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ON THE FRECHET DIFFERENTIABILITY OF FREE BOUNDARY OPERATOR FOR A MUSKAT TYPE PROBLEM①

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1. Introduction and Integral Equations

In this paper we study the following problems:

$$\alpha u_{xx} - u_t = 0 \quad \text{in} \quad S^- = \{(x, t); -1 < x < s(t),\ 0 < t < T\} \tag{1.1}$$

$$u(x, 0) = \varphi(x) \qquad -1 \leq x \leq 0 \tag{1.2}$$

$$u(-1, t) = f(t) \qquad 0 \leq t < T \tag{1.3}$$

$$\beta v_{xx} - v_t = 0 \quad \text{in} \quad S^+ = \{(x, t); s(t) < x < 1,\ 0 < t < T\} \tag{1.4}$$

$$v(x, 0) = \psi(x) \qquad 0 \leq x \leq 1 \tag{1.5}$$

$$v(1, t) = g(t) \qquad 0 \leq t < T \tag{1.6}$$

$$u(s(t), t) = v(s(t), t) \qquad 0 \leq t < T \tag{1.7}$$

$$Ku^2(s(t), t) + \gamma u_x(s(t), t) = \lambda v_x(s(t), t) \qquad 0 \leq t < T \tag{1.8}$$

$$\dot{s}(t) = u(s(t), t) \qquad 0 < t < T \tag{1.9}$$

$$s(0) = 0 \tag{1.10}$$

where $T > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\lambda > 0$ and K are given constants. $f(t)$, $g(t)$, $\varphi(x)$ and $\psi(x)$ are given functions, and the unknowns are $u(x, t)$, $v(x, t)$ and $s(t)$. (1.1) — (1.10) form a simplified mathematical model of the one-dimensional flow of two incompressible and immiscible fluids in a porous medium. $x = s(t)$ is the interface between these two fluids. $u(x, t)$ (resp. $v(x, t)$) is the velocity to the left (resp. right) of the interface. Problems (1.1) — (1.10) is an one-dimensional and parabolic version of a free boundary problem proposed by Muskat. W. Fulks and R. B. Guenther [1] considered the initial problems of this type, and proved the local existence and uniqueness of solution. In the one-dimensional flow within a porous medium of two immiscible fluids, the pressures of two fluids satisfy free boundary problems similar to (1.1) — (1.10). The global existence, uniqueness, regularity and the other properties of solution have been

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discussed by many authors[2], [3], [4], [5], [6], [7].

The main result of this paper is that, the solution operator of (1.1)-(1.10) $S:(f,g)\to s$ is Fréchet differentiable in its domain of definition, and the Fréchet derivative is Lipschitz continuous. The approach of this paper is an extension and modification used in [1].

We assume that the data satisfy the regularity and compatibility conditions:

$$
\begin{aligned}
&\varphi(x),\ \psi(x)\in C^2(R), \quad \text{and} \quad \varphi(x),\ \psi(x)=0 \quad \text{for} \quad |x|\geq 2\\
&f(t),\ g(t)\in C^1[0,\ T],\ \varphi(0)=\psi(0)\doteq a,\ \varphi(-1)=f(0)\\
&\psi(1)=g(0),\ \dot f(0)=\alpha\varphi''(-1),\ \dot g(0)=\beta\psi''(1)\\
&K\varphi^2(0)+\gamma\varphi'(0)=\lambda\psi'(0)
\end{aligned} \tag{1.11}
$$

An integral equation for μ

$$\mu(t)=u(s(t),\ t)=v(s(t),\ t)=\dot s(t) \tag{1.12}$$

which is equivalent to (1.1) — (1.10) can be derived in the same way as in [1]. First of all we define U and V by

$$
\begin{aligned}
U(x,\ t)&=\int_{-\infty}^{\infty}k(x-\xi,\ \alpha t)\varphi(\xi)d\xi,\ U(x,\ 0)=\varphi(x)\\
V(x,\ t)&=\int_{-\infty}^{\infty}k(x-\xi,\ \beta t)\psi(\xi)d\xi,\ V(x,\ 0)=\psi(x)
\end{aligned} \tag{1.13}
$$

where

$$k(x,\ t)=(4\pi t)^{-\frac{1}{2}}\exp(-x^2/4t) \tag{1.14}$$

We will use the standard notations of partial derivatives: $k_1(x,t)=\partial k(x,t)/\partial x$, $k_{11}(x,t)=\partial^2k(x,t)/\partial x^2$, $k_2(x,t)=\partial k(x,t)/\partial t$, etc. Therefore, the solutions of (1.1) — (1.10) are

$$
\begin{aligned}
u(x,\ t)&=U(x,\ t)+2a\int_0^t k_1(x-s(\tau),\ \alpha(t-\tau))y^{(1)}(\tau)d\tau\\
&\quad+2a\int_0^t k_1(x+1,\ \alpha(t-\tau))y^{(2)}(\tau)d\tau\\
v(x,\ t)&=V(x,\ t)+2\beta\int_0^t k_1(x-s(\tau),\ \beta(t-\tau))z^{(1)}(\tau)d\tau\\
&\quad+2\beta\int_0^t k_1(x-1,\ \beta(t-\tau))z^{(2)}(\tau)d\tau
\end{aligned} \tag{1.15}
$$

The functions $y^{(i)}(t)$ and $z^{(i)}(t)$ $(i=1,2)$ in (1.15) satisfy the integral equations

$$
\begin{cases}
y^{(1)}(t)+2a\int_0^t k_1(s(t)-s(\tau),\ \alpha(t-\tau))y^{(1)}(\tau)d\tau\\
+2a\int_0^t k_1(s(t)+1,\ \alpha(t-\tau))y^{(2)}(\tau)d\tau=\mu(t)-U(s(t),\ t)\\
y^{(2)}(t)-2a\int_0^t k_1(-1-s(\tau),\ \alpha(t-\tau))y^{(1)}(\tau)d\tau=U(-1,\ t)-f(t)
\end{cases} \tag{1.16}
$$

and

$$\begin{cases} z^{(1)}(t) - 2\beta \int_0^t k_1(s(t) - s(\tau),\ \beta(t-\tau)) z^{(1)}(\tau) d\tau \\ - 2\beta \int_0^t k_1(s(t) - 1,\ \beta(t-\tau)) z^{(2)}(\tau) d\tau = V(s(t),\ t) - \mu(t) \\ z^{(2)}(t) + 2\beta \int_0^t k_1(1 - s(\tau),\ \beta(t-\tau)) z^{(1)}(\tau) d\tau = g(t) - V(1,\ t) \end{cases} \quad (1.17)$$

Thus we are able to solve (1. 1) — (1. 10) provided $\mu(t)$ is found. The derivatives u_x and v_x are known in terms of μ by differentiating (1. 15) with respect to x, we have

$$\begin{aligned} \mathscr{L}^+\mu(t) &= \gamma u_x(s(t),\ t) = \gamma U_1(s(t),\ t) - \frac{\gamma}{\alpha}\mu(t) y^{(1)}(t) \\ &- 2\gamma \int_0^t k_1(s(t) - s(\tau),\ \alpha(t-\tau)) \mu(\tau) y^{(1)}(\tau) d\tau \\ &+ 2\gamma \int_0^t k(s(t) - s(\tau),\ \alpha(t-\tau)) \dot{y}^{(1)}(\tau) d\tau \\ &+ 2\gamma \int_0^t k(s(t) - 1,\ \alpha(t-\tau)) \dot{y}^{(2)}(\tau) d\tau \end{aligned} \quad (1.18)$$

$$\begin{aligned} \mathscr{L}^-\mu(t) &= \gamma u_x(-1,\ t) = \gamma U_1(-1,\ t) - 2\gamma \int_0^t k_1(-1 - s(\tau),\ \alpha(t-\tau)) \mu(\tau) y^{(1)}(\tau) d\tau \\ &+ 2\gamma \int_0^t k(-1 - s(\tau),\ \alpha(t-\tau)) \dot{y}^{(1)}(\tau) d\tau \\ &+ \frac{\gamma}{\sqrt{\pi\alpha}} \int_0^t \frac{\dot{y}^{(2)}(\tau)}{\sqrt{t-\tau}} d\tau \end{aligned} \quad (1.19)$$

and

$$\begin{aligned} \mathscr{R}^+\mu(t) &= \lambda v_x(1,\ t) = \lambda V_1(1,\ t) - 2\lambda \int_0^t k_1(1 - s(\tau),\ \beta(t-\tau)) \mu(\tau) z^{(1)}(\tau) d\tau \\ &+ 2\lambda \int_0^t k(1 - s(\tau),\ \beta(t-\tau)) \dot{z}^{(1)} d\tau + \frac{\lambda}{\sqrt{\pi\beta}} \int_0^t \frac{\dot{z}^{(2)}(\tau)}{\sqrt{t-\tau}} d\tau \end{aligned} \quad (1.20)$$

$$\begin{aligned} \mathscr{R}^-\mu(t) &= \lambda v_x(s(t),\ t) \\ &= \lambda V_1(s(t),\ t) - \frac{\lambda}{\beta}\mu(t) z^{(1)}(t) \\ &- 2\lambda \int_0^t k_1(s(t) - s(\tau),\ \beta(t-\tau)) \mu(\tau) z^{(1)}(\tau) d\tau \\ &+ 2\lambda \int_0^t k(s(t) - s(\tau),\ \beta(t-\tau)) \dot{z}^{(1)}(\tau) d\tau \\ &+ 2\lambda \int_0^t k(s(t) - 1,\ \beta(t-\tau)) \dot{z}^{(2)}(\tau) d\tau \end{aligned} \quad (1.21)$$

By means of the same calculations as in [1], we conclude that $\mu(t)$ satisfies the following integral equation

$$\mu(t)=a+\frac{2\sqrt{\alpha\beta}}{\lambda\sqrt{\alpha}+\gamma\sqrt{\beta}}\sum_{i=1}^{21}I_i \tag{1. 22}$$

where

$$I_1=\frac{2\gamma\varphi'(0)}{\sqrt{\pi}}\int_0^t\sqrt{t-\theta}k(s(\theta),\ \alpha\theta)\mu(\theta)d\theta$$

$$I_2=-\frac{2\gamma\varphi'(-1)}{\sqrt{\pi}}\int_0^t\sqrt{t-\theta}\ k(s(\theta)+1,\ \alpha\theta)\mu(\theta)d\theta$$

$$I_3=\frac{2\gamma\alpha\varphi'(0)}{\sqrt{\pi}}\int_0^t\sqrt{t-\theta}\ k_1(s(\theta),\ \alpha\theta)d\theta$$

$$I_4=-\frac{2\gamma\alpha\varphi'(-1)}{\sqrt{\pi}}\int_0^t\sqrt{t-\theta}\ k_1(s(\theta)+1,\ \alpha\theta)d\theta$$

$$I_5=\frac{2\lambda\psi'(0)}{\sqrt{\pi}}\int_0^t\sqrt{t-\theta}\ k(s(\theta),\ \beta\theta)\mu(\theta)d\theta$$

$$I_6=-\frac{2\lambda\psi'(1)}{\sqrt{\pi}}\int_0^t\sqrt{t-\theta}\ k(s(\theta)-1,\ \beta\theta)\mu(\theta)d\theta$$

$$I_7=\frac{2\lambda\beta\psi'(0)}{\sqrt{\pi}}\int_0^t\sqrt{t-\theta}\ k_1(s(\theta),\ \beta\theta)d\theta$$

$$I_8=-\frac{2\lambda\beta\psi'(1)}{\sqrt{\pi}}\int_0^t\sqrt{t-\theta}\ k_1(s(\theta)-1,\ \beta\theta)d\theta$$

