\partial Tf=f,\ \partial Tf=\Pi f\qquad(2\cdot 24)$$

Proof. It is sufficient to prove the following results.

$$I=\iint_G t_\zeta D\overline{t(\zeta)}[(Tf)\partial\varphi+\varphi\Pi f]d\sigma_\zeta=0,\ \forall\ \varphi\in D^1_0(G)\qquad(2\cdot 25)$$

$$J=\iint_G t_\zeta D\overline{t(\zeta)}[(Tf)\bar\partial\varphi+f\varphi]d\sigma_\zeta=0,\ \forall\ \varphi\in D^1_0(G)\qquad(2\cdot 26)$$

we assume $f_n\in D^0_\infty(G)$ and $L_p(f_n-f,\bar G)\to 0$, it turns out

$$I_n=\iint_G t_\zeta D\overline{t(\zeta)}[(Tf_n)\partial\varphi+\varphi\Pi f_n]d\sigma_\zeta$$
$$=\iint_G t_\zeta D\overline{t(\zeta)}[\partial(\varphi\cdot Tf_n)-\partial(Tf_n)\cdot\varphi+(\Pi f_n)\cdot\varphi]d\sigma_\zeta$$
$$=\iint_G t_\zeta D\overline{t(\zeta)}\partial(\Phi\cdot Tf_n)d\sigma_\zeta=-\frac{1}{2i}\int_{\partial G}(Tf_n)\cdot\varphi d\overline{t(\zeta)}=0$$

and

$$L_p(T(f_n - f), \bar{G}) \leq \Lambda'_p L_p(f_n - f, \bar{G}) \to 0, \ (n \to \infty)$$
$$L_p(\Pi(f_n - f), \bar{G}) \leq \Lambda''_p L_p(f_n - f, \bar{G}) \to 0, \ (n \to \infty)$$
thus we obtain $I = 0$. By similar method we also obtain $J = 0$.

3. The Generalized Expression of Second Order Hypercomplex Equation

Now we consider the second order hypercomplex equation

$$\bar{\partial}\partial w + \mu_1 \bar{\partial}^2 w + \mu_2 \partial^2 w + \mu_3 \partial^2 \bar{w} + \mu_4 \bar{\partial}^2 \bar{w} + h(z, w, \partial w, \bar{\partial} w) = 0 \qquad (3 \cdot 1)$$

where $h(z, w, \partial w, \bar{\partial} w) = r_1 \bar{\partial} w + r_2 \partial w + r_3 \partial \bar{w} + r_4 \bar{\partial} \bar{w} + s_1 w + s_2 \bar{w} + s_0$, the coefficients $\mu_i(z)$ ($i = 1, 2, 3, 4$) are bounded measurable hypercomplex functions in G; G is a domain in C, $\Gamma \equiv \partial G$ is a smoothly closed curve, w is an unknown hypercomplex function; $r_i(z), s_j(z)$ ($i = 1, 2, 3, 4$; $j = 0, 1, 2$) are hypercomplex functions, belonging to $L_p(\bar{G})$, $p > 2$.

Suppose that

$$\sum_{\substack{z \in \bar{G} \\ i=1,2,3,4}} |\mu_i^k| \leq q_0^k \quad (\text{where we suppose } 0 < q_0^0 < 1) \qquad (3 \cdot 2)$$

where q_0^k ($k = 0, \cdots, r$) are constants, $\mu_i(z) = \sum_{k=0}^{r} \mu_i^k(z) e^k$. According to homomorphic classification method of И. М. Гельфанд and И. Г. Петровский, Б. В. Боярский[8] indicated that the equation is a second order elliptic equation of E_2 class.

Definition 4. *w is a generalized solution of equation* (3.1) *if $w \in C^1(\bar{G}) \cap W_p^2(\bar{G})$, $p > 2$, and it satisfies this equation almost everywhere.*

First let us define e^f and $\log f$ for hypercomplex function, $f = f_0 + F$, F is a nilpotent,

$$e^f = \exp f = e^{f_0}\left(\sum_{k=0}^{r} \frac{1}{k!} F^k \right) \qquad (3 \cdot 3)$$

$$\log f = \log f_0 + \sum_{k=1}^{r} \frac{(-1)^{k-1}}{k} \left(\frac{F}{f_0} \right)^k, \quad (f_0 \neq 0) \qquad (3 \cdot 4)$$

we have easily the following lemma

Lemma 3.1. *If $f(z)$ is a hypercomplex function, $f_0 \neq 0$, then*

$$\overline{\log f} = \log \bar{f}, \ \partial \log f = \frac{\partial f}{f}, \ \bar{\partial} \log f = \frac{\bar{\partial} f}{f} \qquad (3 \cdot 5)$$

