J. Partial Differential Equations
Vol. 1 № 1 (1988), 13–20

A NOTE ON PRESCRIBED GAUSSIAN CURVATURE ON S^2 *

Hong Chongwei
Institute of Mathematics, Academia Sinica
Received June 11, 1987

1. Introduction and Main Results

Given $R(x) \in C^2(S^2)$ where $S^2 = \{x \in R^3 \mid |x| = 1\}$, we want to find a condition on $R(x)$ so that there exists a metric g on S^2 with scalar curvature (i.e. twice the Gaussian curvature) $R(x)$, which is pointwise conformal to the standard metric g_0, so $g = e^u g_0$ for some function u.

This problem is equivalent to the existence of a solution of Eq. (cf. [1])

$$\Delta u(x) - 2 + R(x) e^{u(x)} = 0 \qquad x \in S^2 \tag{1.1}$$

where we use the sign convention for Laplacian Δ so that $\Delta u = u_{xx} + u_{yy}$ on flat R^2.

For known results of this interesting problem, confer [1] - [13]. In this paper we prove

Theorem 1.1. *Assume that* $R(x) \in C^2(S^2)$ *satisfies*

i) $\exists$ *a curve* $\Gamma \in C([0,1], S^2)$, $\Gamma(0) = a \neq b = \Gamma(1)$, $0 < R(b) \leq R(a)$, $b \in S^2$ *is a nondegenerate local maximum point of* $R(x)$.

ii) $\min\limits_{x \in \Gamma} R(x) = m < R(b)$ *and* $\forall\, x \in \Gamma \cap R^{-1}(m)$ *either* $\nabla R(x) \neq \vec{0}$ *or* $\nabla R(x) = \vec{0}$, $\Delta R(x) > 0$.

iii) *There is no critical point of* $R(x)$ *on* $R^{-1}(m, R(b))$ *except a finite number of nondegenerate local maximum points.*

Then Eq. (1.1) *has a solution.*

Remark 1.1. If $\min\limits_{x \in \Gamma} R(x) \leq 0$, assumption ii) can be omitted and assume iii) on $R^{-1}(0, R(b))$, then Theorem 1.1 remains true.

Remark 1.2. Notice that Theorem 1.1 permits $R(b) < R(a) < \max\limits_{x \in S^2} R(x)$, $a \in S^2$ need not be a critical point of $R(x)$, $R(x)$ can be arbitrary on $S^2 \backslash$ a neighborhood of Γ provided iii) holds.

To solve Eq. (1.1), we look for a critical point of

$$J(u) \triangleq \frac{1}{2} \int_{S^2} |\nabla u|^2 + 2 \int_{S^2} u - 8\pi \log \int_{S^2} Re^u \triangleq I(u) - 8\pi \log \int_{S^2} Re^u$$

defined on $H \triangleq \{u \in H^1(S^2) \mid \int_{S^2} Re^u > 0\}$. If $J'(u_0) = 0$, then $u = u_0 +$ some constant C is a solution of Eq. (1.1).

Set $B_r \triangleq \{x \in R^3 \mid |x| < r\}$ and $B_1 \triangleq B$. Define $P(u) \triangleq \int_{S^2} xe^u / \int_{S^2} e^u \in B$, $\forall\, u \in H^1(S^2)$. Throughout this paper we assume $R(x) \in C^2(S^2)$. It is worth while noticing the function $m(x) \triangleq \inf\limits_{u \in H,\, P(u) = x} J(u)$, $x \in B$. In section 2 we prove:

* The project supported by National Natural Science Foundation of China

Lemma 2.1. *If* $R(x)>0$, $\forall x\in S^2$, *then* $m(x)\in C(\bar{B})\cap C^{1-0}(B)$ *and* $m(x)=-8\pi\log(4\pi R(x))$, $\forall x\in S^2$.

In section 3 we prove the following inequality:

Lemma 3.1. *Suppose that* $b\in S^2$ *is a nondegenerate local maximum point of* $R(x)$, $R(b)>0$, *then there exists* $\delta>0$ *depending on* R *such that* $\forall\ 0<\varepsilon\le\delta$, $\exists\ 0<\mu=\mu(R,\delta,\varepsilon)<4\pi$ *so that the following inequality holds:*

$$\int_{S^2}R(x)e^{u(x)}\le\mu R(b)\exp\left(\frac{1}{16\pi}\int_{S^2}|\nabla u|^2+\frac{1}{4\pi}\int_{S^2}u\right),$$

$$\forall u\in H^1(S^2)\quad with\quad \varepsilon\le|P(u)-b|\le\delta \tag{1.2}$$

In section 4 we prove Theorem 1.1 using Lemma 2.1, 3.1 and minimax argument on H.