$$I_9=-\frac{2\gamma}{\sqrt{\pi}}\int_0^t\sqrt{t-\theta}\ \mu(\theta)d\theta\int_{-1}^0 k(s(\theta)-\xi,\ \alpha\theta)\varphi''(\xi)d\xi$$

$$I_{10}=-\frac{2\gamma\alpha}{\sqrt{\pi}}\int_0^t\sqrt{t-\theta}\ d\theta\int_{-1}^0 k_1(s(\theta)-\xi,\ \alpha\theta)\varphi''(\xi)d\xi$$

$$I_{11}=\frac{2\lambda}{\sqrt{\pi}}\int_0^t\sqrt{t-\theta}\ \mu(\theta)d\theta\int_0^1 k(s(\theta)-\xi,\ \beta\theta)\psi''(\xi)d\xi$$

$$I_{12}=\frac{2\lambda\beta}{\sqrt{\pi}}\int_0^t\sqrt{t-\theta}\ d\theta\int_0^1 k_1(s(\theta)-\xi,\ \beta\theta)\psi''(\xi)d\xi$$

$$I_{13}=-\frac{2K}{\sqrt{\pi}}\int_0^t\sqrt{t-\theta}\mu(\theta)\dot{\mu}(\theta)d\theta$$

$$I_{14}=-\frac{\alpha}{\sqrt{\pi}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^\theta k_1(s(\theta)-s(\tau),\ \alpha(\theta-\tau))\mathscr{L}^+\mu(\tau)d\tau$$

$$I_{15}=-\frac{\beta}{\sqrt{\pi}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^\theta k_1(s(\theta)-s(\tau),\ \beta(\theta-\tau))\mathscr{R}^-\mu(\tau)d\tau$$

$$I_{16}=\frac{\alpha}{\sqrt{\pi}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^\theta k_1(s(\theta)+1,\ \alpha(\theta-\tau))\mathscr{L}^-\mu(\tau)d\tau$$

$$I_{17}=\frac{\beta}{\sqrt{\pi}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^\theta k_1(s(\theta)-1,\ \beta(\theta-\tau))\,\mathscr{R}^+\mu(\tau)\,d\tau$$

$$I_{18}=\frac{\gamma}{\sqrt{\pi}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^\theta k(s(\theta)+1,\ \alpha(\theta-\tau))\,\dot{f}(\tau)\,d\tau$$

$$I_{19}=\frac{\lambda}{\sqrt{\pi}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^\theta k(s(\theta)-1,\ \beta(\theta-\tau))\,\dot{g}(\tau)\,d\tau$$

$$I_{20}=\frac{\gamma}{2\pi\sqrt{\alpha}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^\theta\frac{\dot{\mu}(\tau)}{\sqrt{\theta-\tau}}\left[1-\exp\left(-\frac{(s(\theta)-s(\tau))^2}{4\alpha(\theta-\tau)}\right)\right]d\tau$$

$$I_{21}=\frac{\lambda}{2\pi\sqrt{\beta}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^\theta\frac{\dot{\mu}(\tau)}{\sqrt{\theta-\tau}}\left[1-\exp\left(-\frac{(s(\theta)-s(\tau))^2}{4\beta(\theta-\tau)}\right)\right]d\tau$$

Since some I_i depend on $\dot{\mu}(t)$, we also need an integral equation for $\dot{\mu}(t)$.

$$\dot{\mu}(t)=\frac{2\sqrt{\alpha\beta}}{\lambda\sqrt{\alpha}+\gamma\sqrt{\beta}}\sum_{i=1}^{27}J_i \qquad (1.\ 23)$$

where

$$J_1=\frac{dI_1}{dt}=\frac{\gamma\varphi'(0)}{\sqrt{\pi}}\int_0^t\frac{\mu(\theta)}{\sqrt{t-\theta}}\,k(s(\theta),\ \alpha\theta)\,d\theta$$

Similarly, $J_i=dI_i/dt$, $i=2, \ldots, 13$, are obtained from I_i provided that $\sqrt{t-\theta}$ is replaced by $\left(2\sqrt{t-\theta}\right)^{-1}$. But dI_{14}/dt is equal to the sum of four terms:

$$\frac{dI_{14}}{dt}=-\frac{1}{4\pi\sqrt{\alpha}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^\theta\mathscr{L}^+\mu(\tau)\cdot$$

$$\cdot\frac{\partial}{\partial\theta}\left\{\frac{s(\theta)-s(\tau)}{(\theta-\tau)^{3/2}}\left[1-\exp\left(-\frac{(s(\theta)-s(\tau))^2}{4\alpha(\theta-\tau)}\right)\right]\right\}d\tau$$

$$-\frac{1}{4\pi\sqrt{\alpha}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^\theta\mathscr{L}^+\mu(\tau)\frac{\partial}{\partial\theta}\left\{\frac{1}{\sqrt{\theta-\tau}}\left[\mu(\theta)-\frac{s(\theta)-s(\tau)}{\theta-\tau}\right]\right\}d\tau$$

$$+\frac{1}{4\sqrt{\alpha}}\mu(t)\,\mathscr{L}^+\mu(t)+\frac{1}{4\pi\sqrt{\alpha}}\int_0^t\mathscr{L}^+\mu(\tau)\,d\tau\int_0^1\sqrt{\frac{\rho}{1-\rho}}\,\dot{\mu}(\tau+\rho(t-\tau))\,d\rho$$

$$=J_{14}+J_{15}+J_{16}+J_{17},\ \text{respectively.}$$

Replacing α, $\mathscr{L}^+\mu$ in $J_{14}, \ldots, J_{17}$ by β, $\mathscr{R}^-\mu$ respectively, we get $J_{18}, \ldots, J_{21}$, and

$$\frac{dI_{15}}{dt}=J_{18}+J_{19}+J_{20}+J_{21}$$

Moreover,

$$J_{22}=\frac{dI_{16}}{dt}=\frac{\alpha}{\sqrt{\pi}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^\theta\mathscr{L}^-\mu(\tau)\frac{\partial}{\partial\theta}k_1(s(\theta)+1,\ \alpha(\theta-\tau))\,d\tau$$

$$=\frac{\alpha}{\sqrt{\pi}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^\theta\mathscr{L}^-\mu(\tau)\,[k_{11}(s(\theta)+1,\ \alpha(\theta-\tau))\,\mu(\theta)$$

$$+\alpha k_{12}(s(\theta)+1,\ a(\theta-\tau))]d\tau$$

and $J_{23}, \ldots, J_{27}$ can be obtained from $I_{17}, \ldots, I_{21}$ in the same way.

We have the following local existence and uniqueness result, which is an analogue of existence theorem in [1].

Theorem 1 For any given constant $0<\delta<1$, there exist two constants $0<T^*\leq T$ and $N>0$, depending only on the C^2-norms of φ and ψ, the C^1-norms of f and g, the constants $\alpha, \beta, \gamma, \lambda, K$ and δ, such that there exists a unique solution $\mu(t)$ of (1.22), and

$$\begin{aligned} &\mu(t)\in C^1[0,\ T^*],\ \mu(0)=a \\ &|s(t)|\leq 1-\delta \qquad \text{for} \qquad 0\leq t\leq T^* \\ &\|\mu\| = \sup_{0\leq t\leq T^*}|\mu(t)| + \sup_{0\leq t\leq T^*}|\dot{\mu}(t)|\leq N \end{aligned} \tag{1.24}$$

Moreover, for all of f and g in any given compact subset of $C^1[0, T]$, there exist common constants $0<T^*\leq T$ and $N>0$ satisfying the assertions of theorem.

Remark By the equivalence between (1.22) and (1.1) — (1.10), it follows that there exists a unique solution $\{u, v, s\}$ of (1.1) — (1.10) for $0\leq t\leq T^*$, and $\{u, v, s\}$ satisfies

(i) $u,\ u_x,\ u_{xx},\ u_t\in C(\bar{S}_*^-),\ v,\ v_x,\ v_{xx},\ v_t\in C(\bar{S}_*^+)$

(ii) $s(t)\in C^2[0,\ T^*],\ |s(t)|\leq 1-\delta$ for $0\leq t\leq T^*$

where $S_*^-=\{(x,t);-1<x<s(t),0<t<T^*\}$, $S_*^+=\{(x,t);s(t)<x<1,0<t<T^*\}$.

2. The Continuous Dependence

Let $\mu(t)$ and $\bar{\mu}(t)$ be the solutions of (1.22) with the data $\{\varphi, \psi, f, g\}$ and $\{\varphi, \psi, \bar{f}, \bar{g}\}$ for $0\leq t\leq T^*$ respectively, and $\|\mu\|$ and $\|\bar{\mu}\|\leq N$. In this section we derive a C^1-norm estimate of the difference $\Delta\mu(t)=\bar{\mu}(t)-\mu(t)$. We denote that

$$s(t)=\int_0^t \mu(\tau)d\tau,\ \bar{s}(t)=\int_0^t \bar{\mu}(\tau)d\tau$$

$$\Delta s(t)=\bar{s}(t)-s(t),\ \Delta f(t)=\bar{f}(t)-f(t),\ \Delta g(t)=\bar{g}(t)-g(t)$$

and

$$\|\Delta f\| = \|\bar{f}-f\|_{C^1[0,T]},\quad \|\Delta g\| = \|\bar{g}-g\|_{C^1[0,T]},\ \text{etc.}$$

Moreover, throughout this paper we will use C as a generic symbol for positive constants which depend only on the C^2-norms of φ and ψ, the C^1-norms of f and g, the constants $\alpha, \beta, \gamma, \lambda, K, \delta, T^*$ and N.

Lemma 2.1 The solutions of (1.16) and (1.17) $y^{(i)}(t)$ and $z^{(i)}(t)$, $i=1, 2$, satisfy

$$\|y^{(i)}\|,\ \|z^{(i)}\|\leq C \tag{2.1}$$

$$|\Delta y^{(i)}(t)|+|\Delta \dot{y}^{(i)}(t)|\leq C\{\,\|\Delta f\|+|\Delta \dot{\mu}(t)|+\int_0^t|\Delta \dot{\mu}(\tau)|d\tau\} \quad (2.\ 2)$$

$$|\Delta z^{(i)}(t)|+|\Delta \dot{z}^{(i)}(t)|\leq C\{\,\|\Delta g\|+|\Delta \dot{\mu}(t)|+\int_0^t|\Delta \dot{\mu}(\tau)|d\tau\} \quad (2.\ 3)$$

for $0\leq t\leq T^*$.