Now we introduce operator

$$T_0 f = \frac{1}{\pi} \iint_{\bar{G}} t_\zeta D\overline{t(\zeta)} [\log(t(\zeta) - t(z)) \overline{(t(\zeta) - t(z))}] f(\zeta) d\sigma_\zeta \qquad (3 \cdot 6)$$

when $f \in L_p(\bar{G})$, $T_0 f$ have the second derivatives of $\bar{\partial}$ and ∂

$$\begin{cases} \bar{\partial} T_0 f = \bar{T} f, \ \partial T_0 f = T f, \ \bar{\partial}^2 T_0 f = \bar{\Pi} f, \\ \partial^2 T_0 f = \Pi f, \ \partial \bar{\partial} T_0 f = f = \bar{\partial} \partial T_0 f \end{cases} \qquad (3 \cdot 7)$$

Theorem 3.1. *The generalized solution of equation* (3.1) *can be written as*

$$w(z) = \Phi_1(z) + \overline{\Phi_2(z)} + T_0 f \qquad (3 \cdot 8)$$

where $\Phi_i(z)$ ($i = 1, 2$) are hyperanalytic functions in G, $\Phi_i(z) \in C^1(\bar{G}) \cap W_p^2(\bar{G})$, $f \in L_p(\bar{G})$, $p > 2$ and f satisfies

$$f(z) + \mu_1 \bar{\Pi} f + \mu_2 \Pi f + \mu_3 \Pi \bar{f} + \mu_4 \bar{\Pi} \bar{f} + K f = g(z) \qquad (3 \cdot 9)$$

here K is a weak singular integral operator, $g(z)$ can be determined by the coefficients of equation and hyperanalytic functions $\Phi_i(z)$.

Inversly, if there are two hyperanalytic functions $\Phi_i(z) \in C^1(\bar{G}) \cap W_p^2(\bar{G})$, ($i = 1, 2$) and $f(z)$ is solution of equation (3 · 9), *then $w(z)$ given by* (3.8) *is the generalized solution of* (3.1).

Proof. Assume that $w(z)$ is the generalized solution of equation (3.1), then $w(z) \in C^1(\bar{G}) \cap W_p^2(\bar{G})$, so we have $\bar{\partial}\partial w = f \in L_p(\bar{G})$. By use of properties of operator

T_0, we have $\bar\partial\partial T_0 f = f$, let $\omega = T_0 f$, then $\bar\partial\partial(w-\omega)=0$. Set $\hat w = w - \omega$, $\hat w$ satisfies equation $\bar\partial\partial\hat w = 0$, so $\partial\hat w = \Phi(z)$, where $\Phi(z)$ is a hyperanalytic function. On the other hand, $\bar\partial\bar{\hat w} = \bar\Phi$, so $\bar{\hat w} = \Psi(z) + T_0\bar\Phi$ and then $\hat w = \bar\Psi(z) + \bar T\Phi$

Now we compute $T_0\bar\Phi$,

$$
\begin{aligned}
T_0\bar\Phi &= -\frac{1}{\pi}\iint_G \frac{t_\zeta \overline{Dt(\zeta)}\bar\Phi d\sigma_\zeta}{t(\zeta)-t(z)}\\
&= -\frac{1}{\pi}\iint_G t_\zeta\overline{Dt(\zeta)}\bar\Phi\partial[\log(t(\zeta)-t(z))\overline{(t(\zeta)-t(z))}]d\sigma_\zeta\\
&= \lim_{\varepsilon\to 0} -\frac{1}{\pi}\iint_{G_\varepsilon} t_\zeta\overline{Dt(\zeta)}\partial[\bar\Phi\log(t(\zeta)-t(z))\overline{(t(\zeta)-t(z))}]d\sigma_\zeta
\end{aligned}
\quad (3\cdot 10)
$$

by use of lemma 2. 1, we have

$$
\begin{aligned}
T_0\bar\Phi &= \frac{1}{2\pi i}\int_{\partial G}\bar\Phi(\zeta)\log(t(\zeta)-t(z))\overline{(t(\zeta)-t(z))d t(\zeta)} -\\
&\quad -\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{|\zeta-z|=\varepsilon}\bar\Phi(\zeta)\log(t(\zeta)-t(z))\overline{(t(\zeta)-t(z))d t(\zeta)}
\end{aligned}
$$