2. Function $m(x)$ on Unit Ball $\bar{B}$

In what follows we denote various constants by the same C. Set

$$\varphi_{\lambda y}(x)=\log\frac{1-\lambda^2}{(1-\lambda\cos d(x,y))^2}\qquad x,y\in S^2,\qquad 0\le\lambda<1$$

where $d(x,y)$ is the distance on (S^2,g_0) between two points x,y, then (cf. [6]) $u(x)=\varphi_{\lambda y}(x)$ satisfies Eq. (1.1) with $R(x)=2$.

$$\int_{S^2}\exp(\varphi_{\lambda y}(x))=4\pi,\qquad I(\varphi_{\lambda y}(x))=0 \tag{2.1}$$

Direct computation shows

$$P(\varphi_{\lambda y})=C(\lambda)y\in B,\ C(\lambda)=\frac{1}{\lambda}+\frac{1}{2}\left(\frac{1}{\lambda^2}-1\right)\log\frac{1-\lambda}{1+\lambda} \tag{2.2}$$

and there is a homeomorphism $h:B\to B$: $\forall\ \lambda y\in B$, $(\lambda,y)\in[0,1)\times S^2$, $h(\lambda y)\triangleq P(\varphi_{\lambda y})$.

Proof of Lemma 2.1. 1° $J(u)$ is bounded below (cf. [10]) and $J(u)=J(u+C)$ $\forall u\in H^1(S^2)$, $C\in\mathbf{R}$. For fixed $x_0\in B$ choose a minimizing sequence $\{u_i\}\subset H$, $\int_{S^2}u_i=0$, $P(u_i)=x_0$, $J(u_i)\to m(x_0)$. By Aubin [2 Theorem 6], we have

$$\int_{S^2}e^{u_i}\le C\exp\left(\frac{1}{24\pi}\int_{S^2}|\nabla u_i|^2\right) \tag{2.3}$$

C is independent of i. From (2.3) and $J(u_i)\le C$ we derive $\|u_i\|_{H^1}\le C$. We can extract a subsequence, still denoted by $\{u_i\}$, such that $u_i\rightharpoonup u_0$ ($H^1(S^2)$). Since $u\in H^1$: $u\to e^u\in L^1$ is compact (cf. [1 Theorem 2.46]) and J is weakly lower semicontinuous on H, we get $J(u_0)=m(x_0)$, $P(u_0)=x_0$, i. e. $\inf_{u\in H,P(u)=x_0}J(u)=m(x_0)$ is attained by u_0.

2° We prove that $m(x)\in C(B)$. Suppose that $J(u_i)=m(x_i)$, $P(u_i)=x_i\to x\in B$, $\int_{S^2}u_i=0$, using $\varphi_{\lambda y}(x)$ it is easy to see that we can assume $J(u_i)\le C$, again (2.3) holds, the same reasoning as in 1° shows $\underline{\lim}_{x_i\to x}m(x_i)\ge m(x)$. On the other hand, if $J(u_0)=m(x_0)$, $P(u_0)=x_0$, set $P(u)=p=(p_1,p_2,p_3)$, by definition $\int_{S^2}(x-p)e^u=\vec{0}$, using implicit function theorem we see that there exists a neighborhood U of u_0 in $H^1(S^2)$ such that (v,p) is a coordinate system of U, where v is some subspace of $H^1(S^2)$ with codimension 3. Noticing the continuity of J at $u_0\in H$, we obtain $\overline{\lim}_{x_i\to x_0}m(x_i)\le m(x_0)$, hence $m(x)\in C(B)$.

