J. Partial Differential Equations
Vol. 1 № 1 (1988), 1—11

INITIAL VALUE PROBLEMS FOR NONLINEAR HEAT EQUATIONS

Li Tatsien (Li Daqian) * and Chen Yunmei * *
Received August 29, 1986

1. Introduction

In this paper we deal with the global existence and uniqueness of classical solutions to the following initial value problem for nonlinear heat equations

$$\begin{cases} u_t - \Delta u = F(u,\ D_x u,\ D_x^2 u) & (t,\ x) \in R_+ \times R^n & (1\cdot 1) \\ t = 0:\ u = \varphi(x) & x \in R^n & (1\cdot 2) \end{cases}$$

where

$$D_x u = (u_{x_1},\ \dots,\ u_{x_n}), \qquad D_x^2 u = (u_{x_i x_j};\ i,\ j = 1,\ \dots,\ n) \tag{1·3}$$

and

$$\Delta u = \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}\right) u \tag{1·4}$$

Let

$$\widehat{\lambda} = (\lambda;\ (\lambda_i),\ i = 1,\ \dots,\ n;\ (\lambda_{ij}),\ i,\ j = 1,\ \dots,\ n) \tag{1·5}$$

Suppose that in a neighborhood of $\widehat{\lambda} = 0$, say, for $|\widehat{\lambda}| \le 1$, the nonlinear term $F = F(\widehat{\lambda})$ in (1. 1) is suitably smooth and

$$F(\widehat{\lambda}) = O(|\widehat{\lambda}|^{1+\alpha}) \tag{1·6}$$

where α is an integer ≥ 1.

Based on the Nash-Moser-Hormander iteration scheme, S. Klainerman ([1]) first proved the following result in 1982: If

$$\frac{1}{\alpha}\left(1 + \frac{1}{\alpha}\right) < \frac{n}{2} \tag{1·7}$$

then problem (1. 1) - (1. 2) admits a unique global classical solution on $t \geq 0$, provided that the initial data are small. One year later, S. Klainerman and G. Ponce ([2]) reproved the same result, just using the continuation method of local solutions instead of the Nash-Moser-Hormander iteration.

Observing that for the solution to the heat equation, not only its L^∞-norm but also its L^2-norm decay as $t \to +\infty$, Zheng and Chen ([3]) and G. Ponce ([4]) improved almost at the same time the preceding result by replacing hypothesis (1. 7) with

$$\frac{1}{\alpha} < \frac{n}{2} \tag{1·8}$$

To get this improvement, the former still adopted the Nash-Moser-Hormander scheme while the latter used the continuation method of local solutions.

It must be pointed out that in general hypothesis (1. 8) is necessary. As a matter of fact, for the following initial value problem

$$\begin{cases} u_t - \Delta u = u^{1+\alpha}, & (t,\ x) \in R_+ \times R^n & (1\cdot 9) \\ t = 0:\ u = \varphi(x), & x \in R^n & (1\cdot 10) \end{cases}$$

H. Fujita ([5]) and F. B. Weissler ([6]) have proved that if

* Dept. of Math. and Institute of Math., Fudan University, Shanghai, P. R. C.;
Dept. of Math., UCLA, U. S. A.
* * Dept. of Appl. Math., Tongji University, Shanghai, P. R. C.

$$\frac{1}{\alpha} \geq \frac{n}{2} \tag{1·11}$$

then the classical solution may blow up in a finite time even for sufficiently small initial data.

Moreover, in the case that the nonlinear term F in (1. 1) does not explicitly depend on u : $F = F(D_x u, D_x^2 u)$, without any limitation on the dimension $n \geq 1$, Zheng ([7]) has used once again the Nash-Moser-Hormander scheme to get the global existence of classical solutions for small initial data

In this paper, we give a simple proof to the preceding results, which avoids the use of either the Nash-Moser-Hormander technique or the existence of local solutions. Only based on the decay estimates of solutions to the linear homogeneous heat equation and the energy estimates of solutions to linear inhomogeneous heat equations, we can directly obtain the global existence of classical solutions and some more precise asymptotic behaviors of solutions as $t \to +\infty$. For this purpose, all we have to do is to introduce a function space reflecting simultaneously both the properties of decay and the energy estimates of solutions to corresponding linear problems, and to use the ordinary contraction mapping principle in this space to prove for small initial data the global convergence of the sequence of approximate solutions given by an usual iteration. The method mentioned above can be also systematically used to other nonlinear evolution equations.

2. Preliminaries

Consider the initial value problem for inhomogeneous heat equations

$$\begin{cases} u_t - \Delta u = F(t, x), & (t, x) \in R_+ \times R^n \quad (2·1) \\ t = 0: & u = \varphi(x), \quad x \in R^n \quad (2·2) \end{cases}$$

by means of Galerkin' s method we can get

Lemma 2. 1. *For any given $T > 0$, if*

$$\varphi \in H^{s+1}(R^n), \quad F \in L^2(0, T; H^s(R^n)), \tag{2·3}$$

where s is an integer ≥ 0, then problem (2. 1) - (2. 2) admits a unique solution $u = u(t, x)$ satisfying

$$u \in L^2(0, T; H^{s+2}(R^n)) \tag{2·4}$$

$$u_t \in L^2(0, T; H^s(R^n)) \tag{2·5}$$

and

$$\int_0^T \sum_{|k|=2} \| D_x^k u(t, \cdot) \|^2_{H^s(R^n)} dt$$

$$\leq C_0 \left(\| \varphi \|^2_{H^{s+1}(R^n)} + \int_0^T \| F(t, \cdot) \|^2_{H^s(R^n)} dt \right) \tag{2·6}$$

where C_0 is a positive constant independent of T, $k = (k_1, ..., k_n)$ is a multi-index,

$$|k| = k_1 + ... + k_n \tag{2·7}$$

and

$$D_x^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} .... \partial x_n^{k_n}} \tag{2·8}$$