Proof It is clear that

$$0<k(x,\ t)\leq C/\sqrt{t} \quad (2.\ 4)$$

$$|k_1(s(t)-s(\tau),\ a(t-\tau))|=\frac{1}{2a}\cdot\frac{|s(t)-s(\tau)|}{t-\tau}k(s(t)-s(\tau),\ a(t-\tau))$$

$$\leq C/\sqrt{t-\tau} \quad \text{for} \quad 0\leq\tau\leq t \quad (2.\ 5)$$

and the all order partial derivatives of $k(x,t)$ are bounded for $|x|\leq\delta$, the bounds depend on δ. Thus (1.16) and (1.17) are systems of Volterra's integral equations of second kind with kernels possessing weak singularities and continuous inhomogeneous terms, their solutions are the integrals of iterated kernels. Therefore

$$|y^{(i)}(t)|,\ |z^{(i)}(t)|\leq C \quad for \quad 0\leq t\leq T^* \quad (2.\ 6)$$

Differentiating (1.16) with respect to t, we get

$$\dot{y}^{(1)}(t)-\frac{\mu(t)}{4\sqrt{\pi a}}\int_0^t\frac{\dot{y}^{(1)}(\tau)}{\sqrt{t-\tau}}d\tau=\dot{\mu}(t)-U_1(s(t),\ t)\mu(t)-U_2(s(t),\ t)$$

$$-\frac{1}{\sqrt{4\pi a}}\int_0^t y^{(1)}(\tau)\frac{\partial}{\partial t}\left\{\frac{s(t)-s(\tau)}{(t-\tau)^{3/2}}\left[1-\exp\left(-\frac{(s(t)-s(\tau))^2}{4a(t-\tau)}\right)\right]\right\}d\tau$$

$$-\frac{1}{\sqrt{4\pi a}}\int_0^t y^{(1)}(\tau)\frac{\partial}{\partial t}\left\{\frac{1}{\sqrt{t-\tau}}\left[\mu(t)-\frac{s(t)-s(\tau)}{t-\tau}\right]\right\}d\tau$$

$$+\frac{\dot{\mu}(t)}{\sqrt{4\pi a}}\int_0^t\frac{y^{(1)}(\tau)}{\sqrt{t-\tau}}d\tau-2a\mu(t)\int_0^t k_{11}(s(t)+1,\ a(t-\tau))y^{(2)}(\tau)d\tau$$

$$-2a^2\int_0^t k_{12}(s(t)+1,\ a(t-\tau))y^{(2)}(\tau)d\tau \quad (2.\ 7)$$

$$\dot{y}^{(2)}(t)=U_2(-1,\ t)-\dot{f}(t)+2a^2\int_0^t k_{12}(-1-s(\tau),\ a(t-\tau))y^{(1)}(\tau)d\tau \quad (2.\ 8)$$

Introducing notations

$$F(t,\ \tau)=\frac{s(t)-s(\tau)}{(t-\tau)^{3/2}}\left[1-\exp\left(-\frac{(s(t)-s(\tau))^2}{4a(t-\tau)}\right)\right] \quad (2.\ 9)$$

$$G(t,\ \tau)=\frac{1}{\sqrt{t-\tau}}\left[\mu(t)-\frac{s(t)-s(\tau)}{t-\tau}\right] \quad (2.\ 10)$$

we can obtain the estimates

$$\left|\frac{\partial F(t,\ \tau)}{\partial t}\right|,\ \left|\frac{\partial G(t,\ \tau)}{\partial t}\right|\leq\frac{C}{\sqrt{t-\tau}} \quad (2.\ 11)$$

Thus (2.7) is a Volterra' s integral equation of second kind with weak singularity kernel and continuous inhomogeneous term. (2. 8) is the formula of $\dot{y}^{(2)}(t)$. This follows that the bounds of $\dot{y}^{(i)}(t)$ for $0 \leq t \leq T^*$ are estimated by C. $z^{(i)}(t)$ and $\dot{z}^{(i)}(t)$ are estimated as the proceeding ones, the calculations differing only in details.

Observe

$$\Delta y^{(1)}(t) + 2a \int_0^t k_1(s(t) - s(\tau),\ a(t-\tau))\,\Delta y^{(1)}(\tau)\,d\tau$$

$$+ 2a \int_0^t k_1(s(t)+1,\ a(t-\tau))\,\Delta y^{(2)}(\tau)\,d\tau$$

$$= \Delta\mu(t) - \widetilde{U}_1(s(t),\ t)\,\Delta s(t)$$

$$- 2a \int_0^t y^{(1)}(\tau)\,\widetilde{k}_{11}(s(t) - s(\tau),\ a(t-\tau))\,(\Delta s(t) - \Delta s(\tau))\,d\tau$$

$$- 2a\Delta s(t) \int_0^t \widetilde{k}_{11}(s(t)+1,\ a(t-\tau))\,y^{(2)}(\tau)\,d\tau \tag{2. 12}$$

$$\Delta y^{(2)}(t) - 2a \int_0^t k_1(-1-s(\tau),\ a(t-\tau))\,\Delta y^{(1)}(\tau)\,d\tau$$

$$= -\Delta f(t) - 2a \int_0^t y^{(1)}(\tau)\,\widetilde{k}_{11}(-1-s(\tau),\ a(t-\tau))\,\Delta s(\tau)\,d\tau \tag{2. 13}$$

where

$$\widetilde{U}_1(s(t),\ t) = \int_0^1 U_1(\xi\bar{s}(t) + (1-\xi)\,s(t),\ t)\,d\xi \tag{2. 14}$$

the notations $\widetilde{k}_{11}(s(t) - s(\tau), a(t-\tau))$, etc. have the same meaning. Using the inequalities

$$\left|\frac{\partial^{l+m}}{\partial x^l \partial t^m} k(x,\ t)\right| \leq C t^{-(m+\frac{l}{2})} k(x,\ 2t) \tag{2. 15}$$

$$|\Delta\mu(t)| = \left|\int_0^t \Delta\dot{\mu}(\tau)\,d\tau\right| \leq \int_0^t |\Delta\dot{\mu}(\tau)|\,d\tau \tag{2. 16}$$

$$\left|\frac{\Delta s(t) - \Delta s(\tau)}{t-\tau}\right| \leq \frac{1}{t-\tau} \int_\tau^t |\Delta\mu(\theta)|\,d\theta \leq \frac{1}{t-\tau} \int_\tau^t d\theta \int_0^\theta |\Delta\dot{\mu}(\xi)|\,d\xi$$

$$\leq \int_0^t |\Delta\dot{\mu}(\tau)|\,d\tau \qquad \text{for} \qquad 0 \leq \tau < t \tag{2. 17}$$

and the estimates of solution for Volterra's integral equations, we obtain

$$|\Delta y^{(i)}(t)| \leq C\left\{\|\Delta f\| + \int_0^t |\Delta\dot{\mu}(\tau)|\,d\tau\right\} \tag{2. 18}$$

From (2.7) and (2. 8) we get

$$\Delta\dot{y}^{(1)}(t) - \frac{\bar{\mu}(t)}{4\sqrt{\pi a}} \int_0^t \frac{\Delta\dot{y}^{(1)}(\tau)}{\sqrt{t-\tau}}\,d\tau$$

$$= \frac{\Delta\mu(t)}{4\sqrt{\pi a}} \int_0^t \frac{\dot{y}^{(1)}(\tau)}{\sqrt{t-\tau}}\,d\tau + \Delta\dot{\mu}(t) - U_1(\bar{s}(t),\ t)\,\Delta\mu(t)$$

$$- \mu(t)\widetilde{U}_{11}(s(t), t)\Delta s(t) - \widetilde{U}_{21}(s(t), t)\Delta s(t)$$

$$+ \frac{\dot{\mu}(t)}{\sqrt{4\pi a}}\int_0^t \frac{\Delta y^{(1)}(\tau)}{\sqrt{t-\tau}}d\tau + \frac{\Delta\dot{\mu}(t)}{\sqrt{4\pi a}}\int_0^t \frac{y^{(1)}(\tau)}{\sqrt{t-\tau}}d\tau$$

$$- \frac{1}{\sqrt{4\pi a}}\int_0^t \bar{y}^{(1)}(\tau)\frac{\partial}{\partial t}(\bar{F}(t, \tau) - F(t, \tau))d\tau$$

$$- \frac{1}{\sqrt{4\pi a}}\int_0^t \frac{\partial}{\partial t}F(t, \tau)\Delta y^{(1)}(\tau)d\tau$$

$$- \frac{1}{\sqrt{4\pi a}}\int_0^t \bar{y}^{(1)}(\tau)\frac{\partial}{\partial t}(\bar{G}(t, \tau) - G(t, \tau))d\tau$$

$$- \frac{1}{\sqrt{4\pi a}}\int_0^t \frac{\partial}{\partial t}G(t, \tau)\Delta y^{(1)}(\tau)d\tau$$

$$- 2a\int_0^t [\mu(t)k_{11}(s(t)+1, a(t-\tau))$$

$$+ ak_{12}(s(t)+1, a(t-\tau))]\Delta y^{(2)}(\tau)d\tau$$

$$- 2a\Delta\mu(t)\int_0^t k_{11}(\bar{s}(t)+1, a(t-\tau))\bar{y}^{(2)}(\tau)d\tau$$

$$- 2a\Delta s(t)\int_0^t y^{(2)}(\tau)[\mu(t)\widetilde{k}_{111}(s(t)+1, a(t-\tau))$$

$$+ a\widetilde{k}_{121}(s(t)+1, a(t-\tau))]d\tau \tag{2. 19}$$

$$\Delta\dot{y}^{(2)}(t) = -\Delta\dot{f}(t) - 2a^2\int_0^t \widetilde{k}_{121}(-1-s(\tau), a(t-\tau))\bar{y}^{(1)}(\tau)\Delta s(\tau)d\tau$$

$$- 2a^2\int_0^t k_{12}(-1-s(\tau), a(t-\tau))\Delta y^{(1)}(\tau)d\tau \tag{2. 20}$$

where

$$\frac{\partial}{\partial t}(\bar{F}(t, \tau) - F(t, \tau))$$

$$= (\Delta\mu(t) - \Delta\mu(\tau))((t-\tau)^{-\frac{3}{2}} + 4a\sqrt{\pi a}\,\widetilde{k}_{11}(s(t)-s(\tau), a(t-\tau)))$$

$$+ (\Delta s(t) - \Delta s(\tau))\Big\{-\frac{3}{2}(t-\tau)^{-\frac{5}{2}} +$$

$$+ 4a\sqrt{\pi a}\frac{\partial}{\partial t}\widetilde{k}_{11}(s(t)-s(\tau), a(t-\tau))\Big\} \tag{2. 21}$$

Denoting $[\cdot] = [\xi(\bar{s}(t) - \bar{s}(\tau)) + (1-\xi)(s(t) - s(\tau))]$, we have

$$\left|(t-\tau)^{-\frac{3}{2}} + 4a\sqrt{\pi a}\widetilde{k}_{11}(s(t)-s(\tau), a(t-\tau))\right|$$

$$= \Big|\int_0^1 \Big\{(t-\tau)^{-\frac{3}{2}}\Big[1 - \exp\Big(-\frac{[\cdot]^2}{4a(t-\tau)}\Big)\Big]$$

$$+ \frac{1}{2a\sqrt{t-\tau}}\Big(\frac{[\cdot]}{(t-\tau)}\Big)^2\exp\Big(-\frac{[\cdot]^2}{4a(t-\tau)}\Big)\Big\}d\xi\Big|$$

$$\leq C/\sqrt{t-\tau} \qquad \text{for} \qquad 0\leq\tau<t$$

Similarly

$$\left|-\frac{3}{2}(t-\tau)^{-\frac{5}{2}}+4a\sqrt{\pi a}\frac{\partial}{\partial t}\tilde{k}_{11}(s(t)-s(\tau),\ a(t-\tau))\right|\leq C/(t-\tau)^{\frac{3}{2}}$$

Substituting these estimates into (2. 21), we obtain that

$$\left|\frac{\partial}{\partial t}(\bar{F}(t,\ \tau)-F(t,\ \tau))\right|\leq\frac{C}{\sqrt{t-\tau}}\left[|\Delta\mu(t)-\Delta\mu(\tau)|+\frac{|\Delta s(t)-\Delta s(\tau)|}{t-\tau}\right]$$

$$\leq\frac{C}{\sqrt{t-\tau}}\int_0^t|\Delta\dot{u}(\tau)|d\tau \qquad (2.\ 22)$$

and

$$\left|\frac{\partial}{\partial t}(\bar{G}(t,\ \tau)-G(t,\ \tau))\right|=\frac{1}{\sqrt{t-\tau}}\left|\Delta\dot{\mu}(t)-\frac{3}{2(t-\tau)}\left[\Delta\mu(t)-\frac{\Delta s(t)-\Delta s(\tau)}{t-\tau}\right]\right|$$

$$=\frac{1}{\sqrt{t-\tau}}\left|\Delta\dot{\mu}(t)+\frac{3}{2(t-\tau)}\int_\tau^t d\xi\int_\xi^t\Delta\dot{\mu}(\eta)\,d\eta\right|$$

$$\leq\frac{C}{\sqrt{t-\tau}}\left\{|\Delta\dot{\mu}(t)|+\int_0^t|\Delta\dot{\mu}(\tau)|d\tau\right\} \qquad (2.\ 23)$$

By applying (2. 1), (2. 11), (2. 18), (2. 22) and (2. 23), we get the estimate (2. 2), (2. 3) can be obtained in the same way.