$3°$ $\forall$ $\{u_i\} \subset H$ such that $P(u_i) = x_i \to x_0 \in S^2$ and $\int_{S^2} e^{u_i} = 4\pi$, by [5 Lemma 1.1] we have $\int_{S^2} R(x) e^{u_i(x)} \to 4\pi R(x_0)$. Since $I(u_i) \geq 0$ (cf. [12]), we get

$$\underline{\lim}_{x_i \to x_0 \in S^2} m(x_i) \geq -8\pi \log(4\pi R(x_0))$$

On the other hand, by (2.1), (2.2) we have

$$J(\varphi_{\lambda y}) \to -8\pi \log(4\pi R(x_0)) \qquad \text{if} \quad \lambda y \to x_0 \in S^2$$

thus $\overline{\lim}_{x_i \to x_0 \in S^2} m(x_i) \leq -8\pi \log(4\pi R(x_0))$, therefore defining

$$m(x) = -8\pi \log(4\pi R(x)), \ \forall x \in S^2$$

we have $m(x) \in C(\bar{B})$

$4°$ $\forall$ fixed $0 < r < 1$, we prove $m(x) \in C^{1-0}(\bar{B}_r)$, i.e.

$$\frac{|m(x) - m(y)|}{|x - y|} \leq C(r) \qquad \forall x, y \in \bar{B}_r \tag{2.4}$$

If (2.4) were false, then $\exists$ x_i, $y_i \in \bar{B}_r$, $i = 1, 2, \ldots$, $|x_i - y_i| \to 0$, $y_i \to z \in \bar{B}_r$ and

$$0 < \frac{m(x_i) - m(y_i)}{|x_i - y_i|} \to +\infty \tag{2.5}$$

By $1°$ assume $J(u_i) = m(y_i)$, $P(u_i) = y_i$, from $1°$, $2°$ we can assume $u_i \rightharpoonup u_0 (H^1(S^2))$, $P(u_0) = z \in \bar{B}_r$, $J(u_i) \to J(u_0) = m(z)$, thus $\int_{S^2} |\nabla u_i|^2 \to \int_{S^2} |\nabla u_0|^2$ and $\| u_i \|_{H^1} \to \| u_0 \|_{H^1}$, hence $u_i \to u_0 (H^1(S^2))$ strongly. In a neighborhood U of $u_0 \in H^1(S^2)$, as in $2°$, using coordinate system $u = (v, p)$, $P(u) = p$, $u_i = (v_i, y_i)$, noting $J \in C^1(H^1(S^2))$, we should have

$$0 < \frac{m(x_i) - m(y_i)}{|x_i - y_i|} = \frac{m(x_i) - J(u_i)}{|x_i - y_i|}$$

$$\leq \frac{J(v_i, x_i) - J(v_i, y_i)}{|x_i - y_i|} = \frac{\partial J}{\partial P}(v_i, y_i + \theta(x_i - y_i)) \leq C, \ 0 < \theta < 1$$

This contradicts (2.5) and completes the proof.

When $R(x) \leq 0$ somewhere, set $V \triangleq \{x \in S^2 | R(x) > 0\} \cup \{P(u) | u \in H\}$, V is an open set of $\bar{B}$. The same argument as above with slight modification proves the following

Lemma 2.1'. *If $R(x) \in C^1(S^2)$, then $m(x) \in C(\bar{B} \cap V) \cap C^{1-0}(B \cap V)$ and $m(x) = -8\pi \log(4\pi R(x))$ $\forall x \in S^2 \cap V$.*

Remark 2.1. We don't know whether or not $m(x) \in C^1(B \cap V)$, if it were the case, then we could reduce the minimax argument on H to that on $\bar{B} \cap V$ and get more results.

3. An Inequality

To prove Lemma 3.1, first we prove

Lemma 3.2. *Suppose $R(b) > 0$, $b \in S^2$ then $\forall \varepsilon > 0$, $\exists \delta = \delta(R, \varepsilon) > 0$, such that $\forall u \in H$ with $|P(u) - b| \leq \delta$ and $J(u) = m(P(u))$, $\exists \varphi_{\lambda x}$, $(\lambda, x) \in [0, 1) \times S^2$ (see section 2) so that*

$$|P(\varphi_{\lambda x}) - b| \leq \varepsilon \qquad \text{and} \qquad \int_{S^2} |\nabla(u - \varphi_{\lambda x})|^2 \leq \varepsilon \tag{3.1}$$

Proof. $1°$ $\forall$ $\{u_i\} \subset H$ with $P(u_i) \to b$, $\int_{S^2} e^{u_i} = 4\pi$, $J(u_i) = m(P(u_i))$, by proof of Lemma 2.1, $3°$, we have $\int_{S^2} R(x) e^{u_i(x)} \to 4\pi R(b)$. On the other hand, by

Lemma 2. 1. $J(u_i) \to -8\pi\log(4\pi R(b))$, hence $I(u_i) \to 0$.