Corollary 2. 1. *By (2. 4) - (2. 5), we have, with eventual modification on a set of measure zero on $[0, T]$,*

$$u \in C([0, T]; H^{s+1}(R^n)) \tag{2·9}$$

Then, if we suppose furthermore that

$$F \in C([0, T]; H^{s-1}(R^n)) \tag{2·10}$$

by equation (2. 1) we have, with eventual modification on a set of; measure zero on $[0, T]$,

$$u_t \in C([0, T]; H^{s-1}(R^n)) \tag{2·11}$$

We turn now to the decay estimates of solutions to initial value problems for the homogeneous heat equation

$$\begin{cases} u_t - \Delta u = 0, & (t, x) \in R_+ \times R^n \\ t = 0 : u = \varphi(x), & x \in R^n \end{cases} \qquad \begin{matrix}(2 \cdot 12)\\(2 \cdot 13)\end{matrix}$$

The solution to problem (2. 12) - (2. 13) can be denoted as

$$u = S(t)\varphi \qquad (2 \cdot 14)$$

where

$$S(t) : \varphi \to u(t, \cdot) \qquad (2 \cdot 15)$$

is the linear operator defined by

$$u(t, x) = \frac{1}{(2\sqrt{\pi t})^n} \int_{R^n} e^{-|x-\xi|^2/4t} \varphi(\xi)\, d\xi \qquad (2 \cdot 16)$$

in which $\xi = (\xi_1, \ldots, \xi_n)$ and $|x-\xi|^2 = \sum_{i=1}^{n} (x_i - \xi_i)^2$.

By Young's inequality (cf. [8]), it is easy to get

Lemma 2. 2. *For the solution (2. 14) to problem (2. 12) - (2. 13), we have, under the assumption that the norm appearing on the right hand side below is bounded*

$$\| D_x^k (S(t)\varphi) \|_{L^q(R^n)} \le C_0 t^{-\frac{1}{2}(|k| + n(\frac{1}{p} - \frac{1}{q}))} \| \varphi \|_{L^p(R^n)}, \ \forall\, t > 0 \qquad (2 \cdot 17)$$

where k is an arbitrary multi-index, C_0 is a positive constant independent of t and

$$1 < p,\ q < +\infty \qquad (2 \cdot 18)$$

The combination of Lemma 2. 2 and the usual energy estimates gives

Corollary 2. 2. *For the solution (2. 14) to problem (2. 12) - (2. 13), if the norm appearing on the right hand side below makes sense, we have*

$$\| D_x^k (S(t)\varphi) \|_{L^2(R^n)} \le C(1+t)^{-\frac{|k|}{2}} \| \varphi \|_{H^k(R^n)}, \ \forall\, t \ge 0 \qquad (2 \cdot 19)$$

where k is an arbitrary multi-index and C is a positive constant independent of t.

Moreover, directly using the preceding expression (2. 16) and noting the Sobolev embedding theorem, for the L^∞-norm and the L^1-norm of the solution, we can obtain

Lemma 2. 3. *Let N be an arbitrary nonnegative integer. For the solution (2. 14) to problem (2. 12) - (2. 13), if all norms appearing on the right hand side below are bounded, we have*

$$\| D_x^k (S(t)\varphi) \|_{W^{N,\infty}(R^n)} \le C_0 (1+t)^{-\frac{n+|k|}{2}} \| \varphi \|_{W^{N+n+|k|+1,1}(R^n)}, \ \forall\, t \ge 0 \ (|k| = 0, 1) \qquad (2 \cdot 20)$$

and

$$\| D_x^k (S(t)\varphi) \|_{W^{N,1}(R^n)} \le C_0 (1+t)^{-\frac{|k|}{2}} \| \varphi \|_{W^{N+|k|,1}(R^n)}, \ \forall\, t \ge 0 \ (|k| = 0, 1) \qquad (2 \cdot 21)$$

where C_0 is a positive constant independent of t.

Now we give some estimates about product functions and composite functions, which can be proved (cf. [1]) by means of Holder's inequality and Nirenberg's inequality ([9]).

Lemma 2. 4. *Suppose that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \quad 1 \le p,\ q,\ r \le +\infty \qquad (2 \cdot 22)$$

If all norms appearing on the right hand side below are bounded, then for any given integer $s \ge 0$, we have

$$\| D^s (fg) \|_{L^r(R^n)} \le C_s (\| f \|_{L^p(R^n)} \| D^s g \|_{L^q(R^n)} + \| D^s f \|_{L^q(R^n)} \| g \|_{L^p(R^n)}) \qquad (2 \cdot 23)$$

and for any given integer $s \ge 1$, we have

$$\| D^s (fg) - f D^s g \|_{L^r(R^n)} \le C_s (\| Df \|_{L^p(R^n)} \| D^{s-1} g \|_{L^q(R^n)} + \| D^s f \|_{L^q(R^n)} \| g \|_{L^p(R^n)}) \qquad (2 \cdot 24)$$

where C_s is a positive constant only depending on s.

Corollary 2. 3. *Under assumption (2. 22) if all norms appearing on the right hand side below are bounded, then for any given integer $s \ge 0$, we have*

$$\| fg \|_{W^{s,r}(R^n)} \leq C_s (\| f \|_{L^p(R^n)} \| g \|_{W^{s,q}(R^n)} + \| f \|_{W^{s,q}(R^n)} \| g \|_{L^p(R^n)}) \tag{2·25}$$

Lemma 2.5. *Suppose that $F = F(w)$ is a sufficiently smooth function of $w = (w_1, \dots, w_N)$ with*

$$F(0) = 0 \tag{2·26}$$

For any given integer $s \geq 0$, if a vector function $w = w(x)$ satisfies

$$w \in W^{s,p}(R^n), \quad 1 \leq p \leq +\infty \tag{2·27}$$

and

$$\| w \|_{L^\infty(R^n)} \leq M \tag{2·28}$$

where M is a positive constant, then the composite function

$$F(w) \in W^{s,p}(R^n) \tag{2·29}$$

and

$$\| F(w) \|_{W^{s,p}(R^n)} \leq C(M) \| w \|_{W^{s,p}(R^n)} \tag{2·30}$$

where $C(M)$ denotes a positive constant only depending on M.