Lemma 2. 2 The functions $\mathscr{L}\pm\mu$, $\mathscr{R}\pm\mu$ defined by (1. 18) — (1. 21) satisfy

$$|\mathscr{L}\pm\mu(t)|,\ |\mathscr{R}\pm\mu(t)|\leq C \qquad (2.\ 24)$$

$$|\Delta\mathscr{L}\pm\mu(t)|\leq C\left\{\|\Delta f\|+\int_0^t\left(1+\frac{1}{\sqrt{t-\tau}}\right)|\Delta\dot{\mu}(\tau)|d\tau\right\} \qquad (2.\ 25)$$

$$|\Delta\mathscr{R}\pm\mu(t)|\leq C\left\{\|\Delta g\|+\int_0^t\left(1+\frac{1}{\sqrt{t-\tau}}\right)|\Delta\dot{\mu}(\tau)|d\tau\right\} \qquad (2.\ 26)$$

for $0\leq t\leq T^*$.

Proof It is a straightforward conclusion of Lemma 2. 1.

Lemma 2. 3 Suppose the nonnegative continuous function $x(t)$ satisfies

$$x(t)\leq A+B\int_0^t d\tau\int_0^1\sqrt{\frac{\rho}{1-\rho}}\,x(\tau+\rho(t-\tau))\,d\rho+C\int_0^t\left(1+\frac{1}{\sqrt{t-\tau}}\right)x(\tau)\,d\tau$$

$$(2.\ 27)$$

for $0\leq t\leq T$. Then there exists a constant D, depending only on B, C and T, such that

$$x(t)\leq AD \qquad \text{for} \qquad 0\leq t\leq T \qquad (2.\ 28)$$

Proof Let us take

$$C_1=\frac{\pi}{2}B+C\left(1+\frac{2}{\sqrt{T}}\right),\ C_2=\max\{1,\ C_1T\},\ m=1+[4(4B+C_1)^2T^2]$$

where $[x]$ denotes the integer part of number x, then the desired constant

$$D = \frac{2(2^m C_2^m - 1)}{2C_2 - 1} \tag{2. 29}$$

In fact, let $h = T/m$, it is clear that $h \leq T$ and $(4B + C_1)\sqrt{Th} < 1/2$. Deviding $[0, T]$ to m subintervals: $t_k \leq t \leq t_{k+1}$, $t_k = kh$, $k = 0, 1, \ldots, m-1$, and denoting $\max_{0 \leq t \leq t_k} x(t) = X_k$, $\max_{t_k \leq t \leq t_{k+1}} x(t) = X_{k,k+1}$, we can inductively prove

$$X_k \leq 2A \frac{2^k C_2^k - 1}{2C_2 - 1}, \quad k = 1, 2, \ldots, m \tag{2. 30}$$

For $k = 1$ and $t \in [0, t_1] = [0, h]$, we have

$$\int_0^t d\tau \int_0^1 \sqrt{\frac{\rho}{1-\rho}}\, x(\tau + \rho(t - \tau))\, d\tau \leq \frac{\pi t}{2} X_1 \leq \frac{\pi h}{2} X_1 \leq \frac{\pi}{2}\sqrt{Th} X_1$$

$$\int_0^t \left(1 + \frac{1}{\sqrt{t-\tau}}\right) x(\tau)\, d\tau \leq (t + 2\sqrt{t}) X_1 \leq (\sqrt{T} + 2)\sqrt{h}\, X_1$$

it follows that

$$x(t) \leq A + \left[\frac{\pi}{2} B\sqrt{T} + C(\sqrt{T} + 2)\right]\sqrt{h}\, X_1$$

$$= A + C_1\sqrt{Th}\, X_1 \leq A + \frac{1}{2} X_1, \quad t \in [0, h]$$

This immediately leads to $X_1 \leq 2A$.

Now suppose (2. 30) holds for $k = l \geq 1$, to obtain that (2. 30) also holds for $k = l + 1$, we only need to prove

$$X_{l+1} \leq 2A + 2C_2 X_l \tag{2. 31}$$

Observe that for $t \in (t_l, t_{l+1}]$

$$\int_0^t d\tau \int_0^1 \sqrt{\frac{\rho}{1-\rho}}\, x(\tau + \rho(t-\tau))\, d\rho$$

$$= \int_0^{t_l} d\tau \int_0^{\rho_l} \sqrt{\frac{\rho}{1-\rho}}\, x(\tau + \rho(t-\tau))\, d\rho + \int_0^{t_l} d\tau \int_{\rho_l}^1 \sqrt{\frac{\rho}{1-\rho}}\, x(\tau + \rho(t-\tau))\, d\rho$$

$$+ \int_{t_l}^t d\tau \int_0^1 \sqrt{\frac{\rho}{1-\rho}}\, x(\tau + \rho(t-\tau))\, d\rho$$

$= I_1 + I_2 + I_3$, respectively,

where $\rho_l = \frac{t_l - \tau}{t - \tau}$, $\tau \in [0, t_l]$. It is easy to see that

$$I_1 \leq \frac{\pi}{2} t_l X_l \leq \frac{\pi T}{2} X_l$$

$$I_2 \leq X_{l,l+1} \int_0^{t_l} d\tau \int_{\rho_l}^1 \sqrt{\frac{\rho}{1-\rho}}\, d\rho$$

$$= \left[t_l\left(\frac{\pi}{2} - \operatorname{arctg}\sqrt{\frac{t_l}{t-t_l}}\right) + \frac{3}{2}\sqrt{t-t_l}\int_0^{t_l}\frac{\sqrt{t_l-\tau}}{t-\tau}d\tau\right]X_{l,l+1}$$

$$\leq \left[t_l\left(\frac{\pi}{2} - \operatorname{arctg}\sqrt{l}\right) + 3\sqrt{t_l h}\right]X_{l,l+1}$$

$$\leq \left[\sqrt{l}\left(\frac{\pi}{2} - \operatorname{arctg}\sqrt{l}\right) + 3\right]\sqrt{t_l h}\, X_{l,l+1}$$

$$< 4\sqrt{Th}X_{l,l+1}$$

$$I_3 \leq \frac{\pi}{2}(t-t_l)X_{l,l+1} \leq \frac{\pi h}{2}X_{l,l+1} \leq \frac{\pi}{2}\sqrt{Th}\,X_{l,l+1}$$

Moreover, we also have

$$\int_0^t\left(1+\frac{1}{\sqrt{t-\tau}}\right)x(\tau)\,d\tau = \int_0^{t_l}\left(1+\frac{1}{\sqrt{t-\tau}}\right)x(\tau)\,d\tau + \int_{t_l}^t\left(1+\frac{1}{\sqrt{t-\tau}}\right)x(\tau)\,d\tau$$

and

$$\int_0^{t_l}\left(1+\frac{1}{\sqrt{t-\tau}}\right)x(\tau)\,d\tau \leq (t_l + 2\sqrt{t_l})X_l \leq T\left(1+\frac{2}{\sqrt{T}}\right)X_l$$

$$\int_{t_l}^t\left(1+\frac{1}{\sqrt{t-\tau}}\right)x(\tau)\,d\tau \leq \left(t-t_l+2\sqrt{t-t_l}\right)X_{l,l+1} \leq \sqrt{Th}\left(1+\frac{2}{\sqrt{T}}\right)X_{l,l+1}$$

Substituting all of these estimates into (2. 27), we get

$$x(t) \leq A + C_1TX_l + (4B+C_1)\sqrt{Th}X_{l,l+1}$$

$$< A + C_1TX_l + \frac{1}{2}X_{l,l+1} \qquad \text{for} \qquad t\in(t_l, t_{l+1}]$$

It implies the desired inequality (2. 31).

In the next section we need a global existence result for the following integral equations.

Lemma 2. 4 Suppose $A(t)\in C[0,T]$, $B(t)$ and $C(t)\in L^\infty(0,T)$. Then there exists a unique continuous function $x(t):[0,T]\to R$ satisfies the linear integral equation

$$x(t) = A(t) + \int_0^t B(\tau)\,d\tau\int_0^1\sqrt{\frac{\rho}{1-\rho}}\,x(\tau+\rho(t-\tau))\,d\rho +$$

$$+ \int_0^t\left(1+\frac{1}{\sqrt{t-\tau}}\right)C(\tau)x(\tau)\,d\tau \qquad (2.\ 32)$$

for $0\leq t\leq T$. Moreover, $x(t)$ depends linearly on $A(t)$.

Proof Let $\|B\|_{L^\infty}=B$, $\|C\|_{L^\infty}=C$, $\|A\|_C=A$, and define C_1, C_2, m, and h as in Lemma 2. 3, and devide $[0,T]$ to m subintervals: $t_k\leq t\leq t_{k+1}$, $t_k=kh$, $k=0,1,\dots,m-1$. It is easy to find the solution $x(t)$ in every subintervals $[t_k,t_{k+1}]$, $k=0,1,\dots,m-1$, successively by means of contraction mapping principle.

Theorem 2. 5 There exists a constant $N_1>0$, depending only on the C^2-norms of φ and ψ, the upper bound of C^1-norms of f, g, $\bar f$ and $\bar g$, the constants α, β, γ, λ, K, δ, T^*

and N, such that

$$\| \bar{s} - s \|_{C^2[0,T^*]} \leq N_1 \{ \| \bar{f} - f \| + \| \bar{g} - g \| \} \tag{2. 33}$$

Proof Since

$$\Delta \dot{\mu}(t) = \frac{2\sqrt{\alpha\beta}}{\lambda\sqrt{\alpha} + \gamma\sqrt{\beta}} \sum_{i=1}^{27} \Delta J_i \tag{2. 34}$$

where $\Delta J_i = \bar{J}_i - J_i$, we estimate every ΔJ_i in terms of $\| \Delta f \|$, $\| \Delta g \|$, $\Delta \dot{\mu}(t)$ and the integral of $\Delta \dot{\mu}$. From

$$\Delta J_1 = \frac{\gamma\varphi'(0)}{\sqrt{\pi}} \left\{ \int_0^t \frac{1}{\sqrt{t-\theta}} k(\bar{s}(\theta), \alpha\theta) \Delta\mu(\theta) d\theta + \int_0^t \frac{\mu(\theta)}{\sqrt{t-\theta}} \tilde{k}_1(s(\theta), \alpha\theta) \Delta s(\theta) d\theta \right\}$$

it follows that

$$|\Delta J_1| \leq C \left\{ \int_0^t \frac{|\Delta\mu(\theta)|}{\sqrt{\theta(t-\theta)}} d\theta + \int_0^t \frac{1}{\sqrt{t-\theta}} \frac{|\Delta s(\theta)|}{\theta} d\theta \right\}$$

$$\leq C \left\{ \int_0^t \frac{d\theta}{\sqrt{\theta(t-\theta)}} \int_0^\theta |\Delta\dot{\mu}(\tau)| d\tau + \int_0^t \frac{d\theta}{\sqrt{t-\theta}} \int_0^\theta |\Delta\dot{\mu}(\tau)| d\tau \right\}$$

$$\leq C \int_0^t |\Delta\dot{\mu}(\tau)| \, d\tau$$

Similarly,

$$|\Delta J_2|, \ \ldots, \ |\Delta J_8| \leq C \int_0^t |\Delta\dot{\mu}(\tau)| d\tau$$