2° Use spherical coordinates $x = (\theta, \psi) \in S^2$, $0 \le \theta \le \pi$, $0 \le \psi < 2\pi$ with North Pole $y = (0, \psi)$, consider the transformation (cf. [6] proof of Lemma 2. 3)

$$T_{\lambda y}: H^1(S^2) \to H^1(S^2), \ T_{\lambda y}u(\theta, \psi) = u \circ F_{\lambda y}(\theta, \psi) + \varphi_{\lambda y}, \quad (\lambda, y) \in [0, 1) \times S^2$$

where $F_{\lambda y}(\theta, \psi) = \left(2\mathrm{arctg}\left(\sqrt{\frac{1+\lambda}{1-\lambda}}\mathrm{tg}\frac{\theta}{2}\right), \psi\right)$.

By [5] Prop. 3. 3 and proof of Prop. 3. 1, $\exists\ T_{\lambda_i y_i}$, $i = 1, 2, \ldots$ such that

$$I(T_{\lambda_i y_i}u_i) = I(u_i) \to 0, \quad \int_{S^2}\exp(T_{\lambda_i y_i}u_i) = \int_{S^2}e^{u_i} = 4\pi \ \ and \ \ P(T_{\lambda_i y_i}u_i) = \vec{0}$$

The argument similar to the proof of Lemma 2. 1. 1° shows $\| T_{\lambda_i y_i}u_i \|_{H^1} \le C$ and we can assume $T_{\lambda_i y_i}u_i \rightharpoonup u_0 (H^1(S^2))$. Since $\inf\limits_{\int e^u = 4\pi,\ P(u) = \vec{0}} I(u)$ is attained by unique $u = 0$, we have $T_{\lambda_i y_i}u_i \rightharpoonup 0 (H^1(S^2))$ thus $\int_{S^2}|\nabla(T_{\lambda_i y_i}u_i)|^2 \to 0$. From conformal invariance of $\int_{S^2}|\nabla v|^2$, we get

$$\int_{S^2}|\nabla(u_i + \varphi_{\lambda_i y_i} \cdot F^{-1}_{\lambda_i y_i})|^2 \to 0$$

Dircet computation shows $\varphi_{\lambda(-y)} \circ F_{\lambda y} + \varphi_{\lambda y} = 0$, hence

$$\int_{S^2}|\nabla(u_i - \varphi_{\lambda_i x_i})|^2 \to 0, \qquad \text{where} \quad x_i = -y_i \tag{3. 2}$$

3° We prove $P(\varphi_{\lambda_i x_i}) \to b$. Otherwise, choose a subsequence, if necessary, we should have either a) $\varphi_{\lambda_i x_i} \to \varphi_{\lambda_0 x_0}$ or b) $P(\varphi_{\lambda_i x_i}) \to e \in S^2$, $e \ne b$. The case a) contradicts (3. 2) and the fact $\int_{S^2}|\nabla u_i|^2 \to +\infty$. By [5] Prop. 4. 4 with $J(u)$ replaced by $K(u) \triangleq I(u) - 8\pi\log\int_{S^2}e^u$, we get $e = b$ i. e. $P(\varphi_{\lambda_i x_i}) \to b$. Lemma 3. 2 follows from 2° and 3°.

Proof of Lemma 3. 1.

1° Choose small $\varepsilon > 0$ in Lemma 3. 2 ($\varepsilon > 0$ to be determined later). $\forall\ u \in H$ with $J(u) = m(P(u))$ and $|P(u) - b| \le \delta(\varepsilon)$, $\int_{S^2}e^u = 4\pi$ (see Lemma 3. 2) $\exists\ C = C(u, \varphi_{\lambda x})$, such that

$$\int_{S^2}(u - \varphi_{\lambda x} - C) = 0 \tag{3. 3}$$

Using spherical coordinates on S^2 as above with North Pole $x = (0, \psi)$, then $\exp\varphi_{\lambda x} = \frac{1-\lambda^2}{(1-\lambda\cos\theta)^2}$, we have

$$\int_{S^2}R(y)e^{u(y)} - 4\pi R(b) = \int_{S^2}(R(y) - R(b))e^{\varphi_{\lambda x}} \cdot e^C \cdot e^{u - \varphi_{\lambda x} - C}$$

$$= (1-\lambda^2)e^C\int_0^{2\pi}\int_0^{\pi}(R(y) - R(b))\frac{\sin\theta \cdot e^{u-\varphi_{\lambda x}-C}}{(1-\lambda\cos\theta)^2}d\theta d\psi$$

$$= (1-\lambda^2)e^C\left(\int_0^{2\pi}\int_0^{\mu} + \int_0^{2\pi}\int_{\mu}^{\pi}\right) \triangleq (1-\lambda^2)e^C(\mathrm{I} + \mathrm{II})$$