In virtue of

$$
\begin{aligned}
|\log(t(\zeta)-t(z))| &= |\log(\zeta-z)| + \left|\sum_{k=1}^{r}\frac{(-1)^{k-1}}{k}\left(\frac{T(\zeta)-T(z)}{\zeta-z}\right)^k\right|\\
&\le M^* + |\log\varepsilon|
\end{aligned}
$$

we obtain

$$
\begin{aligned}
T_0\bar\Phi &= \frac{1}{2\pi i}\int_{\partial G}\bar\Phi(\zeta)\log(t(\zeta)-t(z))d\overline{t(\zeta)} + \frac{1}{2\pi i}\int_{\partial G}\overline{\Phi(\zeta)\log(t(\zeta)-t(z))dt(\zeta)}\\
&= \Psi_1 + \bar\Phi_1
\end{aligned}
$$

where Ψ_1 and Φ_1 are hyperanalytic functions. Set $\Phi_2 = \Psi + \Psi_1$, $w(z)$ can be expressed in the form of (3. 8). Obviously $\Phi_i(z)\in C^1(\bar G)\cap W_p^2(\bar G)$. Substituting w into (3. 1), we know that $f(z)$ satisfies (3. 7), where K is a weak singular operator from $L_p(\bar G)$ to itself and is linear combination of $T, \bar T, T_0$ and $\bar T_0$. $g(z)$ is linear combination of coefficients of equation (3. 1), $\Phi_i(z)(i=1,2)$ and their derivatives up to 2-th order, so $g(z)\in L_p(\bar G)$.

Inversly, for arbitrary hyperanalytic functions $\Phi_i(z)\in C^1(\bar G)\cap W_p^2(\bar G)$ and $f\in L_p(G)$ satisfying equation (3. 9), then $w(z) = \Phi_1 + \bar\Phi_2 + T_0 f$ must be generalized solution of equation (3. 1) since (3. 7).

Theorem 3. 2. *Suppose the coefficients of equation* (3. 1) *satisfy* (3. 2), *then that equation* (3. 9) *has unique solution* $f(z)\in L_p(\bar G)$, $p>2$ *for any* $g(z)\in L_p(\bar G)$, *i. e. there exists the generalized solution* $w(z)$ *of equation* (3. 1), *which depends on two arbitrary hyperanalytic functions.*

Proof. By use of theorem 2. 2, we know that Π and $\bar\Pi$ are the linear bounded operators from $L_p(\bar G)$ into itself and K is also the linear bounded operator from $L_p(\bar G)$ into itself.

For the coefficients μ_i, r_i, s_j $(i=1,\cdots,4, j=0,1,2)$ satisfying the following

$$
\Lambda_p \sup_{i=1,\cdots,4}|\mu_i| + \Lambda_k \le \Lambda_p\sum_{k=0}^{4} q_0^k + \Lambda_k < \delta < 1 \quad (3\cdot 11)
$$

where Λ_p is the norm of operator Π on $L_p(\bar G)$, Λ_k is the norm of operator K, δ is a positive constant, by use of Schauder's fixed-point theorem, we can get the result of the theorem directly.

4. A Priori Estimate

We denote the Schwartz operator $S\gamma$, γ is a Holder continuous real hypercomplex function defined on G, $S\gamma$ is a hyperanalytic function and satisfies

$$\lim_{z \to \tau^+ \in \Gamma} \operatorname{Re}(S\gamma)(z) = \gamma(\tau) \tag{4·1}$$

The equation $\bar{\partial} w = 0$ can be written as the following

$$w_{0\bar{z}} = 0 \tag{4·2}$$

$$w_{k\bar{z}} = -\sum_{j=0}^{k-1} q_{k-j} w_{jz}, \ (k = 1 \cdots, r) \tag{4·2'}$$

we consider the boundary value problem

$$\operatorname{Re} w(z) = \gamma(z) = \sum_{k=0}^{r} \gamma_k(z) e^k, \ z \in \Gamma \tag{4·3}$$