From the expression

$$\Delta J_9 = -\frac{\gamma}{\pi} \left\{ \int_0^t \frac{\Delta\mu(\theta)}{\sqrt{t-\theta}} d\theta \int_{-1}^0 k(\bar{s}(\theta) - \xi, \alpha\theta) \varphi''(\xi) d\xi \right.$$

$$\left. + \int_0^t \frac{\mu(\theta)\Delta s(\theta)}{\sqrt{t-\theta}} d\theta \int_{-1}^0 \tilde{k}_1(s(\theta) - \xi, \alpha\theta) \varphi''(\xi) d\xi \right\}$$

it is easy to get

$$|\Delta J_9| \leq C \left\{ \int_0^t \frac{|\Delta\mu(\theta)|}{\sqrt{\theta(t-\theta)}} d\theta + \int_0^t \frac{1}{\sqrt{t-\theta}} \frac{|\Delta s(\theta)|}{\theta} d\theta \right\}$$

$$\leq C \int_0^t |\Delta\dot{\mu}(\tau)| d\tau$$

The estimates

$$|\Delta J_{10}|, \ |\Delta J_{11}|, \ |\Delta J_{12}| \leq C \int_0^t |\Delta\dot{\mu}(\tau)| d\tau$$

can be obtained similarly, and then we find that

$$|\Delta J_{13}| = \frac{|K|}{\sqrt{\pi}} \left| \int_0^t \frac{1}{\sqrt{t-\theta}} (\dot{\bar{\mu}}(\theta) \Delta\mu(\theta) + \mu(\theta) \Delta\dot{\mu}(\theta)) d\theta \right|$$

$$\leq C \int_0^t \frac{1}{\sqrt{t-\theta}} (|\Delta\mu(\theta)| + |\Delta\dot{\mu}(\theta)|) d\theta$$

$$\leq C\int_0^t\left(1+\frac{1}{\sqrt{t-\tau}}\right)|\Delta\dot{\mu}(\tau)|\,d\tau$$

Since

$$\Delta J_{14}=-\frac{a}{\sqrt{\pi}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\left\{\int_0^\theta \Delta\mathscr{L}+\mu(\tau)\frac{\partial}{\partial\theta}\bar{F}(\theta,\ \tau)\,d\tau\right.$$

$$\left.+\int_0^\theta \mathscr{L}+\mu(\tau)\frac{\partial}{\partial\theta}(\bar{F}(\theta,\ \tau)-F(\theta,\ \tau))\,d\tau\right\}$$

therefore

$$|\Delta J_{14}|\leq C\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\left\{\int_0^\theta\frac{1}{\sqrt{\theta-\tau}}\left(\|\Delta f\|+|\Delta\dot{\mu}(\tau)|+\int_0^\tau|\Delta\dot{\mu}(\xi)|\,d\xi\right)d\tau\right.$$

$$\left.+\int_0^\theta\frac{d\tau}{\sqrt{\theta-\tau}}\int_0^\tau|\Delta\dot{\mu}(\xi)|\,d\xi\right\}$$

$$\leq C\int_0^t d\tau\int_\tau^t\frac{d\theta}{\sqrt{(t-\theta)(\theta-\tau)}}\left(\|\Delta f\|+|\Delta\dot{\mu}(\tau)|+\int_0^\tau|\Delta\dot{\mu}(\xi)|\,d\xi\right)$$

$$\leq C\left\{\|\Delta f\|+\int_0^t|\Delta\dot{\mu}(\tau)|\,d\tau\right\}$$

The estimates

$$|\Delta J_{15}|,\ |\Delta J_{16}|\leq C\left\{\|\Delta f\|+\int_0^t|\Delta\dot{\mu}(\tau)|\,d\tau\right\}$$

can be found in the same way. But the estimate of ΔJ_{17} is different from them, we have

$$|\Delta J_{17}|=\frac{1}{4\pi\sqrt{a}}\left|\int_0^t\Delta\mathscr{L}+\mu(\tau)\,d\tau\int_0^1\sqrt{\frac{\rho}{1-\rho}}\dot{\bar{\mu}}(\tau+\rho(t-\tau))\,d\rho\right.$$

$$\left.+\int_0^t\mathscr{L}+\mu(\tau)\,d\tau\int_0^1\sqrt{\frac{\rho}{1-\rho}}\Delta\dot{\mu}(\tau+\rho(t-\tau))\,d\rho\right|$$

$$\leq C\left\{\|\Delta f\|+\int_0^t d\tau\int_0^1\sqrt{\frac{\rho}{1-\rho}}|\Delta\dot{\mu}(\tau+\rho(t-\tau))|\,d\rho+\int_0^t|\Delta\dot{\mu}(\tau)|\,d\tau\right\}$$

It is clear that the estimates

$$|\Delta J_{18}|,\ |\Delta J_{19}|,\ |\Delta J_{20}|\leq C\left\{\|\Delta g\|+\int_0^t|\Delta\dot{\mu}(\tau)|\,d\tau\right\}$$

$$|\Delta J_{21}|\leq C\left\{\|\Delta g\|+\int_0^t d\tau\int_0^1\sqrt{\frac{\rho}{1-\rho}}|\Delta\dot{\mu}(\tau+\rho(t-\tau))|\,d\rho+\int_0^t|\Delta\dot{\mu}(\tau)|\,d\tau\right\}$$

$$|\Delta J_{22}|\leq C\left\{\|\Delta f\|+\int_0^t|\Delta\dot{\mu}(\tau)|\,d\tau\right\}$$

$$|\Delta J_{23}|\leq C\left\{\|\Delta g\|+\int_0^t|\Delta\dot{\mu}(\tau)|\,d\tau\right\}$$

hold for $0\leq t\leq T^*$. By calculation we have

$$\Delta J_{24}=\frac{\gamma}{\sqrt{\pi}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\left\{\int_0^\theta k(\bar{s}(\theta)+1.\ a(\theta-\tau))\,\Delta f(\tau)\,d\tau\right.$$

$$+\Delta s(\theta)\int_0^{\theta}\tilde{k}_1(s(\theta)+1,\ a(\theta-\tau))f(\tau)\,d\tau\Big\}$$

therefore

$$|\Delta J_{24}|\leq C\Big\{\|\Delta f\|+\int_0^t|\Delta\dot{\mu}(\tau)|\,d\tau\Big\}$$

Similarly, we also have

$$|\Delta J_{25}|\leq C\Big\{\|\Delta g\|+\int_0^t|\Delta\dot{\mu}(\tau)|\,d\tau\Big\}$$

Finally, to estimate ΔJ_{26} and ΔJ_{27}, we denote

$$H(\theta,\ \tau)=\frac{1}{\sqrt{\theta-\tau}}\Big[1-\exp\Big(-\frac{(s(\theta)-s(\tau))^2}{4a(\theta-\tau)}\Big)\Big]$$

it follows that

$$\Big|\frac{\partial}{\partial\theta}H(\theta,\ \tau)\Big|\leq\frac{C}{\sqrt{\theta-\tau}}$$

$$\Big|\frac{\partial}{\partial\theta}(\overline{H}(\theta,\ \tau)-H(\theta,\ \tau))\Big|\leq\frac{C}{\sqrt{\theta-\tau}}\int_0^{\theta}|\Delta\dot{\mu}(\xi)|\,d\xi$$

thus

$$|\Delta J_{26}|=\frac{\gamma}{2\pi\sqrt{a}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^{\theta}\Big|\Delta\dot{\mu}(\tau)\frac{\partial\overline{H}(\theta,\ \tau)}{\partial\theta}+\dot{\mu}(\tau)\frac{\partial}{\partial\theta}(\overline{H}(\theta,\ \tau)-H(\theta,\ \tau))\Big|\,d\tau$$

$$\leq C\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^{\theta}\frac{d\tau}{\sqrt{\theta-\tau}}\Big(|\Delta\dot{\mu}(\tau)|+\int_0^{\tau}|\Delta\dot{\mu}(\xi)|\,d\xi\Big)$$

$$\leq C\int_0^t|\Delta\dot{\mu}(\tau)|\,d\tau$$

and

$$|\Delta J_{27}|\leq C\int_0^t|\Delta\dot{\mu}(\tau)|\,d\tau$$

Substituting the all estimates of ΔJ_i into (2. 34), we find that $\Delta\dot{\mu}(t)$ satisfies an integral inequality

$$|\Delta\dot{\mu}(t)|\leq C\Big\{\|\Delta f\|+\|\Delta g\|+\int_0^t d\tau\int_0^1\sqrt{\frac{\rho}{1-\rho}}|\Delta\dot{\mu}(\tau+\rho(t-\tau))|\,d\rho$$

$$+\int_0^t\Big(1+\frac{1}{\sqrt{t-\tau}}\Big)|\Delta\dot{\mu}(\tau)|\,d\tau\Big\}\tag{2. 35}$$

It follows from Lemma 2. 3 that there exists a constant $C>0$, such that

$$|\Delta\dot{\mu}(t)|\leq C\{\|\Delta f\|+\|\Delta g\|\}\tag{2. 36}$$

for $0\leq t\leq T^*$.

Remark As a conclusion of Theorem 2. 5, the results of Lemma 2. 1, 2. 2 can be rewritten as

$$\| \Delta y^{(i)} \|, \quad \| \Delta z^{(i)} \| \leq C\{ \| \Delta f \| + \| \Delta g \| \} \tag{2. 37}$$

$$|\Delta \mathscr{L}^{\pm} \mu(t)|, \quad |\Delta R^{\pm} \mu(t)| \leq C\{ \| \Delta f \| + \| \Delta g \| \} \tag{2. 38}$$

for $0 \leq t \leq T^*$.

3. Fréchet Differentiability

Define an operator

$$S: (f, g) \to s \tag{3. 1}$$

from the set

$$\sum = \{ (f, g) : f \text{ and } g \in C^1[0, T] \text{ satisfying } (1. 11) \} \tag{3. 2}$$

to $C^2[0, T^*]$, where $x = s(t)$ is the free boundary in (1. 1) — (1. 10). It is proved that S is Lipschitz continuous operator from $\sum$ to $C^2[0, T^*]$ in the previous section. In this section it will be proved that S is Fréchet differentiable from $\sum$ to $C^2[0, T^*]$.