Since b is a nondegenerate local maximum point of $R(y)$, $\exists\ a, r > 0$ such that $R(y) - R(b) \le -a(d(y, b))^2$ if $d(y, b) = \mathrm{dist}(y, b) \le r$. Let $\mu = \frac{r}{2}$ be fixed. It is easy to see that $|\mathrm{II}| \le C_1$. we have $B(t, \lambda, b) \triangleq \int_0^{2\pi}\int_t^{\mu}\frac{\sin\theta \cdot (d(y, b))^2}{(1-\lambda\cos\theta)^2} \to +\infty$, as

$\lambda\to 1,\ t\to 0$, uniformly in $b\in S^2$. Otherwise $\exists\ \lambda_i\to 1,\ t_i\to 0,\ b_i\to b_0\in S^2,\ k>0$ such that $B(t_i,\ \lambda_i,\ b_i)\le k,\ \forall\ i$ then $\forall\ 0<\nu<\mu$, we should have

$$\int_0^{2\pi}\int_\nu^{\mu}\frac{\sin\theta\cdot(d(y,\ b_0))^2}{(1-\cos\theta)^2}\le k$$

This is impossible since the integral

$$\int_0^{2\pi}\int_0^{\mu}\frac{\sin\theta(d(y,\ b_0))^2}{(1-\cos\theta)^2}d\theta d\psi$$

is divergent. We claim that $\forall\ M>0,\ \exists\ \varepsilon_0>0$ such that $\mathrm{I}\le -M$ if $\varepsilon\le\varepsilon_0$ (see Lemma 3. 2). In fact choose $t_0,\ \lambda_0$ so that

$$B(t,\ \lambda,\ b)\ge\frac{2M}{a}\qquad \text{if}\quad 0<t\le t_0,\quad \lambda_0\le\lambda<1 \tag{3. 4}$$

Let t_0 be fixed, when $\varepsilon\le\frac{r}{4}$ we have

$$-\frac{1}{a}\mathrm{I}\ge\int_0^{2\pi}\int_0^{\mu}\frac{\sin\theta(d(y,\ b))^2}{(1-\lambda\cos\theta)^2}\cdot e^{u-\varphi_{\lambda x}-C}d\theta d\psi$$

$$\ge B(t_0,\ \lambda,\ b)+\int_0^{2\pi}\int_{t_0}^{\mu}\frac{\sin\theta(d(y,\ b))^2}{(1-\lambda\cos\theta)^2}(e^{u-\varphi_{\lambda x}-C}-1)$$

$$\triangleq B(t_0,\ \lambda,\ b)+\mathrm{III} \tag{3. 5}$$

From (3. 1), (3. 3) noting $v\in H^1(S^2):v\to e^v\in L^1$ is compact, we get

$$|\mathrm{III}|\le C(t_0)\int_{S^2}|e^{u-\varphi_{\lambda x}-C}-1|\to 0\qquad \text{as}\quad \varepsilon\to 0$$

Hence $\exists\ \varepsilon_1>0$ such that

$$|\mathrm{III}|\le\frac{M}{a}\qquad \text{as}\qquad \varepsilon\le\varepsilon_1 \tag{3. 6}$$

It is easy to see that $\exists\ \varepsilon_2>0$ such that $\lambda\ge\lambda_0$ if $|P(\varphi_{\lambda x})-b|\le\varepsilon_2$. From (3. 4), (3. 5), (3. 6) we get $\mathrm{I}\le -M$ if $\varepsilon\le\varepsilon_0\triangleq\min(\varepsilon_1,\ \varepsilon_2)$. Therefore, letting $M=C_1+1$ we obtain

$$\int_{S^2}R(y)e^u<4\pi R(b)\qquad \forall\ u\in H^1(S^2)\quad \text{with}\quad J(u)=m(P(u)),\ \int e^u=4\pi$$
$$\text{and}\quad |P(u)-b|\le\delta_0\triangleq\delta(R,\ \varepsilon_0) \tag{3. 7}$$