Obviously the solution of (4. 1), (4. 2), (4. 3) can be expressed by $S\gamma$, i. e.

$$w_k(z) = -\int_\Gamma \gamma_k(d_n G^{\mathrm{I}}(z, \tau) - i dG^{\mathrm{II}}(z, \tau))$$

$$+ \frac{1}{2}\iint_G \sum_{j=0}^{k-1} q_{k-j} w_{j\zeta}(G^{\mathrm{I}}_\zeta(z, \zeta) + G^{\mathrm{II}}_\zeta(z, \zeta)) d\sigma_\zeta$$

$$+ \frac{1}{2}\iint_G \sum_{j=0}^{k-1} \bar{q}_{k-j} \bar{w}_{j\zeta}(G^{\mathrm{I}}_\zeta(z, \zeta) - G^{\mathrm{II}}_\zeta(z, \zeta)) d\sigma_\zeta$$

$$(k = 0, 1, \cdots, r) \tag{4·4}$$

where $G^{\mathrm{I}}(z,\zeta)$ and $G^{\mathrm{II}}(z,\zeta)$ are the first and second Green functions.

Let $\varphi(z)$ be a conformal mapping from G into unit desk, then $G^{\mathrm{I}}(z,\zeta)$ and $G^{\mathrm{II}}(z,\zeta)$ can be written as

$$G^{\mathrm{I}}(z, \zeta) = -\frac{1}{2\pi}\log\left|\frac{\varphi(z) - \varphi(\zeta)}{1 - \varphi(z)\overline{\varphi(\zeta)}}\right|$$

$$G^{\mathrm{II}}(z, \zeta) = -\frac{1}{2\pi}\log|(\varphi(\zeta) - \varphi(z))\overline{(\varphi(\zeta) - \varphi(z))}|$$

when G is unit desk, (4. 4) can be written as

$$w_k(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\gamma_k(\zeta)(\zeta + z)}{\zeta(\zeta - z)} d\zeta - \frac{1}{\pi}\iint_G \left[\frac{w_{k\zeta}}{\zeta - z} - \frac{z\bar{w}_{k\zeta}}{1 - \bar{\zeta} z}\right] d\sigma_\zeta$$

$$\triangleq \Gamma\gamma_k + P(w_\zeta), \ (k = 0, 1, \cdots, r) \tag{4·5}$$

The following two lemmas are obvious.

Lemma 4. 1. *Let G be unit desk, Γ a boundary of G, $\gamma \in C_\alpha(\Gamma)$, $\frac{1}{2} < \alpha < 1$, $2 < p < \frac{2}{1-\alpha}$, then $S\gamma$ has the following estimation*

$$\| S\gamma \|_{W^1_p(\bar{G})} \leq M \| \gamma \|_{C_\alpha(\Gamma)} \tag{4·6}$$

Lemma 4. 2. *Let G be a bounded domain, Γ its smooth boundary, $f \in L_p(\bar{G})$, $\varphi \in C^1_\alpha(\Gamma)$ and set*

$$Rf = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\zeta) Tf}{\zeta - z} d\zeta$$

then we have the estimation

$$\| Rf \|_{W^1_p(\bar{G})} \leq M^* \| f \|_p \tag{4·7}$$

Theorem 4. 1. *Let G be unit desk, Γ its boundary, $f \in L_p(\bar{G})$, $\varphi \in C^1_\alpha(\Gamma)$, $q \in B^{1,\alpha}(C)$, $\frac{1}{2} < \alpha < 1$, $2 < p < \frac{2}{1-\alpha}$, then*

$$\| S(\varphi Tf) \|_{W^1_p(\bar{G})} \leq A_r \| f \|_p \tag{4·8}$$

Proof. Let $w = S(\varphi Tf)$, we have the following estimate for each component

$$\| w_0 \|_p \leq \| \Gamma(\varphi Tf) \|_p \leq C^1_0 \| f \|_p,$$

$$\| \partial_{\bar{z}} w_0 \|_p \leq \| \partial_z \Gamma(\varphi Tf) \|_p = \| 2\partial_z Rf \|_p \leq C^2_0 \| f \|_p,$$