Let (f, g) and $(f + \delta f, g + \delta g) \in \sum$, and

$$\frac{d^2}{dt^2} S(f, g)(t) = \ddot{s}(t) = \dot{\mu}(t), \quad \frac{d^2}{dt^2} S(f + \delta f, g + \delta g)(t) = \ddot{\bar{s}}(t) = \dot{\bar{\mu}}(t) \tag{3. 3}$$

we derive the expression of $\delta \dot{\mu}(t)$ which is the linear principal part of $\Delta \dot{\mu}(t) = \dot{\bar{\mu}}(t) - \dot{\mu}(t)$ with respect to δf and δg. Throughout this section we will use the notations

$$\delta \mu(t) = \int_0^t \delta \dot{\mu}(\tau) d\tau, \quad \delta s(t) = \int_0^t \delta \mu(\tau) d\tau \tag{3. 4}$$

Lemma 3. 1 *Suppose $\delta y^{(i)}(t)$, $i = 1, 2$, satisfy the system of linear integral equations*

$$\begin{aligned} \delta y^{(1)}(t) &+ 2a \int_0^t k_1(s(t) - s(\tau), a(t-\tau)) \delta y^{(1)}(\tau) d\tau \\ &+ 2a \int_0^t k_1(s(t) + 1, a(t-\tau)) \delta y^{(2)}(\tau) d\tau \\ &= \delta \mu(t) - U_1(s(t), t) \delta s(t) \\ &- 2a \int_0^t y^{(1)}(\tau) k_{11}(s(t) - s(\tau), a(t-\tau)) (\delta s(t) - \delta s(\tau)) d\tau \\ &- 2a \delta s(t) \int_0^t y^{(2)}(\tau) k_{11}(s(t) + 1, a(t-\tau)) d\tau \end{aligned} \tag{3. 5}$$

$$\begin{aligned} \delta y^{(2)}(t) &- 2a \int_0^t k_1(-1 - s(\tau), a(t-\tau)) \delta y^{(1)}(\tau) d\tau \\ &= -\delta f(t) - 2a \int_0^t y^{(1)}(\tau) k_{11}(-1 - s(\tau), a(t-\tau)) \delta s(\tau) d\tau \end{aligned} \tag{3. 6}$$

Then $\delta y^{(i)}(t)$ and $\delta \dot{y}^{(i)}(t) = d\delta y^{(i)}(t)/dt$, $i = 1, 2$. depend linearly on $\delta \dot{\mu}$, $\delta \mu$, δs and δf, and

$$|\delta y^{(i)}(t)|, \quad |\delta \dot{y}^{(i)}(t)| \leq C\left\{ \| \delta f \| + |\delta \dot{\mu}(t)| + \int_0^t |\delta \dot{\mu}(\tau)| d\tau \right\} \tag{3. 7}$$

$$\begin{aligned} &|\Delta y^{(i)}(t) - \delta y^{(i)}(t)|, \quad |\Delta \dot{y}^{(i)}(t) - \delta \dot{y}^{(i)}(t)| \\ &\leq C\left\{ \| \delta f \|^2 + \| \delta g \|^2 + \int_0^t |\Delta \dot{\mu}(\tau) - \delta \dot{\mu}(\tau)| d\tau \right\} \end{aligned} \tag{3. 8}$$

for $0 \leq t \leq T^*$.

Proof Differentiating (3. 5) and (3. 6) with respect to t, we find that $\delta \dot{y}^{(i)}(t)$ satisfy a system of linear integral equations

$$\delta \dot{y}^{(1)}(t) - \frac{\mu(t)}{4\sqrt{\pi a}} \int_0^t \frac{\delta \dot{y}^{(1)}(\tau)}{\sqrt{t-\tau}} d\tau = \frac{\delta\mu(t)}{4\sqrt{\pi a}} \int_0^t \frac{\dot{y}^{(1)}(\tau)}{\sqrt{t-\tau}} d\tau$$

$$+ \delta\dot{\mu}(t) - U_1(s(t), t)\delta\mu(t) - U_{11}(s(t), t)\mu(t)\delta s(t) - U_{21}(s(t), t)\delta s(t)$$

$$+ \frac{\dot{\mu}(t)}{\sqrt{4\pi a}} \int_0^t \frac{\delta y^{(1)}(\tau)}{\sqrt{t-\tau}} d\tau + \frac{\delta\dot{\mu}(t)}{\sqrt{4\pi a}} \int_0^t \frac{y^{(1)}(\tau)}{\sqrt{t-\tau}} d\tau$$

$$- \frac{1}{\sqrt{4\pi a}} \int_0^t y^{(1)}(\tau) \frac{\partial}{\partial t}\delta F(t, \tau) d\tau - \frac{1}{\sqrt{4\pi a}} \int_0^t \frac{\partial}{\partial t} F(t, \tau) \delta y^{(1)}(\tau) d\tau$$

$$- \frac{1}{\sqrt{4\pi a}} \int_0^t y^{(1)}(\tau) \frac{\partial}{\partial t}\delta G(t, \tau) d\tau - \frac{1}{\sqrt{4\pi a}} \int_0^t \frac{\partial}{\partial t} G(t, \tau) \delta y^{(1)}(\tau) d\tau$$

$$- 2a \int_0^t [\mu(t) k_{11}(s(t)+1, a(t-\tau)) + a k_{12}(s(t)+1, a(t-\tau))] \delta y^{(2)}(\tau) d\tau$$

$$- 2a\delta\mu(t) \int_0^t k_{11}(s(t)+1, a(t-\tau)) y^{(2)}(\tau) d\tau$$

$$- 2a\delta s(t) \int_0^t y^{(2)}(\tau) [\mu(t) k_{111}(s(t)+1, a(t-\tau))$$

$$+ a k_{121}(s(t)+1, a(t-\tau))] d\tau \qquad (3.\ 9)$$

$$\delta \dot{y}^{(2)}(t) = -\delta \dot{f}(t) - 2a^2 \int_0^t k_{121}(-1-s(\tau), a(t-\tau)) y^{(1)}(\tau) \delta s(\tau) d\tau$$

$$- 2a^2 \int_0^t k_{12}(-1-s(\tau), a(t-\tau)) \delta y^{(2)}(\tau) d\tau \qquad (3.\ 10)$$

where

$$\frac{\partial}{\partial t}\delta F(t, \tau) = (\delta\mu(t) - \delta\mu(\tau)) \left\{ (t-\tau)^{-\frac{3}{2}} + 4a\sqrt{\pi a} k_{11}(s(t)-s(\tau), a(t-\tau)) \right\}$$

$$+ (\delta s(t) - \delta s(\tau)) \left(-\frac{3}{2}(t-\tau)^{-\frac{5}{2}} + 4a\sqrt{\pi a} \frac{\partial}{\partial t} k_{11}(s(t)-s(\tau), a(t-\tau)) \right)$$

$$(3.\ 11)$$

$$\frac{\partial}{\partial t}\delta G(t, \tau) = \frac{1}{\sqrt{t-\tau}} \left\{ \delta\dot{\mu}(t) - \frac{3}{2(t-\tau)} \left[\delta\mu(t) - \frac{\delta s(t) - \delta s(\tau)}{t-\tau} \right] \right\} \qquad (3.\ 12)$$

It is clear that $\delta y^{(i)}(t)$ and $\delta \dot{y}^{(i)}(t)$, $i = 1, 2$, depend linearly on $\delta\dot{\mu}$, $\delta\mu$, δs and δf. (3. 7) can be obtained in the same way as it in Lemma 2. 1.

To estimate the difference $dy^{(i)}(t) = \Delta y^{(i)}(t) - \delta y^{(i)}(t)$, subtracting (3. 5) and (3. 6) from (2. 12) and (2. 13) respectively, and denoting $d\dot{\mu} = \Delta\dot{\mu} - \delta\dot{\mu}$, $d\mu = \Delta\mu - \delta\mu$, $ds = \Delta s - \delta s$, we have

$$dy^{(1)}(t) + 2a \int_0^t k_1(s(t)-s(\tau), a(t-\tau)) dy^{(1)}(\tau) d\tau$$

$$+ 2a \int_0^t k_1(s(t)+1, a(t-\tau)) dy^{(2)}(\tau) d\tau$$

$$= d\mu(t) - U_1(s(t),\ t)\,ds(t) - 2\alpha ds(t)\int_0^t y^{(2)}(\tau)k_{11}(s(t)+1,\ \alpha(t-\tau))\,d\tau$$

$$-2\alpha\int_0^t y^{(1)}(\tau)k_{11}(s(t)-s(\tau),\ \alpha(t-\tau))\,(ds(t)-ds(\tau))\,d\tau + R^{(1)} \quad (3.\ 13)$$

$$dy^{(2)}(t) - 2\alpha\int_0^t k_1(-1-s(\tau),\ \alpha(t-\tau))\,dy^{(1)}(\tau)\,d\tau$$

$$= -2\alpha\int_0^t y^{(1)}(\tau)k_{11}(-1-s(\tau),\ \alpha(t-\tau))\,ds(\tau)\,d\tau + R^{(2)} \quad (3.\ 14)$$

where

$$R^{(1)} = -(\widetilde{U}_1(s(t),\ t) - U_1(s(t),\ t))\Delta s(t)$$
$$-2\alpha\int_0^t y^{(1)}(\tau)(\Delta s(t) - \Delta s(\tau))[\widetilde{k}_{11}(s(t)-s(\tau),\ \alpha(t-\tau))$$
$$-k_{11}(s(t)-s(\tau),\ \alpha(t-\tau))]d\tau$$
$$-2\alpha\Delta s(t)\int_0^t y^{(2)}(\tau)[\widetilde{k}_{11}(s(t)+1,\ \alpha(t-\tau))$$
$$-k_{11}(s(t)+1,\ \alpha(t-\tau))]d\tau$$
$$= -\widetilde{\widetilde{U}}_{11}(s(t),\ t)(\Delta s(t))^2$$
$$-2\alpha\int_0^t y^{(1)}(\tau)(\Delta s(t)-\Delta s(\tau))^2\widetilde{\widetilde{k}}_{111}(s(t)-s(\tau),\ \alpha(t-\tau))\,d\tau$$
$$-2\alpha(\Delta s(t))^2\int_0^t y^{(2)}(\tau)\widetilde{\widetilde{k}}_{111}(s(t)+1,\ \alpha(t-\tau))\,d\tau \quad (3.\ 15)$$

$$R^{(2)} = -2\alpha\int_0^t y^{(1)}(\tau)\Delta s(\tau)[\widetilde{k}_{11}(-1-s(\tau),\ \alpha(t-\tau))$$
$$-k_{11}(-1-s(\tau),\ \alpha(t-\tau))]d\tau$$
$$= -2\alpha\int_0^t y^{(1)}(\tau)\widetilde{\widetilde{k}}_{111}(-1-s(\tau),\ \alpha(t-\tau))(\Delta s(\tau))^2 d\tau \quad (3.\ 16)$$

The fact is used here: for any function $\Phi(x,t)\in C^1$

$$\widetilde{\Phi}(x,\ t) - \Phi(x,\ t) = \int_0^1[\Phi(x+\xi\Delta x,\ t) - \Phi(x,\ t)]d\xi$$
$$= \Delta x\int_0^1\xi d\xi\int_0^1\Phi_1(x+\xi\eta\Delta x,\ t)\,d\eta = \widetilde{\widetilde{\Phi}}_1(x,\ t)\Delta x$$

It is easy to find

$$|R^{(1)}|,\ |R^{(2)}| \le C\{\|\delta f\|^2 + \|\delta g\|^2\} \quad (3.\ 17)$$

$$|dy^{(i)}(t)| \le C\left\{\|\delta f\|^2 + \|\delta\xi\|^2 + \int_0^t |d\dot{\mu}(\tau)|\,d\tau\right\} \quad (3.\ 18)$$

for $0\le t\le T^*$ in the same way as in Section 2. The expressions and estimates of $d\dot{y}^{(i)}(t),\ i=1,\ 2$, can be derived similarly.