Lemma 2.6. *Suppose that $F = F(w)$ is a sufficiently smooth function of $w = (w_1, \dots, w_N)$ satisfying that when*

$$|w| \leq \nu_0 \tag{2·31}$$

then

$$F(w) = O(|w|^{1+\alpha}) \quad (\alpha \geq 1 \ \text{integer}) \tag{2·32}$$

For any given integer $s \geq 0$, if a vector function $w = w(x)$ satisfies

$$\| w \|_{L^\infty(R^n)} \leq \nu_0 \tag{2·33}$$

and such that all norms appearing on the right hand side below are bounded, then

$$\| F(w) \|_{W^{s,r}(R^n)} \leq C_s \| w \|_{W^{s,q}(R^n)} \| w \|_{L^p(R^n)} \| w \|^{\alpha-1}_{L^\infty(R^n)} \tag{2·34}$$

where C_s is a positive constant (depending on ν_0) and p, q, r satisfy (2.22).

Lemma 2.7. *Suppose that $F = F(w)$ is a sufficiently smooth function of $w = (w_1, \dots, w_N)$ satisfying (2.31)-(2.32). For any given integer $s \geq 0$, if vector functions $w = \overline{w}(x)$ and $w = \overline{\overline{w}}(x)$ satisfy (2.33) respectively and such that all norms appearing on the right hand side below are bounded, then*

$$\begin{aligned} &\| F(\overline{w}) - F(\overline{\overline{w}}) \|_{W^{s,r}(R^n)} \\ &\leq C_s \{ \| w^* \|_{L^p(R^n)} (\| \overline{w} \|_{W^{s,q}(R^n)} + \| \overline{\overline{w}} \|_{W^{s,q}(R^n)}) \\ &\quad + \| w^* \|_{W^{s,q}(R^n)} (\| \overline{w} \|_{L^p(R^n)} + \| \overline{\overline{w}} \|_{L^p(R^n)}) \} \\ &\quad \cdot (\| \overline{w} \|_{L^\infty(R^n)} + \| \overline{\overline{w}} \|_{L^\infty(R^n)})^{\alpha-1} \end{aligned} \tag{2·35}$$

in which

$$w^* = \overline{w} - \overline{\overline{w}} \tag{2·36}$$

p, q, r satisfy (2.22) and C_s is a positive constant depending on ν_0.

3. Initial Value Problem (1.1)-(1.2) for Nonlinear Heat Equations

For any given integer $s \geq n + 5$ and positive constant E, introduce the following set of functions

$$X_{s,E} = \{ v = v(t, x) \mid D_s(v) \leq E \} \tag{3·1}$$

where

$$D_s(v) = \sup_{t \geq 0} (1+t)^{n/2} \| v(t, \cdot) \|_{W^{s-n-3,\infty}(R^n)} + \sup_{t \geq 0} \| v(t, \cdot) \|_{W^{s,1}(R^n)} + \left(\int_0^\infty \sum_{|k|=2} \| D_x^k (v(t, \cdot)) \|^2_{H^s(R^n)} dt \right)^{1/2} \tag{3·2}$$

It is easy to prove

Lemma 3.1. *Endowed with the metric*

$$\rho(\overline{v}, \overline{\overline{v}}) = D_s(\overline{v} - \overline{\overline{v}}), \ \forall \ \overline{v}, \overline{\overline{v}} \in X_{s,E} \tag{3·3}$$

$X_{s,E}$ is a nonempty complete metric space.

By the definition of $X_{s,E}$, if $E \leq 1$, then

$$\sup_{t\geq 0} \| v(t, \cdot) \|_{W^{2,\infty}(R^n)} \leq 1, \qquad \forall\, v \in X_{s,E} \tag{3·4}$$

Hence, for any $v \in X_{s,E}$ (with $E \leq 1$), we can always use hypothesis (1.6) for $F(v, D_x v, D_x^2 v)$. The main result in this section is

Theorem 3.1. *Suppose that the nonlinear function F on the right hand side of (1.1) satisfies (1.6), and (1.8) holds. For any given integer $s \geq n+5$, there exist suitably small positive constants δ and $E (E \leq 1)$ such that if*

$$\varphi \in W^{s,1}(R^n) \cap H^{s+1}(R^n) \tag{3·5}$$

and

$$\| \varphi \|_{W^{s,1}(R^n)} + \| \varphi \|_{H^{s+1}(R^n)} \leq \delta E \tag{3·6}$$

then the initial value problem (1.1)-(1.2) admits a unique global solution $u \in X_{s,E}$ on $t \geq 0$. Moreover, with eventual modification on a set of measure zero on $[0, \infty)$, for any $T > 0$, we have

$$u \in L^2(0, T; H^{s+2}(R^n)) \cap C([0, T]; H^{s+1}(R^n)) \tag{3·7}$$

$$u_t \in L^2(0, T; H^s(R^n)) \cap C([0, T]; H^{s-1}(R^n)) \tag{3·8}$$

Remark 3.1. Using the Sobolev embedding theorem (observe that $s \geq n+5$), it easily follows from (3.7) - (3.8) that the solution given in Theorem 3.1 is the global classical solution to problem (1.1) - (1.2). Besides, according to the definition of $X_{s,E}$, it can be seen that the solution possesses the same decay rate when $t \to +\infty$ as the solution to the initial value problem (2.12) - (2.13) for the linear homogeneous heat equation.

We now prove Theorem 3.1

Let s be a given integer $\geq n+5$ and $E (E \leq 1)$ a suitably small positive number to be determined later on. For any function

$$v \in X_{s,E} \tag{3·9}$$

we define a map

$$\hat{T} : v \to u = \hat{T} v \tag{3·10}$$

by solving the following initial value problem for inhomogeneous heat equations

$$\begin{cases} u_t - \Delta u = F(\Lambda v), \quad (t, x) \in R_+ \times R^n & (3·11) \\ t = 0 : u = \varphi(x), \quad x \in R^n & (3·12) \end{cases}$$

where

$$\Lambda v = (v, D_x v, D_x^2 v) \tag{3·13}$$

We shall prove that when δ and E are chosen to be suitably small, $\hat{T}$ is a contraction map from $X_{s,E}$ into itself, then the Banach fixed-point theorem can be used to get the desired conclusion.