thus

$$\| w_0 \|_{W_p^1(\bar{G})} \leq C_0 \| f \|_p,$$

$$\| w_1 \|_p \leq \| \Gamma(\varphi T f) \|_p + \| P(w_{1\xi}) \|_p$$
$$\leq C_1^1 \| f \|_p + \Lambda_p \| q_1 w_{0z} \|_p \leq C_1^2 \| f \|_p,$$
$$\| \partial_{\bar{z}} w_1 \|_p = \| q_1 w_{0z} \|_p \leq C_1^3 \| f \|_p,$$
$$\| \partial_z w_1 \|_p \leq \| 2\partial_z R f \|_p + \| \partial_z P(w_{1\xi}) \|_p \leq C_1^4 \| f \| p + \bar{\Lambda}_p \| q_1 w_{0z} \|_p$$
$$\leq C_1^5 \| f \|_p$$

and then

$$\| w_1 \|_{W_p^1(\bar{G})} \leq C_1 \| f \|_p,$$

where $\bar{\Lambda}_p$ is the norm of $\partial_z P$ on $L_p(\bar{G})$ and recusively we have

$$\| w_k \|_{W_p^1(\bar{G})} \leq C_k \| f \|_p,$$

so we obtain

$$\| S(\varphi T f) \|_{W_p^1(\bar{G})} \leq \Lambda_s \| f \|_p,$$

From (4. 8), we get

$$\| \partial S(\varphi T f) \|_p \leq k \| f \|_p, \tag{4 · 9}$$

where k is a constant.

5. Pseudo-Neumann Problem

In this section we consider the boundary value problem

$$\lambda \partial w + \bar{\lambda} \bar{\partial} w = \gamma = \gamma_1 - i\gamma_2, \quad z \in \Gamma, \ (\lambda_0 \neq 0) \tag{5 · 1}$$

where λ, γ are Holder continuous hypercomplex functions, γ_1 and γ_2 are real hypercomplex functions. From section 3, we know that the generalized solution of equation (3. 1) can be written as (3. 8). Substituting (3. 8) into (5. 1), we obtain

$$\lambda(\partial \Phi_1 + T f) + \bar{\lambda}(\partial \bar{\Phi}_2 + \bar{T} f) = \gamma_1 - i\gamma_2 \tag{5 · 2}$$

so we can change (5. 1) into the boundary value problem of hyperanalytic function:

$$\text{Re}[\lambda(\partial \Phi_1 + \partial \Phi_2)] = \gamma_1 - \text{Re}[\lambda(T(f + \bar{f}))] \tag{5 · 3}$$

$$\text{Re}[i\lambda(\partial \Phi_1 - \partial \Phi_2)] = \gamma_2 - \text{Re}[i\lambda(T(f - \bar{f}))] \tag{5 · 4}$$

Assume $t(z)$ is a normal generating solution (i. e. $t(0) = 0$) and set

$$\theta(\tau) = \frac{1}{i} \log \frac{\lambda(\tau)}{[\lambda(\tau)\overline{\lambda(\tau)}]^{\frac{1}{2}}}$$

$$\psi(\tau) = \frac{1}{i} \log \frac{[t(\tau)]^{\varkappa}}{[t(\tau)\overline{t(\tau)^{\varkappa}}]^{\frac{1}{2}}} \tag{5 · 5}$$

where $\varkappa$ is an index of problem (5. 1), $\varkappa = ind_\Gamma \bar{\lambda} = \frac{1}{2\pi} \Delta_\Gamma \arg \bar{\lambda}_0$ where $\lambda = \lambda_0 + \Lambda$, Λ is a nilpotent function. θ and ψ are real hypercomplex functions, function $\theta(\tau) - \psi(\tau)$ is single-valued. Let

$$\omega(z) = iS(\theta - \psi) - i(\theta - \psi) \tag{5 · 6}$$

$$\varphi(z) = t(z)^{\varkappa} \exp\{iS(\theta - \psi)\} \triangleq t(z)^{\varkappa} \tilde{\varphi}(z) \tag{5 · 7}$$

we know that $\tilde{\varphi}(z)$ is a hyperanalytic function, $\varphi(z)$ satisfies

$$\varphi^+(\tau) = p(\tau)\overline{\lambda(\tau)}, \quad (\tau \in \Gamma) \tag{5 · 8}$$

where

$$p(\tau) = \left[\frac{t(\tau)^{\varkappa}\overline{t(\tau)^{\varkappa}}}{\lambda(\tau)\overline{\lambda(\tau)}}\right]^{\frac{1}{2}} \exp \omega_+(\tau) \tag{5 · 9}$$

here $p(\tau)$ is real hypercomplex function on Γ and $p_0 > 0$.