Lemma 3.2 *Suppose $\delta z^{(i)}(t)$, $i=1,\ 2$, satisfy the system of linear integral equations*

$$\delta z^{(1)}(t) - 2\beta\int_0^t k_1(s(t)-s(\tau),\ \beta(t-\tau))\,\delta z^{(1)}(\tau)\,d\tau$$
$$-2\beta\int_0^t k_1(s(t)-1,\ \beta(t-\tau))\,\delta z^{(2)}(\tau)\,d\tau$$
$$= V_1(s(t),\ t)\,\delta s(t) - \delta\mu(t)$$

$$+2\beta\int_0^t z^{(1)}(\tau)k_{11}(s(t)-s(\tau),\ \beta(t-\tau))(\delta s(t)-\delta s(\tau))d\tau$$
$$+2\beta\delta s(t)\int_0^t z^{(2)}(\tau)k_{11}(s(t)-1,\ \beta(t-\tau))d\tau \tag{3. 19}$$

$$\delta z^{(2)}(t)+2\beta\int_0^t k_1(1-s(\tau),\ \beta(t-\tau))\delta z^{(1)}(\tau)d\tau$$
$$=\delta g(t)+2\beta\int_0^t k_{11}(1-s(\tau),\ \beta(t-\tau))z^{(1)}(\tau)\delta s(\tau)d\tau \tag{3. 20}$$

Then $\delta z^{(i)}(t)$ *and* $\delta\dot{z}^{(i)}(t)=d\delta z^{(i)}(t)/dt,\ i=1,2$, *depend linearly on* $\delta\dot{\mu}$, $\delta\mu$, δs *and* δg, *and*

$$|\delta z^{(i)}(t)|,\ |\delta\dot{z}^{(i)}(t)|\leq C\{\|\delta g\|+|\delta\dot{\mu}(t)|+\int_0^t|\delta\dot{\mu}(\tau)|d\tau\} \tag{3. 21}$$
$$|\Delta z^{(i)}(t)-\delta z^{(i)}(t)|,\ |\Delta\dot{z}^{(i)}(t)-\delta\dot{z}^{(i)}(t)|$$
$$\leq C\{\|\delta f\|^2+\|\delta g\|^2+\int_0^t|\Delta\dot{\mu}(\tau)-\delta\dot{\mu}(\tau)|d\tau\} \tag{3. 22}$$

Proof The proof follows the same outline as that of Lemma 3. 1.

Lemma 3. 3 *There exist* $\delta\mathscr{L}^{\pm}\mu$ *and* $\delta\mathscr{R}^{\pm}\mu$, *depending linearly on* $\delta\dot{\mu}$, $\delta\mu$, δs, δf *and* δg, *such that for* $0\leq t\leq T^*$

$$|\delta\mathscr{L}^{\pm}\mu(t)|\leq C\left\{\|\delta f\|+\int_0^t\left(1+\frac{1}{\sqrt{t-\tau}}\right)|\delta\dot{\mu}(\tau)|d\tau\right\} \tag{3. 23}$$

$$|\delta\mathscr{R}^{\pm}\mu(t)|\leq C\left\{\|\delta g\|+\int_0^t\left(1+\frac{1}{\sqrt{t-\tau}}\right)|\delta\dot{\mu}(\tau)|d\tau\right\} \tag{3. 24}$$

$$|\Delta\mathscr{L}^{\pm}\mu(t)-\delta\mathscr{L}^{\pm}\mu(t)|,\ |\Delta\mathscr{R}^{\pm}\mu(t)-\delta\mathscr{R}^{\pm}\mu(t)|$$
$$\leq C\left\{\|\delta f\|^2+\|\delta g\|^2+\int_0^t|\Delta\dot{\mu}(\tau)-\delta\dot{\mu}(\tau)|d\tau\right\} \tag{3. 25}$$

Proof Let

$$\delta\mathscr{L}^{+}\mu(t)=\gamma U_{11}(s(t),\ t)\delta s(t)-\frac{\gamma}{\alpha}y^{(1)}(t)\delta\mu(t)-\frac{\gamma}{\alpha}\mu(t)\delta y^{(1)}(t)$$
$$-2\gamma\int_0^t\mu(\tau)y^{(1)}(\tau)k_{11}(s(t)-s(\tau),\ \alpha(t-\tau))(\delta s(t)-\delta s(\tau))d\tau$$
$$-2\gamma\int_0^t y^{(1)}(\tau)k_1(s(t)-s(\tau),\ \alpha(t-\tau))\delta\mu(\tau)d\tau$$
$$-2\gamma\int_0^t\mu(\tau)k_1(s(t)-s(\tau),\ \alpha(t-\tau))\delta y^{(1)}(\tau)d\tau$$
$$+2\gamma\int_0^t\dot{y}^{(1)}(\tau)k_1(s(t)-s(\tau),\ \alpha(t-\tau))(\delta s(t)-\delta s(\tau))d\tau$$
$$+2\gamma\int_0^t k(s(t)-s(\tau),\ \alpha(t-\tau))\delta\dot{y}^{(1)}(\tau)d\tau$$
$$+2\gamma\delta s(t)\int_0^t k_1(s(t)+1,\ \alpha(t-\tau))\dot{y}^{(2)}(\tau)d\tau$$
$$+2\gamma\int_0^t k(s(t)+1,\ \alpha(t-\tau))\delta\dot{y}^{(2)}(\tau)d\tau \tag{3. 26}$$

replacing $\alpha, \gamma, U, y^{(i)}, \delta y^{(i)}, \dot{y}^{(i)}, \delta\dot{y}^{(i)}, s(t)+1$ in (3. 26) by $\beta, \lambda, V, z^{(i)}, \delta z^{(i)}, \dot{z}^{(i)}, \delta\dot{z}^{(i)}, s(t)-1$ respectively, we call the resulting expression $\delta\mathscr{R}^{-}\mu(t)$. Let

$$\delta\mathscr{L}^{-}\mu(t)=2\gamma\int_0^t \mu(\tau)y^{(1)}(\tau)k_{11}(-1-s(\tau),\ \alpha(t-\tau))\delta s(\tau)d\tau$$

$$-2\gamma\int_0^t y^{(1)}(\tau)k_1(-1-s(\tau),\ \alpha(t-\tau))\delta\mu(\tau)d\tau$$

$$-2\gamma\int_0^t \mu(\tau)k_1(-1-s(\tau),\ \alpha(t-\tau))\delta y^{(1)}\mu(\tau)d\tau$$

$$-2\gamma\int_0^t k_1(-1-s(\tau),\ \alpha(t-\tau))\dot{y}^{(1)}(\tau)\delta s(\tau)d\tau$$

$$+2\gamma\int_0^t k(-1-s(\tau),\ \alpha(t-\tau))\delta\dot{y}^{(1)}(\tau)d\tau$$

$$+\frac{\gamma}{\sqrt{\pi\alpha}}\int_0^t\frac{\delta\dot{y}^{(2)}(\tau)}{\sqrt{t-\tau}}d\tau \qquad (3.\ 27)$$

replacing $\alpha, \gamma, y^{(i)}, \delta y^{(i)}, \dot{y}^{(i)}, \delta\dot{y}^{(i)}, -1-s(\tau)$ in (3. 27) by $\beta, \lambda, z^{(i)}, \delta z^{(i)}, \dot{z}^{(i)}, \delta\dot{z}^{(i)}, 1-s(\tau)$ respectively, we call the resulting expression $\delta\mathscr{R}^{+}\mu(t)$.

It is easy to check that $\delta\mathscr{L}^{\pm}\mu$ and $\delta\mathscr{R}^{\pm}\mu$ satisfy the assertions of this lemma.

Theorem 3. 4 *There exists a unique continuous function $\delta\dot{\mu}(t):\ [0, T^*]\rightarrow R$ satisfies the linear integral equation for $\delta\dot{\mu}$,*

$$\delta\dot{\mu}(t)=\frac{2\sqrt{\alpha\beta}}{\lambda\sqrt{\alpha}+\gamma\sqrt{\beta}}\sum_{i=1}^{27}\delta J_i \qquad (3.\ 28)$$

for $0\leq t\leq T^*$, where δJ_i are defined below. Moreover, there exists a constant $N_2>0$, depending only on the C^2-norms of φ and ψ, the C^1-norms of f and g, the constants $\alpha, \beta, \gamma, \lambda, K, \delta, T^*$, N and N_1, such that

$$|\delta\dot{\mu}(t)|\leq N_2\{\|\delta f\|+\|\delta g\|\} \qquad (3.\ 29)$$

$$|\Delta\dot{\mu}(t)-\delta\dot{\mu}(t)|\leq N_2\{\|\delta f\|^2+\|\delta g\|^2\} \qquad (3.\ 30)$$

for $0\leq t\leq T^*$.

Proof Without loss of generality we restrict that the increments δf and δg satisfy

$$\|\delta f\|,\ \|\delta g\|\leq\max\{\|f\|,\ \|g\|\} \qquad (3.\ 31)$$

in our proof. Thus T^*, N and N_1 are independent on δf and δg. Let

$$\delta J_1=\frac{\gamma\varphi'(0)}{\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t-\theta}}[k(s(\theta),\ \alpha\theta)\delta\mu(\theta)+k_1(s(\theta),\ \alpha\theta)\mu(\theta)\delta s(\theta)]d\theta$$

$$\delta J_2=-\frac{\gamma\varphi'(-1)}{\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t-\theta}}[k(s(\theta)+1,\ \alpha\theta)\delta\mu(\theta)$$

$$+k_1(s(\theta)+1,\ \alpha\theta)\mu(\theta)\delta s(\theta)]d\theta$$

$$\delta J_3 = \frac{\gamma a \varphi'(0)}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\theta}} k_{11}(s(\theta),\ a\theta)\,\delta s(\theta)\,d\theta$$

$$\delta J_4 = -\frac{\gamma a \varphi'(-1)}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\theta}} k_{11}(s(\theta)+1,\ a\theta)\,\delta s(\theta)\,d\theta$$

Replacing $a, \gamma, \varphi'(0), \varphi'(-1), s(\theta)+1$ in the expressions of $\delta J_1, \ldots, \delta J_4$ by $\beta, \lambda, \psi'(0), \psi'(1), s(\theta)-1$ respectively, we obtain $\delta J_5, \ldots, \delta J_8$. Next we let

$$\delta J_9 = -\frac{\gamma}{\sqrt{\pi}}\Big[\int_0^t \frac{\delta\mu(\theta)\,d\theta}{\sqrt{t-\theta}} \int_{-1}^0 k(s(\theta)-\xi,\ a\theta)\,\varphi''(\xi)\,d\xi + \int_0^t \frac{\mu(\theta)\,\delta s(\theta)\,d\theta}{\sqrt{t-\theta}} \int_{-1}^0 k_1(s(\theta)-\xi,\ a\theta)\,\varphi''(\xi)\,d\xi\Big]$$

$$\delta J_{10} = -\frac{\gamma a}{\sqrt{\pi}} \int_0^t \frac{\delta s(\theta)\,d\theta}{\sqrt{t-\theta}} \int_{-1}^0 k_{11}(s(\theta)-\xi,\ a\theta)\,\varphi''(\xi)\,d\xi$$