First of all, using Lemma 2.1 we can easily prove

Lemma 3.2. *For any $v \in X_{s,E}$, with eventual modification on a set of measure zero on $[0, \infty)$, for any $T > 0$, we have*

$$u = \hat{T} v \in L^2(0, T; H^{s+2}(R^n)) \cap C([0, T]; H^{s+1}(R^n)) \tag{3·14}$$

$$u_t \in L^2(0, T; H^s(R^n)) \tag{3·15}$$

Lemma 3.3. *$\hat{T}$ maps $X_{s,E}$ into itself, provided that δ and E are suitably small.*

Proof. By Duhamel's principle, the solution to problem (3.11) - (3.12) can be expressed in the form

$$u = \hat{T} v = S(t)\varphi + \int_0^t S(t-\tau) F(\Lambda v(\tau, \cdot))\, d\tau \tag{3·16}$$

Note that it follows from (2.34) and the definition of $X_{s,E}$ that

$$\| F(\Lambda v(\tau, \cdot)) \|_{W^{s-2,1}(R^n)} \leq C \| v(\tau, \cdot) \|_{W^{s,1}(R^n)} \| v(\tau, \cdot) \|^{\alpha}_{W^{2,\infty}(R^n)}$$

$$\leq C E^{1+\alpha} (1+\tau)^{-\frac{n\alpha}{2}} \tag{3·17}$$

here and hereafter C denotes a constant. Moreover, under hypothesis (1.8) we have

$$\int_0^t (1+t-\tau)^{-\frac{n}{2}}(1+\tau)^{-\frac{n\alpha}{2}}d\tau \leq C(1+t)^{-\frac{n}{2}} \tag{3·18}$$

Then, by means of estimate (2. 20) (in which we take $|k|=0$, $N=s-n-3$) and noting (3. 6), we get from (3. 16) that

$$\begin{aligned}
&\| u(t, \cdot) \|_{W^{s-n-3,\infty}(R^n)} \\
&\leq C(1+t)^{-\frac{n}{2}} \| \varphi \|_{W^{s-2,1}(R^n)} \\
&\quad + C\int_0^t (1+t-\tau)^{-\frac{n}{2}} \| F(\Lambda v(\tau, \cdot)) \|_{W^{s-2,1}(R^n)} d\tau \\
&\leq C\delta E(1+t)^{-\frac{n}{2}} + CE^{1+\alpha}\int_0^t (1+t-\tau)^{-\frac{n}{2}}(1+\tau)^{-\frac{n\alpha}{2}}d\tau \\
&\leq C_1(1+t)^{-\frac{n}{2}}(\delta E + E^{1+\alpha}), \quad \forall\, t \geq 0
\end{aligned} \tag{3·19}$$

where C_1 is a positive constant.

Similarly, note that it follows from (2. 34) and the definition of $X_{s,E}$ that

$$\begin{aligned}
\| F(\Lambda v(\tau, \cdot)) \|_{W^{s,1}(R^n)} &\leq C \| v(\tau, \cdot) \|^2_{H^{s+2}(R^n)} \| v(\tau, \cdot) \|^{\alpha-1}_{W^{2,\infty}(R^n)} \\
&\leq CE^{\alpha-1}(1+\tau)^{-\frac{n(\alpha-1)}{2}} \| v(\tau, \cdot) \|^2_{H^{s+2}(R^n)}
\end{aligned} \tag{3·20}$$

and

$$\begin{aligned}
\| v(\tau, \cdot) \|^2_{H^{s+2}(R^n)} &\leq C \| v(\tau, \cdot) \|_{W^{s-n-3,\infty}(R^n)} \| v(\tau, \cdot) \|_{W^{s,1}(R^n)} + \\
&\quad + \sum_{|k|=2} \| D_x^k v(\tau, \cdot) \|^2_{H^s(R^n)} \\
&\leq CE^2(1+\tau)^{-\frac{n}{2}} + \sum_{|k|=2} \| D_x^k v(\tau, \cdot) \|^2_{H^s(R^n)}
\end{aligned} \tag{3·21}$$

Moreover, under hypothesis (1. 8) we have

$$\int_0^t (1+\tau)^{-\frac{n\alpha}{2}}d\tau \leq C \tag{3·22}$$

Then, by means of estimate (2. 21) (in which we take $|k|=0$, $N=s$) and noting (3. 6), we get again from (3. 16) that

$$\| u(t, \cdot) \|_{W^{s,1}(R^n)} \leq \delta E + C_2 E^{1+\alpha}, \ \forall\, t \geq 0 \tag{3·23}$$

where C_2 is a positive constant.

Finally, according to (2. 34) and the definition of $X_{s,E}$, it holds that

$$\begin{aligned}
\| F(\Lambda v(\tau, \cdot)) \|_{H^s(R^n)} &\leq C \| v(\tau, \cdot) \|_{H^{s+2}(R^n)} \| v(\tau, \cdot) \|^{\alpha}_{W^{2,\infty}(R^n)} \\
&\leq CE^{\alpha}(1+\tau)^{-\frac{n\alpha}{2}} \| v(\tau, \cdot) \|_{H^{s+2}(R^n)}
\end{aligned} \tag{3·24}$$

Furthermore, under hypothesis (1. 18) we have

$$\int_0^\infty (1+\tau)^{-\frac{(1+2\alpha)}{2}n}d\tau \leq C \tag{3·25}$$

Then, noticing (3. 6), it follows from (2. 6) that

$$\left(\int_0^\infty \sum_{|k|=2} \| D_x^k u(t, \cdot) \|^2_{H^s(R^n)} dt \right)^{1/2} \leq C_3(\delta E + E^{1+\alpha}) \tag{3·26}$$

where C_3 is a positive constant.

The combination of (3. 19), (3. 23) and (3. 26) gives $u \in X_{s,E}$, provided that δ and E are suitably small. The proof of Lemma 3. 2 is complete.