2° By (3. 7) noting $I(u)\ge 0\quad \forall\ u\in H^1$ with $\int_{S^2}e^u=4\pi$ (cf. [12]) we have $J(u)>-8\pi\log(4\pi R(b))\ \ \forall\ u\in H^1(S^2)$ with $J(u)=m(P(u)),\ |P(u)-b|\le\delta_0$

Using Lemma 2. 1′ we have $\forall\ 0<\varepsilon\le\delta_0,\ \exists\ \ \beta=\beta(R,\ \delta_0,\ \varepsilon)>0$ such that

$$J(u)\ge -8\pi\log(4\pi R(b))+\beta\qquad \forall\ u\in H^1(S^2)$$
$$\text{with}\quad J(u)=m(P(u)),\ 0<\varepsilon\le|P(u)-b|\le\delta_0 \tag{3. 8}$$

Noting the definition of $m(x)$, we see that (3. 8) holds $\forall\ u\in H^1(S^2)$ with $0<\varepsilon\le|P(u)-b|\le\delta_0$. This is equivalent to (1. 2).

Remark 3. 1. By the way, we have

Lemma 3. 3. *Suppose that $b\in S^2$ is a nondegenerate local minimum point of $R(x)$, $R(b)>0$, then $\exists\ \ \delta>0$ depending on R, such that $\forall\ 0<\varepsilon\le\delta\quad \exists\ \beta=\beta(R,\ \varepsilon,\ \delta)>0$ such that*

$$J(\varphi_{\lambda x})\le -8\pi\log(4\pi R(b))-\beta\qquad \text{if}\quad \varepsilon\le|P(\varphi_{\lambda x})-b|\le\delta$$

Proof. It's similar to the proof of Lemma 3. 1, 1°and simpler, notice that $\Phi(y)\triangleq J(\varphi_{\lambda x})\ (P(\varphi_{\lambda x})=y)$ is continuous in a neighborhood of b in $\overline{B}$. We omit the detailed proof since we don't use it in this paper.

4. Proof of Theorem 1. 1

Under the assumptions of Theorem 1. 1, using Lemma 3. 1 for $b\in S^2$ we obtain

$\delta, \rho > 0$ such that

$$J(u) \geq -8\pi\log(4\pi R(b)) + \rho \triangleq s + \rho \qquad (4.1)$$
$$\forall u \in H^1(S^2) \quad \text{with} \quad |P(u) - b| = \delta$$

Choose $0 < \lambda_0 < 1$ sufficiently close to 1 so that

$$J(\varphi_{\lambda_0 b}) \leq s + \frac{\rho}{2}, \ 0 < |P(\varphi_{\lambda_0 b}) - b| < \delta, \ |P(\varphi_{\lambda_0 a}) - b| > \delta \qquad (4.2)$$

and $J(\varphi_{\lambda_0 a}) \leq s + \frac{\rho}{2}$. Let $\varphi_{\lambda_0 a} \triangleq p$, $\varphi_{\lambda_0 b} \triangleq q$ be fixed.

$$\Lambda \triangleq \{h \in C([0, 1], H) \mid h(0) = p, \ h(1) = q\}$$

Because $R(x)$ may be negative somewhere on S^2, we need

Lemma 4. 1. *Λ is nonempty.*

Proof. e. g.

$$h_0(t) \triangleq \log[(1-t)\exp\varphi_{\lambda_0 a} + t\exp\varphi_{\lambda_0 b}] \in \Lambda$$

Define

$$\nu = \inf_{h \in \Lambda} \max_{t \in [0, 1]} J(h(t)) \qquad (4.3)$$

By (4. 1), (4. 2), $\nu \geq s + \rho$

Lemma 4. 2. *If $m = \min_{x \in \Gamma} R(x) > 0$, then $\nu < -8\pi\log(4\pi m)$*