We discuss the two cases as follows:

(I) Case 1, $\varkappa < 0$.

Reducing problems (5. 3), (5. 4) to the following

$$\text{Re}\left\{\frac{t(\tau)^{-\varkappa}}{\tilde{\varphi}(\tau)}(\partial \Phi_1 + \partial \Phi_2)\right\} = \frac{p^{-1}\gamma_1}{\lambda\bar{\lambda}} - \text{Re}\left[\frac{1}{p\bar{\lambda}} T(f + \bar{f})\right] \triangleq \Gamma_1 + F_1 \tag{5 · 10}$$

$$\mathrm{Re}\left\{\frac{it(\tau)^{-\varkappa}}{\tilde{\varphi}(\tau)}(\partial\Phi_1-\partial\Phi_2)\right\}=\frac{p^{-1}\gamma_2}{\lambda\bar{\lambda}}-\mathrm{Re}\left[\frac{i}{p\bar{\lambda}}T(f-\bar{f})\right]\triangleq\Gamma_2+F_2 \quad (5\cdot 11)$$

and solving (5.10), (5.11), we obtain

$$\partial\Phi_1+\partial\Phi_2=t(z)^{\varkappa}\varphi(z)[S(\Gamma_1+F_1)+ic_1] \quad (5\cdot 12)$$

$$\partial\Phi-\partial\Phi_2=-it(z)^{\varkappa}\tilde{\varphi}(z)[S(\Gamma_2+F_2)+ic_2] \quad (5\cdot 13)$$

where c_i (i=1, 2) are real arbitrary hypercomplex values. Because $\partial\Phi_i$ (i=1, 2) are continuous at $z=0$, pseudo-Neumann problem is solvable iff two functions

$$[S(\Gamma_1+F_1)+ic_1](z),\ [S(\Gamma_2+F_2)+ic_2](z) \quad (5\cdot 14)$$

have $-\varkappa$ order zero at $z=0$ when $\varkappa<0$. In particular when $\varkappa=-1$, we get two solvable conditions

$$\mathrm{Re}S(\Gamma_1+F_1)(0)=0,\ \mathrm{Re}S(\Gamma_2+F_2)(0)=0 \quad (5\cdot 15)$$

Let $$H_1^*=\frac{1}{2}\tilde{\varphi}(z)t(z)^{-\varkappa}\{S(\Gamma_1+F_1)-iS(\Gamma_2+F_2)+ic_1+c_2\} \quad (5\cdot 16)$$

$$H_2^*=\frac{1}{2}\tilde{\varphi}(z)t(z)^{\varkappa}\{S(\Gamma_1+F_1)+iS(\Gamma_2+F_2)+ic_1-c_2\} \quad (5\cdot 17)$$

Those above solvable conditions are equivalent to the following conditions

$$S(\Gamma_1+F_1)-iS(\Gamma_2+F_2)+ic_1+c_2,\ S(\Gamma_1+F_1)+iS(\Gamma_2+F_2)+ic_1-c_2$$

have $-\varkappa$ order zero at $z=0$

So $f(z)$ satisfies $-2\varkappa-1$ complex relations. Thus solution of (3.1), (5.1) can be expressed by

$$w(z)=\bar{T}H_1^*+T\bar{H}_2^*+T_0f+c_1^* \quad (5\cdot 18)$$

where c_1^* is an arbitrary hypercomplex value.

Substituting (5.18) into (3.1), we obtain

$$f+\mu_1\bar{\Pi}f+\mu_2\Pi f+\mu_3\Pi\bar{f}+\mu_4\overline{\Pi f}+Q^*(f)=\rho \quad (5\cdot 19)$$

where $Q^*(f)$ is the linear combination of the operators Tf, $\bar{T}f$, T_0f, $\bar{T}_0f$ and $S(F_1)$, $S(F_2)$, $\partial S(F_1)$, $\partial S(F_2)$. From section 4, we have

$$\|SF_1\|_p=\|S\{\mathrm{Re}(\varphi T(f+\bar{f}))\}\|_p\le M_1\|f\|_p$$

$$\|\partial SF_1\|_p\le M_2\|f\|_p,\quad \|SF_2\|\le M_3\|f\|_p,\quad \|\partial SF_2\|_p\le M_4\|f\|_p$$

where $\varphi=\frac{1}{p\lambda}\in C_\alpha^1(\Gamma)$, so $Q^*(f)$ is linear bounded operator on $L_p(\bar{G})$. Denoting its norm Λ_Q^*, when

$$\Lambda_p\sum_{k=0}^{r}q_0^k+\Lambda_Q^*<\delta_0^*<1 \quad (5\cdot 20)$$

then equation (5.19) has a unique solution $f(z)$ in $L_p(\bar{G})$.