Lemma 3. 3. *$\hat{T}$ is a contraction map in $X_{s,E}$, provided that δ and E are suitably small.*

Proof: For any $\bar{v}$, $\bar{\bar{v}} \in X_{s,E}$, by Lemma 3. 2, if δ and E are suitably small, we have

$$\bar{u} = \hat{T}\bar{v}, \ \bar{\bar{u}} = \hat{T}\bar{\bar{v}} \in X_{s,E} \tag{3·27}$$

Let

$$v^* = \bar{v} - \bar{\bar{v}}, \ u^* = \bar{u} - \bar{\bar{u}} \tag{3·28}$$

we want to prove that if δ and E are suitably small, then there exists a positive constant $\eta < 1$ such that

$$D_s(u^*) \leq \eta D_s(v^*) \tag{3·29}$$

By the definition of $\widehat{T}$, we have

$$\begin{cases} u_t^* - \Delta u^* = F(\Lambda \bar{v}) - F(\Lambda \bar{\bar{v}}), & (t, x) \in R_+ \times R^n \\ t = 0:\ u^* = 0, & x \in R^n \end{cases} \qquad \begin{matrix} (3 \cdot 30) \\ (3 \cdot 31) \end{matrix}$$

It follows from Lemma 2. 7 that

$$\begin{aligned} &\| F(\Lambda \bar{v}(\tau, \cdot)) - F(\Lambda \bar{\bar{v}}(\tau, \cdot)) \|_{W^{s-2,1}(R^n)} \\ &\leq C\{ \| v^*(\tau, \cdot) \|_{W^{s,1}(R^n)} \\ &\quad \cdot \| \tilde{v}(\tau, \cdot) \|^{\alpha-1}_{W^{2,\infty}(R^n)} \\ &\quad + \| v^*(\tau, \cdot) \|_{W^{s-n-3,\infty}(R^n)} \| \tilde{v}(\tau, \cdot) \|_{W^{s,1}(R^n)} \\ &\quad \cdot \| \tilde{v}(\tau, \cdot) \|^{\alpha-1}_{W^{2,\infty}(R^n)} \} \end{aligned} \qquad (3 \cdot 32)$$

$$\begin{aligned} &\| F(\Lambda \bar{v}(\tau, \cdot)) - F(\Lambda \bar{\bar{v}}(\tau, \cdot)) \|_{W^{s,1}(R^n)} \\ &\leq C\{ \| v^*(\tau, \cdot) \|_{H^{s+2}(R^n)} \| \tilde{v}(\tau, \cdot) \|_{H^2(R^n)} \\ &\quad + \| v^*(\tau, \cdot) \|_{H^2(R^n)} \| \tilde{v}(\tau, \cdot) \|_{H^{s+2}(R^n)} \} \\ &\quad \cdot \| \tilde{v}(\tau, \cdot) \|^{\alpha-1}_{W^{2,\infty}(R^n)} \end{aligned} \qquad (3 \cdot 33)$$

$$\begin{aligned} &\| F\Lambda \bar{v}(\tau, \cdot)) - F(\Lambda \bar{\bar{v}}(\tau, \cdot)) \|_{H^s(R^n)} \\ &\leq C\{ \| v^*(\tau, \cdot) \|_{H^{s+2}(R^n)} \| \tilde{v}(\tau, \cdot) \|_{W^{2,\infty}(R^n)} \\ &\quad + \| v^*(\tau, \cdot) \|_{W^{2,\infty}(R^n)} \| \tilde{v}(\tau, \cdot) \|_{H^{s+2}(R^n)} \} \\ &\quad \cdot \| \tilde{v}(\tau, \cdot) \|^{\alpha-1}_{W^{2,\infty}(R^n)} \end{aligned} \qquad (3 \cdot 34)$$

here and hereafter, for abbreviating, we write

$$\| D_x^k \tilde{v}(\tau, \cdot) \|_{W^{m,p}(R^n)} = \| D_x^k \bar{v}(\tau, \cdot) \|_{W^{m,p}(R^n)} + \| D_x^k \bar{\bar{v}}(\tau, \cdot) \|_{W^{m,p}(R^n)} \qquad (3 \cdot 35)$$

Moreover, noting the definition of $X_{s,E}$ and $D_s(v)$, we have

$$\| v^*(\tau, \cdot) \|_{H^{s+2}(R^n)} \leq C(1+\tau)^{-\frac{n}{4}} D_s(v^*) + \Big(\sum_{|k|=2} \| D_x^k v^*(\tau, \cdot) \|^2_{H^s(R^n)} \Big)^{1/2} \qquad (3 \cdot 36)$$

$$\| v^*(\tau, \cdot) \|_{H^2(R^n)} \leq C(1+\tau)^{-\frac{n}{4}} D_s(v^*) \qquad (3 \cdot 37)$$

and some similar estimates for $\bar{v}$ and $\bar{\bar{v}}$

Using estimates (2. 20) and (3. 32), (2. 21) and (3. 33), and (2. 6) and (3. 34) respectively to problem (3. 30) - (3. 31), similar to the proof of Lemma 3. 2 we can get

$$\sup_{t \geq 0} (1+t)^{\frac{n}{2}} \| u^*(t, \cdot) \|_{W^{s-n-3,\infty}(R^n)} \leq C_1 E^\alpha D_s(v^*) \qquad (3 \cdot 38)$$

$$\sup_{t \geq 0} \| u^*(t, \cdot) \|_{W^{s,1}(R^n)} \leq C_2 E^\alpha D_s(v^*) \qquad (3 \cdot 39)$$

$$\Big(\int_0^\infty \sum_{|k|=2} \| D_x^k u^*(\tau, \cdot) \|^2_{H^s(R^n)} d\tau \Big)^{1/2} \leq C_3 E^\alpha D_s(v^*) \qquad (3 \cdot 40)$$

where C_1, C_2 and C_3 are positive constants. The combination of (3. 38) - (3. 40) then leads to (3. 29), if $E > 0$ is suitably small. Lemma 3. 3 is proved.

Now we use Lemmas 3. 2 and 3. 3 to finish the proof of Theorem 3. 1. In fact, according to the Banach fixed point theorem, if δ and E are suitably small, $\widehat{T}$ should have a unique fixed point $u = \widehat{T}u \in X_{s,E}$ which is nothing but the unique solution to the problem (1. 1) - (1. 3). Moreover, by (3. 14) - (3. 15) we have

$$F(\Lambda u) \in C([0, T];\ H^{s-1}(R^n)),\ \forall\ T > 0 \qquad (3 \cdot 41)$$

Hence, it comes from Corollary 2. 1 that

$$u_t \in C([0, T];\ H^{s-1}(R^n)),\ \forall\ T > 0 \qquad (3 \cdot 42)$$

This finishes the proof of Theorem 3. 1.