Proof. Set

$$\eta(x) \in C^{1-0}(S^2): \eta(x) = \begin{cases} 0 & \text{if} \quad R(x) \geq R(b) \\ 1 & \text{if} \quad R(x) \leq m \end{cases}$$

using the flow $\frac{dx}{dt} = \eta(x) \nabla R(x)$ on S^2, noting that $\Gamma \cap R^{-1}(m)$ is compact, we see that Γ can be deformed a little with endpoints a, b fixed, which is still denoted by $\Gamma: [0, 1] \to S^2$, so that $\Delta R(x) > 0 \ \forall x \in \Gamma \cap R^{-1}(m)$. The same argument as in [5] proof of Lemma 4. 5 shows that $\exists\ 0 < \lambda_1 < 1$ such that if $\lambda_1 < \lambda < 1$ then $J(\varphi_{\lambda\Gamma(t)}) < -8\pi\log(4\pi m) \ \forall t \in [0, 1]$. Connecting p with $\varphi_{\lambda a}$, q with $\varphi_{\lambda b}$, together with $\varphi_{\lambda\Gamma(t)}$ we get $h \in \Lambda$ such that

$$J(u) < -8\pi\log(4\pi m) \qquad \forall u \in h(t)$$

Hence $\nu < -8\pi\log(4\pi m)$.

Lemma 4. 3. *If $J'(u) \neq \theta$, $\forall u \in H$, then $\exists\ h_n \in \Lambda$, $n = 2, 3, \ldots$ such that*

$$\max_{t \in [0, 1]} J(h_n(t)) \leq \nu + \frac{\rho}{n^2} \qquad (4.4)$$

and $\forall w \in h_n(t)$, $t \in [0, 1]$ if $J(w) \geq \nu - \frac{\rho}{n^2}$ then $\exists\ \bar{w} \in H$ such that

$$\| w - \bar{w} \|_{H^1} \leq \frac{1}{n}, \qquad |J(\bar{w}) - \nu| \leq \frac{\rho}{n^2}, \qquad \| J'(\bar{w}) \| \leq \frac{2\rho}{n} \qquad (4.5)$$

Proof. Since $\inf_{h \in \Lambda} \max_{t \in [0, 1]} J(h(t)) = \nu$, $\exists\ k_n \in \Lambda$ such that

$$\max_{t \in [0, 1]} J(k_n(t)) \leq \nu + \frac{\rho}{n^2}$$

Consider the Eq. on H:

$$\frac{du}{dt} = -\eta(u) \frac{J'(u)}{\| J'(u) \|} \qquad u(0, v) = v$$

where $0 \leq \eta(u) \leq 1$,

$$\eta(u) \in C^{1-0}(H): \eta(u) = \begin{cases} 1 & \text{if} \quad \| u - p \| \geq 2\varepsilon \quad \text{and} \quad \| u - q \| \geq 2\varepsilon \\ 0 & \text{if} \quad \| u - p \| \leq \varepsilon \quad \text{or} \quad \| u - q \| \leq \varepsilon, \end{cases}$$

where $\varepsilon > 0$ is small enough so that $J(u) \leq s + \frac{5}{8}\rho$ if $\| u - p \| \leq 2\varepsilon$ or $\| u - q \| \leq 2\varepsilon$, then it's not difficult to see that $h_n(t) \triangleq u\left(\frac{1}{n}, \ k_n(t)\right)$, $n = 2, 3, \ldots$ satisfy (4. 4) and (4. 5) with $\bar{w} = u(\tau, k_n(t_0))$ for some $0 \leq \tau \leq \frac{1}{n}$, where $w = h_n(t_0)$.

Conclusion of the proof of Theorem 1. 1

1° Assume $m > 0$. If $J'(u) \neq 0$, $\forall u \in H$, by Lemma 4. 3, $\exists h_n \in \Lambda$, $n = 2, 3, \ldots$, satisfying (4. 4) (5. 5).

Set

$$A \triangleq \{x \in S^2 \mid -8\pi \log(4\pi R(x)) = \nu,\ \vec{\nabla} R(x) \neq 0\}$$

we claim that $\forall \varepsilon > 0$, $\exists n_0 \in \mathbf{Z}_+$ such that

$$\text{dist}(P(w), A) \leq \varepsilon \quad \forall w = h_n(t),\ n \geq n_0,\ 0 < t < 1 \quad \text{with} \quad J(w) \geq \nu - \frac{\rho}{n^2} \tag{4. 6}$$

If (4. 6) were false, then $\exists \varepsilon_0 > 0$, $w_i = h_{n_i}(t_i)$, $i = 2, 3, \ldots$, $n_i \to +\infty$, $0 < t_i < 1$

$$J(w_i) \geq \nu - \frac{\rho}{n_i^2}, \qquad \text{dist}(P(w_i), A) > \varepsilon_0 \tag{4. 7}$$