(II) Case 2, $\varkappa\ge 0$

In this case, problems (5.3), (5.4) can be reduced to

$$\mathrm{Re}\left\{\frac{(\partial\Phi_1+\partial\Phi_2)\tilde{\varphi}^{-1}(\tau)}{t(\tau)^{\varkappa}}\right\}=\Gamma_1+F_1 \quad (5\cdot 21)$$

$$\mathrm{Re}\left\{\frac{i(\partial\Phi_1-\partial\Phi_2)\tilde{\varphi}^{-1}(\tau)}{t(\tau)^{\varkappa}}\right\}=\Gamma_2+F_2 \quad (5\cdot 22)$$

Setting $P_m(z)=\sum_{k=0}^{m}c_kt(z)^k$ be a hypercomplex polynomial of degree m and solving (5.21), (5.22), we obtain

$$\begin{aligned}\partial\Phi_1+\partial\Phi_2&=\tilde{\varphi}(z)\{P_{\varkappa-1}(z)+t(z)^{\varkappa}ic\\&+t(z)^{\varkappa}S(\Gamma_1+F_1)-t(z)^{\varkappa}S(\mathrm{Re}t(z)^{-\varkappa}P_{\varkappa-1}(\tau))\}\triangleq R_1\end{aligned}$$

$$\begin{aligned}\partial\Phi_1-\partial\Phi_2&=-i\tilde{\varphi}(z)\{\tilde{P}_{\varkappa-1}(z)+t(z)^{\varkappa}i\bar{c}\\&+t(z)^{\varkappa}S(\Gamma_2+F_2)-t(z)^{\varkappa}S(\mathrm{Re}t(z)^{-\varkappa}\tilde{P}_{\varkappa-1}(\tau))\}\triangleq R_2\end{aligned}$$

thus

$$\partial\Phi_1=\frac{R_1+R_2}{2}\triangleq H_1,\ \partial\Phi_2=\frac{R_1-R_2}{2}\triangleq H_2 \tag{5·23}$$

where c_k, $\tilde{c}_k$ $(k=0,\cdots,\varkappa-1)$ are arbitrary hypercomplex values, c and $\tilde{c}$ are arbitrary real hypercomplex values.

Solving (5.23), we obtain

$$\Phi_1=\bar{T}H_1+c',\qquad \Phi_2=\bar{T}H_2+c''$$

where c' and c'' are arbitrary hypercomplex values, set $c_0=c'+c''$, then the solution $w(z)$ can be expressed by

$$w(z)=\bar{T}H_1+T\bar{H}_2+T_0f+c_0 \tag{5·24}$$

so the solution $w(z)$ of the problem depends on $2\varkappa+2$ arbitrary hypercomplex values.

Substituting (5.24) into (3.1), we also obtain a sigular integral equation for f, we only replace $Q^*(f)$ by $Q(f)$ and H_1^*, H_2^* by H_1, H_2. Denoting the norm of $Q(f)$ on $L_p(G)$ by Λ_Q, we know that there exists a unique solution of that equation in $L_p(\bar{G})$ if

$$\Lambda_p\sum_{k=0}^{r}q_0^k+\Lambda_Q\leq\delta_0<1 \tag{5·20*}$$

so we prove the following theorem:

Theorem 5.1. (I) *Case* 1, $\varkappa<0$. *Suppose the coefficients satisfy inequality* (5.20), *then the sufficient and necessary condition for the solvability of pseudo-Neumann problem* (5.1) *consists of* $-2\varkappa-1$ *complex relations and its solution is dependent on an arbitrary hypercomplex constant.*

(II) *Case* 2, $\varkappa\geq 0$. *Suppose that coefficients satisfy inequality* (5.20*), *the pseudo-Neumann problem* (5.1) *is always solvable and its solution is dependent on* $2\varkappa+2$ *arbitrary hypercomplex constants.*

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