4. Special Cases in Which the Nonlinear Term F Does Not Explicitly Depend on u

In this section we consider the following initial value problem

$$\begin{cases} u_t - \Delta u = F(D_x u,\ D_x^2 u), & (t,\ x) \in \mathbf{R}_+ \times \mathbf{R}^n \quad (4\cdot 1) \\ t = 0:\ u = \varphi(x), & x \in \mathbf{R}^n \quad (4\cdot 2) \end{cases}$$

in which the nonlinear term $F = F(D_x u,\ D_x^2 u)$ does not explicitly depend on u.

Let

$$\hat{\lambda} = ((\lambda_i),\ i = 1,\ \dots,\ n;\ (\lambda_{ij}),\ i,\ j = 1,\ \dots,\ n) \qquad (4\cdot 3)$$

Still suppose that in a neighborhood of $\hat{\lambda} = 0$, say, for $|\hat{\lambda}| \leq 1$, $F = F(\hat{\lambda})$ is a sufficiently smooth function with

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}) \qquad (\alpha \geq 1,\ \text{integer}) \qquad (4\cdot 4)$$

We want to prove that in this special case, for any dimension $n \geq 1$, the initial value problem (4. 1) - (4. 2) with small initial data always admits a unique global classical solution on $t \geq 0$ and the solution possesses some corresponding decay properties as $t \to +\infty$. In order to get this result, we need some more refined estimates.

For any given integer $s \geq n+7$ and positive constant E, introduce the following set of functions

$$Y_{s,E} = \{v = v(t,\ x)\ |\ D_s(v) \leq E\} \qquad (4\cdot 5)$$

where

$$\begin{aligned} D_s(v) = & \sup_{t \geq 0} (1+t)^{\frac{n+1}{2}} \| D_x v(t,\ \cdot) \|_{W^{s-n-6,\infty}(R^n)} \\ & + \sup_{t \geq 0} (1+t)^{\frac{1}{2}} \| D_x v(t,\ \cdot) \|_{W^{s-3,1}(R^n)} \\ & + \sup_{t \geq 0} \sum_{|k| \leq s} (1+t)^{\beta(k)} \| D_x v(t,\ \cdot) \|_{L^2(R^n)} \\ & + \left(\int_0^\infty \sum_{|k|=2} \| D_x^k v(t,\ \cdot) \|^2_{H^s(R^n)} dt \right)^{\frac{1}{2}} \end{aligned} \qquad (4\cdot 6)$$

in which

$$\beta(k) = \begin{cases} \dfrac{|k|}{2}, & \text{if } \ |k| \leq n+2 \\ \dfrac{n}{2} + 1, & \text{if } \ n+2 < |k| \leq s \end{cases} \qquad (4\cdot 7)$$

Endowed with the metric

$$\rho(\bar{v},\ \bar{\bar{v}}) = D_s(\bar{v} - \bar{\bar{v}}),\ \forall\ \bar{v},\ \bar{\bar{v}} \in Y_{s,E} \qquad (4\cdot 8)$$

it is easy to see that $Y_{s,E}$ is a nonempty complete metric space.

By the definition of $Y_{s,E}$ and noting that $s \geq n+7$, if $E \leq 1$, then for any $v \in Y_{s,E}$ we have

$$\| D_x v(t,\ \cdot) \|_{W^{1,\infty}(R^n)} \leq 1,\ \forall\ t \geq 0 \qquad (4\cdot 9)$$

Hence, for any $v \in Y_{s,E}$ (with $E \leq 1$) we can always use hypothesis (4. 4) to $F(D_x v,\ D_x^2 v)$.

Theorem 4. 1. *Suppose that the nonlinear function F is sufficiently smooth and satisfies (4. 4). Without any limitation on the dimension $n \geq 1$, for any given integer $s \geq n+7$, there exist positive numbers δ and E ($E \leq 1$) so small that if*

$$\varphi \in H^{s+1}(R^n) \cap W^{s-2,1}(R^n) \qquad (4\cdot 10)$$

and

$$\| \varphi \|_{H^{s+1}(R^n)} + \| \varphi \|_{W^{s-2,1}(R^n)} \leq \delta E \qquad (4\cdot 11)$$

then problem (4. 1) - (4. 2) admits a unique global solution $u \in Y_{s,E}$ on $t \geq 0$. Moreover, with eventual modification on a set of measure zero on $[0,\ \infty)$, for any $T > 0$, we have

$$u \in L^2(0,\ T;\ H^{s+2}(R^n)) \cap C([0,\ T];\ H^{s+1}(R^n)) \qquad (4\cdot 12)$$

$$u_t \in L^2(0,\ T;\ H^s(R^n)) \cap C([0,\ T];\ H^{s-1}(R^n)) \qquad (4\cdot 13)$$

Obviously, in order to prove Theorem 4. 1, we only need to consider the case $\alpha = 1$. For any $v \in Y_{s,E}$, we define a map

$$\hat{T}:\ v \to u = \hat{T} v \qquad (4\cdot 14)$$

by solving the following initial value problem for inhomogeneous heat equations

$$\begin{cases} u_t - \Delta u = F(\Lambda v), & (t, x) \in \mathbf{R}_+ \times \mathbf{R}^n \\ t = 0 : u = \varphi(x), & x \in \mathbf{R}^n \end{cases} \quad \begin{matrix} (4 \cdot 15) \\ (4 \cdot 16) \end{matrix}$$

where

$$\Lambda v = (D_x v, D_x^2 v) \quad (4 \cdot 17)$$

We want to prove that $\widehat{T}$ is a contraction map from $Y_{s,E}$ into itself.