By Lemma 4. 3, $\exists \bar{w}_i$, $i = 2, 3, \ldots$, satisfying (4. 5), then $|P(\bar{w}_i)| \to 1$, otherwise we should have a subsequence, still denoted by $\{\bar{w}_i\}$ with $|P(\bar{w}_i)| \leq 1 - \delta < 1$, using [5] Prop. 2. 1 for $\{\bar{w}_i\}$ we should have

$$\tilde{w}_i \triangleq \bar{w}_i - \frac{1}{4\pi} \int_{S^2} \bar{w}_i \to w_0 (H^1(S^2))$$

strongly and $J'(w_0) = \theta$ a contradiction.

Using [5] Prop. 4. 3. for $\{\bar{w}_i\}$ we obtain

$$P(\bar{w}_i) \to E \triangleq \{x \in S^2 \mid -8\pi \log(4\pi R(x)) = \nu\} \qquad \text{as} \quad i \to \infty$$

From $\| w_i - \bar{w}_i \|_{H^1} \leq \frac{1}{n_i}$ and (4. 7) noting $\int_{S^2} |\nabla \bar{w}_i|^2 \to +\infty$, similar argument as in the proof of Lemma 2. 1, 1° shows $|P(w_i)| \to 1$. Using [5] Prop 4. 4 for $\{w_i\}$, $\{\bar{w}_i\}$, we can derive $P(w_i) \to E$ as $i \to \infty$. $\forall$ nondegenerate local maximum point $y \in E$, applying Lemma 3. 1, noting (4. 4) we see that no subsequence of $P(w_i)$ can tend to y. Thus $P(w_i) \to A$ as $i \to \infty$, contradicting (4. 7). Therefore (4. 6) is true. From (4. 6) applying [5] Lemma 5. 1 (a continuous flow on H) with slight modification (i. e. keep p, q fixed) for some $h_n \in \Lambda$, n being sufficiently large, we get $\bar{h}_n \in \Lambda$ with $\max_{t \in [0,1]} J(\bar{h}_n(t)) < \nu$ contradicting (4. 3). Hence J must have a critical point.

2° When $m \leq 0$, instead of Lemma 4. 2, using Lemma 4. 1, we see that $\nu < +\infty$, we don't need assumption ii) and assume iii) only on $R^{-1}(0, R(b))$, the rest of proof is the same as above.

References

[1] Th. Aubin, Nonlinear analysis on manifolds, Monge-Ampere Equations, Springer-Verlag, New York, 1982.

[2] Th. Aubin, Meilleures constants dans le théorème d'inclusion de Sobolev, *J. Funct. Anal.* 32 (1979), 148—174.

[3] S. Y. A. Chang, Extremal functions in a sharp form of Sobolev inequality, preprint.

[4] Chen Wen-xiong, A problem concerning the scalar curvature on S^2, preprint.

[5] Chen Wen-xiong & Ding Weiyue, Scalar curvature on S^2, to appear in Trans. Amer. Math. Soc.

[6] Hong Chongwei, A best constant and the Gaussian curvature, *Proc. A. M. S.* Vol. 97, *No.* 4 (1986), 737—747.

[7] J. Kazdan and F. Warner, Curvature functions for compact 2-manifolds, *Ann. of Math.* (2) 99 (1974), 14—47.

[8] J. Kazdan and F. Warner, Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature, *Ann. of Math.* (2) 101 (1975), 317—331.

[9] J. Kazdan, Gaussian and scalar curvature, an update, Seminar on Differential

Geometry (S. T. Yau, ed.), Princeton Univ. Press, Princeton, N. J., (1982), 185—191.
〔10〕 J. Moser, A sharp form of an inequality of N. Trudinger, Indiana Univ. *Math. J.* 20 (1971), 1077—1092.
〔11〕 J. Moser, On a nonlinear problem in differential geometry, Dynamical Systems (M. Peixoto ed.), Academic Press, New York, 1973.
〔12〕 E. Onofri, On the positivity of the effective action in a theory of random surface, *Comm. Math. Phys.* 86 (1982), 321—326.
〔13〕 S. T. Yau, Survey on partial differential equations in differential geometry, Seminar on Differential Geometry (S. T. Yau ed.), Princeton Univ. Press, Princeton, N. J., (1982), 3-71.