By the definition of $Y_{s,E}$, it is easy to get

Lemma 4. 1: *For any $v \in Y_{s,E}$, with eventual modification on a set of measure zero on $[0, \infty)$, for any $T > 0$ we have*

$$u = \widehat{T}v \in L^2(0, T; H^{s+2}(R^n)) \cap C([0, T]; H^{s+1}(R^n)) \quad (4 \cdot 18)$$

$$u_t \in L^2(0, T; H^s(R^n)) \quad (4 \cdot 19)$$

Lemma 4. 2. *$\widehat{T}$ maps $Y_{s,E}$ into itself, provided that δ and E are suitably small.*

Proof. For the solution $u = \widehat{T}v$ to problem (4. 15) - (4. 16), we still have (3. 16) and then

$$D_x u(t, \cdot) = D_x S(t)\varphi + \int_0^t D_x S(t-\tau) F(\Lambda v(\tau, \cdot)) d\tau \quad (4 \cdot 20)$$

By (2. 20) (in which we take $|k| = 1$, $N = s - n - 6$) and (2. 34) (in which we take $\alpha = 1$), noticing (4. 11) and the definition of $Y_{s,E}$, we can get from (4. 20) that

$$\begin{aligned} &\| D_x u(t, \cdot) \|_{W^{s-n-6,\infty}(R^n)} \\ &\leq C(1+t)^{-\frac{n+1}{2}} \| \varphi \|_{W^{s-6,1}(R^n)} \\ &\quad + C \int_0^t (1+t-\tau)^{-\frac{n+1}{2}} \| D_x v(\tau, \cdot) \|_{W^{s-3,1}(R^n)} \| D_x v(\tau, \cdot) \|_{W^{1,\infty}(R^n)} d\tau \\ &\leq C_1 (1+t)^{-\frac{n+1}{2}} (\delta E + E^2) \end{aligned} \quad (4 \cdot 21)$$

Similarly, by (2. 21) (in which we take $|k| = 1$, $N = s - 3$) and (2. 34) (in which we take $\alpha = 1$) and noting that

$$\begin{aligned} &\| D_x v(\tau, \cdot) \|^2_{H^{s-1}(R^n)} \\ &\leq C\{ \| D_x v(\tau, \cdot) \|_{L^1(R^n)} \| D_x v(\tau, \cdot) \|_{L^\infty(R^n)} \\ &\quad + \sum_{|k|=2}^{s} \| D_x^k v(\tau, \cdot) \|^2_{L^2(R^n)} \} \end{aligned} \quad (4 \cdot 22)$$

we have

$$\sup_{t \geq 0} (1+t)^{1/2} \| D_x u(t, \cdot) \|_{W^{s-3,1}(R^n)} \leq C_2 (\delta E + E^2) \quad (4 \cdot 23)$$

Next, using (2. 19) and noting that

$$\begin{aligned} &\| F(\Lambda v(\tau, \cdot)) \|_{H^{|k|}(R^n)} \\ &\leq C \| D_x v(\tau, \cdot) \|_{W^{1,\infty}(R^n)} (\| D_x v(\tau, \cdot) \|_{L^2(R^n)} + \sum_{|k|=2} \| D_x^k v(\tau, \cdot) \|_{H^{|k|}(R^n)}) \\ &\leq CE^2 (1+\tau)^{(-\frac{n}{2}+1)} + CE(1+\tau)^{-\frac{n+1}{2}} \sum_{|k|=2} \| D_x^k v(\tau, \cdot) \|_{H^s(R^n)} \end{aligned} \quad (4 \cdot 24)$$

we get

$$\begin{aligned} &\| D_x^k u(t, \cdot) \|_{L^2(R^n)} \\ &\leq C\delta E(1+t)^{-\beta(k)} + C \int_0^t (1+t-\tau)^{-\beta(k)} \| F(\Lambda v(\tau, \cdot)) \|_{H^{|k|}(R^n)} d\tau \\ &\leq C(\delta E + E^2)(1+t)^{-\beta(k)} \\ &\quad + CE \left(\int_0^t (1+t-\tau)^{-2\beta(k)} (1+\tau)^{-(n+1)} d\tau \right)^{1/2} \\ &\quad \cdot \left(\int_0^t \sum_{|k|=2} \| D_x^k v(\tau, \cdot) \|^2_{H^s(R^n)} d\tau \right)^{1/2} \\ &\leq C_3 (1+t)^{-\beta(k)} (\delta E + E^2), \qquad |k| \leq s, \ \forall\, t \geq 0 \end{aligned} \quad (4 \cdot 25)$$

Finally, by (2. 6) and noting (4. 24) we have

$$\left(\int_0^\infty \sum_{|k|=2} \| D_x^k u(t,\ \cdot) \|^2_{H^s(R^n)} dt\right)^{1/2} \leq C_4(\delta E + E^2) \tag{4·26}$$

The combination of (4. 21) (4. 23) and (4. 25) - (4. 26) leads to the desired result: $u = \widehat{T}v \in Y_{s,E}$, provided that δ and E are chosen to be small. The proof of Lemma 4. 2 is complete.

Lemma 4. 3. *If δ and E are suitably small, then $\widehat{T}$ is a contraction map in $Y_{s,E}$.*

Proof. For any $\bar{v}, \bar{\bar{v}} \in Y_{s,E}$, by Lemma 4. 2, if δ and E are small, we have

$$\bar{u} = T\bar{v},\ \bar{\bar{u}} = T\bar{\bar{v}} \in Y_{s,E} \tag{4·27}$$

Let

$$v^* = \bar{v} - \bar{\bar{v}},\ u^* = \bar{u} - \bar{\bar{u}} \tag{4·28}$$

we have

$$\begin{cases} u_t^* - \Delta u^* = F(\Lambda\bar{v}) - F(\Lambda\bar{\bar{v}}),\ (t,\ x) \in R_+ \times R^n & (4·29)\\ t = 0 : u^* = 0,\ x \in R^n & (4·30)\end{cases}$$

By (2. 20) and (noting (3. 35))

$$\begin{aligned}
&\| F(\Lambda\bar{v}(\tau,\ \cdot)) - F(\Lambda\bar{\bar{v}}(\tau,\ \cdot)) \|_{W^{s-4,1}(R^n)}\\
&\leq C(\| D_x v^*(\tau,\ \cdot) \|_{W^{s-3,1}(R^n)} \| D_x\tilde{v}(\tau,\ \cdot) \|_{W^{1,\infty}(R^n)}\\
&\quad + \| D_x v^*(\tau,\ \cdot) \|_{W^{1,\infty}(R^n)} \| D_x\tilde{v}(\tau,\ \cdot) \|_{W^{s-3,1}(R^n)})\\
&\leq CE(1+\tau)^{-(n/2+1)} D_s(v^*)
\end{aligned} \tag{4·31}$$

Similar to the proof of (4. 21), we can get

$$\sup_{t\geq 0} (1+t)^{\frac{n+1}{2}} \| D_x u^*(t,\ \cdot) \|_{W^{s-n-6,\infty}(R^n)} \leq C_1 E D_s(v^*) \tag{4·32}$$

Likewise, by (2. 21) and

$$\begin{aligned}
&\| F(\Lambda\bar{v}(\tau,\ \cdot)) - F(\Lambda\bar{\bar{v}}(\tau,\ \cdot)) \|_{W^{s-2,1}(R^n)}\\
&\leq C(\| D_x v^*(\tau,\ \cdot) \|_{H^1(R^n)} \| D_x\tilde{v}(\tau,\ \cdot) \|_{H^{s-1}(R^n)}\\
&\quad + \| D_x v^*(\tau,\ \cdot) \|_{H^{s-1}(R^n)} \| D_x\tilde{v}(\tau,\ \cdot) \|_{H^1(R^n)})\\
&\leq C \| D_x v^*(\tau,\ \cdot) \|_{H^{s-1}(R^n)} \| D_x\tilde{v}(\tau,\ \cdot) \|_{H^{s-1}(R^n)}\\
&\leq CE\{(1+\tau)^{-(\frac{n}{2}+1)} + (1+\tau)^{-2}\} D_s(v^*)
\end{aligned} \tag{4·33}$$

similar to the proof of (4. 23), we have

$$\sup_{t\geq 0} (1+t)^{1/2} \| D_x u^*(t,\ \cdot) \|_{W^{s-3,1}(R^n)} \leq C_2 E D_s(v^*) \tag{4·34}$$

Moreover, by (2. 19) and

$$\begin{aligned}
&\| F(\Lambda\bar{v}(\tau,\ \cdot)) - F(\Lambda\bar{\bar{v}}(\tau,\ \cdot)) \|_{H^{|k|}(R^n)}\\
&\leq C(\| D_x v^*(\tau,\ \cdot) \|_{W^{1,\infty}(R^n)} \| D_x\tilde{v}(\tau,\ \cdot) \|_{H^{|k|+1}(R^n)}\\
&\quad + \| D_x v^*(\tau,\ \cdot) \|_{H^{|k|+1}(R^n)} \| D_x\tilde{v}(\tau,\ \cdot) \|_{W^{1,\infty}(R^n)})\\
&\leq CE(1+\tau)^{-(\frac{n}{2}+1)} D_s(v^*)\\
&\quad + CE(1+\tau)^{-\frac{n+1}{2}} \sum_{|k|=2} \| D_x^k v^*(\tau,\ \cdot) \|_{H^s(R^n)}\\
&\quad + C(1+\tau)^{-\frac{n+1}{2}} D_s(v^*) \sum_{|k|=2} \| D_x^k\tilde{v}(\tau,\ \cdot) \|_{H^s(R^n)}
\end{aligned} \tag{4·35}$$

similar to the proof of (4. 25), we have

$$\sup_{t\geq 0} \sum_{|k|\leq s} (1+t)^{\beta(k)} \| D_x^k u^*(t,\ \cdot) \|_{L^2(R^n)} \leq C_3 E D_s(v^*) \tag{4·36}$$

Finally, by means of (2. 6) and

$$\begin{aligned}
&\| F(\Lambda\bar{v}(\tau,\ \cdot)) - F(\Lambda\bar{\bar{v}}(\tau,\ \cdot)) \|^2_{H^s(R^n)}\\
&\leq CE^2(1+\tau)^{-(n+2)} D_s^2(v^*) + CE^2 \sum_{|k|=2} \| D_x^k v^*(\tau,\ \cdot) \|^2_{H^s(R^n)}\\
&\quad + CD_s^2(v^*) \sum_{|k|=2} \| D_x^k\tilde{v}(\tau,\ \cdot) \|^2_{H^s(R^n)}
\end{aligned} \tag{4·37}$$

we get

$$\left(\int_0^\infty \sum_{|\lambda|=2} \| D_x^\lambda u^* (t, \cdot) \|_{H^s(R^n)}^2 dt \right)^{1/2} \leq C_4 E D_s(v^*) \tag{4·38}$$

The combination of (4. 32), (4. 34), (4. 36) and (4. 38) gives

$$D_s(u^*) \leq \eta D_s(v^*) \tag{4·39}$$

where η is a positive constant with $\eta < 1$, provided that E is small. This completes the proof of Lemma 4. 3.

The remainder in the proof of Theorem 4. 1 can be completely repeated as in the proof of Theorem 3. 1.

References

[1] S. Klainerman, Long-time behavior of solutions to nonlinear evolution equations, *Arch. Rat. Mech. Anal.*, 78 (1982), 73-96.

[2] S. Klainerman and G. Ponce, Global, small amplitude solutions to nonlinear evolution equations, *Comm. Pure Appl. Math.*, 36 (1983), 133-141.

[3] Zheng Songmu and Chen Yunmei, Global existence for nonlinear parabolic equations, *Chin. Ann. of Math.*, 7B (1986), 57-73.

[4] G. Ponce, Global existence of small solutions to a class of nonlinear evolution equations, *Nonlinear Analysis, Theory, Methods and Applications*, 9 (1985), 399-418.

[5] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo, Sec.* I, 13 (1966), 109-124.

[6] F. B. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, *Israel J. Math.*, 38 (1981), 29-40.

[7] Zheng Songmu, Remarks on global existence for nonlinear parabolic equations, *Nonlinear Analysis, Theory Methods and Applications*, 10 (1986), 107-114.

[8] J. Bergh and J. Lofstom, Interpolation spaces, An introduction, Springer-Verlag, Berlin, 1976.

[9] L. Nirenberg, On elliptic partial differential equations, *Ann. Scu. Norm. Sup. Pisa*, 13 (1959), 115-162.