Replacing a, γ, φ'' and the integral interval $[-1, 0]$ in the expressions of δJ_9 and δJ_{10} by β, λ, ψ'' and the integral interval $[0, 1]$ respectively, we get $-\delta J_{11}$ and $-\delta J_{12}$. And then let

$$\delta J_{13} = -\frac{K}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\theta}} [\mu(\theta)\,\delta\dot{\mu}(\theta) + \dot{\mu}(\theta)\,\delta\mu(\theta)]\,d\theta$$

$$\delta J_{14} = -\frac{1}{4\pi\sqrt{a}} \int_0^t \frac{d\theta}{\sqrt{t-\theta}}\Big[\int_0^\theta \frac{\partial F(\theta,\ \tau)}{\partial\theta}\,\delta\mathscr{L}^+\mu(\tau)\,d\tau + \int_0^\theta \mathscr{L}^+\mu(\tau)\,\frac{\partial}{\partial\theta}\delta F(\theta,\ \tau)\,d\tau\Big]$$

$$\delta J_{15} = -\frac{1}{4\pi\sqrt{a}} \int_0^t \frac{d\theta}{\sqrt{t-\theta}}\Big[\int_0^\theta \frac{\partial G(\theta,\ \tau)}{\partial\theta}\,\delta\mathscr{L}^+\mu(\tau)\,d\tau + \int_0^\theta \mathscr{L}^+\mu(\tau)\,\frac{\partial}{\partial\theta}\delta G(\theta,\ \tau)\,d\tau\Big]$$

$$\delta J_{16} = \frac{1}{4\sqrt{a}}[\mathscr{L}^+\mu(t)\,\delta\mu(t) + \mu(t)\,\delta\mathscr{L}^+\mu(t)]$$

$$\delta J_{17} = \frac{1}{4\pi\sqrt{a}}\Big[\int_0^t \delta\mathscr{L}^+\mu(\tau)\,d\tau \int_0^1 \sqrt{\frac{\rho}{1-\rho}}\,\dot{\mu}(\tau+\rho(t-\tau))\,d\rho + \int_0^t \mathscr{L}^+\mu(\tau)\,d\tau \int_0^1 \sqrt{\frac{\rho}{1-\rho}}\,\delta\dot{\mu}(\tau+\rho(t-\tau))\,d\rho\Big]$$

Replacing a and $\mathscr{L}^+\mu$ in the expressions of $\delta J_{14}, \ldots, \delta J_{17}$ (including F and G) by β and $\mathscr{R}^-\mu$, we find $\delta J_{18}, \ldots, \delta J_{21}$ respectively. Let

$$\delta J_{22} = \frac{a}{\sqrt{\pi}} \int_0^t \frac{d\theta}{\sqrt{t-\theta}}\Big\{\int_0^\theta \delta\mathscr{L}^-\mu(\tau)\,[\mu(\theta)\,k_{11}(s(\theta)+1,\ a(\theta-\tau)) + a k_{12}(s(\theta)+1,\ a(\theta-\tau))]\,d\tau + \delta\mu(\theta)\int_0^\theta \mathscr{L}^-\mu(\tau)\,k_{11}(s(\theta)+1,\ a(\theta-\tau))\,d\tau + \delta(s(\theta)\int_0^\theta \mathscr{L}^-\mu(\tau)\,[\mu(\theta)\,k_{111}(s(\theta)+1,\ a(\theta-\tau)) + a k_{121}(s(\theta)+1,\ a(\theta-\tau))]\,d\tau\Big\}$$

Replacing α, $\mathscr{L}-\mu$, $s(\theta)+1$ in the formula of δJ_{22} by β, $\mathscr{R}+\mu$, $s(\theta)-1$ respectively, we have δJ_{23}, and then replacing α, $\mathscr{L}-\mu$, k_{11}, k_{12}, k_{111} and k_{121} in the expression of δJ_{22} by γ, $\dot{f}$, k_1, k_2, k_{11} and k_{21} respectively and replacing β, $\mathscr{R}^{+}\mu$, k_{11}, k_{12}, k_{111} and k_{121} respectively in the expression of δJ_{23} by λ, $\dot{g}$, k_1, k_2, k_{11} and k_{21} respectively, we get δJ_{24} and δJ_{25}. Finally let

$$\delta J_{26}=\frac{\gamma}{2\pi\sqrt{\alpha}}\int_0^t\frac{d\theta}{\sqrt{t-\theta}}\int_0^{\theta}\left[\delta\dot{\mu}(\tau)\frac{\partial}{\partial\theta}H(\theta,\ \tau)+\dot{\mu}(\tau)\frac{\partial}{\partial\theta}\delta H(\theta,\ \tau)\right]d\tau$$

where

$$\frac{\partial}{\partial\theta}\delta H(\theta,\ \tau)=\frac{\partial}{\partial\theta}\left\{\frac{s(\theta)-s(\tau)}{2\alpha(\theta-\tau)^{3/2}}\exp\left(-\frac{(s(\theta)-s(\tau))^2}{4\alpha(\theta-\tau)}\right)(\delta s(\theta)-\delta s(\tau))\right\} \tag{3. 32}$$

replacing α and γ in δJ_{26} and (3. 32) by β and λ we obtain δJ_{27}.

By the definitions of δJ_i and Lemma 3. 3 it follows that every δJ_i depends linearly on $\delta\dot{\mu}$, $\delta\mu$, δs, δf and δg, and since the dependence upon $\delta\dot{\mu}$ in $\delta\mu$ and δs is linear, (3. 28) is a linear integral equation for $\delta\dot{\mu}(t)$ with an inhomogeneous term depending linearly on δf and δg. Moreover this equation is one of that types discussed in Lemma 2. 4. Therefore by Lemma 2. 4 and 2. 3 there exists unique solution $\delta\dot{\mu}(t)$ of (3. 28) for $0\leq t\leq T''$ which depends linearly on δf and δg, and $\delta\dot{\mu}(t)$ satisfies the estimate

$$|\delta\dot{\mu}(t)|\leq C\{\|\delta f\|+\|\delta g\|\}\qquad\text{for}\qquad 0\leq t\leq T'' \tag{3. 33}$$

To prove the estimate (3. 30), we need to calculate every difference $\Delta J_i-\delta J_i$. For this we first replace $\delta\dot{\mu}$, $\delta\mu$, δs, $\delta y^{(i)}$, $\delta\dot{y}^{(i)}$, $\delta z^{(i)}$ and $\delta\dot{z}^{(i)}$ in $\delta\mathscr{L}\pm\mu$, $\delta\mathscr{R}\pm\mu$, δF, δG and δH by $d\dot{\mu}$, $d\mu$, ds, $dy^{(i)}$, $d\dot{y}^{(i)}$, $dz^{(i)}$ and $d\dot{z}^{(i)}$ respectively and call the resulting expressions $d\mathscr{L}\pm\mu$, $d\mathscr{R}\pm\mu$, dF, dG and dH. And then we replace $\delta\dot{\mu}$, $\delta\mu$, δs, $\delta\mathscr{L}\pm\mu$, $\delta\mathscr{R}\pm\mu$, δF, δG and δH in every δJ_i by $\delta\dot{\mu}$, $d\mu$, ds, $d\mathscr{L}\pm\mu$, $d\mathscr{R}\pm\mu$, dF, dG and dH respectively, and call the resulting expression dJ_i corresponding to δJ_i. Now let

$$\Delta J_i-\delta J_i=dJ_i+K_i,\quad i=1,\ 2,\ \dots,\ 27 \tag{3. 34}$$

It follows that

$$\begin{aligned}|dJ_1|&=\left|\frac{\gamma\varphi'(0)}{\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t-\theta}}(k(s(\theta),\ \alpha\theta)\,d\mu(\theta)+k_x(s(\theta),\ \alpha\theta)\,\mu(\theta)\,ds(\theta))\,d\theta\right|\\&\leq C\int_0^t\frac{1}{\sqrt{t-\theta}}\left[\frac{|d\mu(\theta)|}{\sqrt{\theta}}+\frac{|ds(\theta)|}{\theta}\right]d\theta\\&\leq C\int_0^t|d\dot{\mu}(\tau)|\,d\tau\end{aligned} \tag{3. 35}$$

$$|K_1|=\left|\frac{\gamma\varphi'(0)}{\sqrt{\pi}}\int_0^t\frac{\mu(\theta)}{\sqrt{t-\theta}}\bar{k}_{11}(s(\theta),\ \alpha\theta)\,(\Delta s(\theta))^2\,d\theta\right|$$

$$\leq C\left(\int_0^t |\Delta\dot{\mu}(\tau)|\delta\tau\right)^2$$

$$\leq C\{ \| \delta f \|^2 + \| \delta g \|^2\} \tag{3. 36}$$

Similarly

$$|dJ_i| \leq C \int_0^t |d\dot{\mu}(\tau)|d\tau, \ i=2, 3, ..., 12 \tag{3. 37}$$

$$|dJ_{13}| \leq C \int_0^t \left(1+\frac{1}{\sqrt{t-\tau}}\right)|d\dot{\mu}(\tau)|d\tau \tag{3. 38}$$

$$|K_i| \leq C\{ \| \delta f \|^2 + \| \delta g \|^2\}, \ i=2, 3, ..., 13 \tag{3. 39}$$

To estimate the other dJ_i and K_i, we need

$$|d\mathscr{L}^{\pm}\mu(t)|, \ |d\mathscr{R}^{\pm}\mu(t)| \leq C\{ \| \delta f \|^2 + \| \delta g \|^2 + \int_0^t |d\dot{\mu}(\tau)|d\tau\} \tag{3. 40}$$

$$|\Delta\mathscr{L}^{\pm}\mu(t) - \delta\mathscr{L}^{\pm}\mu(t) - d\mathscr{L}^{\pm}\mu(t)|, \ |\Delta\mathscr{R}^{\pm}\mu(\tau) - \delta\mathscr{R}^{\pm}\mu(t) - d\mathscr{R}^{\pm}\mu(t)|$$
$$\leq C\{ \| \delta f \|^2 + \| \delta g \|^2\} \tag{3. 41}$$

for $0 \leq t \leq T^*$, and

$$\left|\frac{\partial}{\partial\theta}dF(\theta, \tau)\right|, \ \left|\frac{\partial}{\partial\theta}dH(\theta, \tau)\right| \leq \frac{C}{\sqrt{\theta-\tau}} \int_0^\theta |d\dot{\mu}(\tau)|d\tau \tag{3. 42}$$

$$\left|\frac{\partial}{\partial\theta}dG(\theta, \tau)\right| \leq \frac{C}{\sqrt{\theta-\tau}}\left\{|d\dot{\mu}(\theta)| + \int_0^\theta |d\dot{\mu}(\tau)|d\tau\right\} \tag{3. 43}$$

$$\left|\frac{\partial}{\partial\theta}[\bar{F}(\theta, \tau) - F(\theta, \tau) - \delta F(\theta, \tau) \quad dF(\theta, \tau)]\right|,$$

$$\left|\frac{\partial}{\partial\theta}[\bar{H}(\theta, \tau) - H(\theta, \tau) - \delta H(\theta, \tau) - dH(\theta, \tau)]\right| \leq \frac{C}{\sqrt{\theta-\tau}}\{ \| \delta f \|^2 + \| \delta g \|^2\} \tag{3. 44}$$

and an identity

$$\bar{G}(\theta, \tau) - G(\theta, \tau) - \delta G(\theta, \tau) - dG(\theta, \tau) = 0 \tag{3. 45}$$

for $0 \leqslant \tau < \theta$.

Using these estimates we find that for $i \geq 14$, $i \neq 17$ and 21

$$|dJ_i| \leq C\left\{ \| \delta f \|^2 + \| \delta g \|^2 + \int_0^t |d\dot{\mu}(\tau)|d\tau\right\} \tag{3. 46}$$

and

$$|dJ_{17}|, \ |dJ_{21}| \leq C\left\{ \| \delta f \|^2 + \| \delta g \|^2 + \int_0^t d\tau \int_0^1 \sqrt{\frac{\rho}{1-\rho}}|d\mu(\tau+\rho(t-\tau))|d\tau\right.$$
$$\left. + \int_0^t |d\dot{\mu}(\tau)|d\tau\right\} \tag{3. 47}$$

moreover, all of K_i satisfy

$$|K_i| \leq C\{\|\delta f\|^2 + \|\delta g\|^2\} \tag{3.48}$$

Substituting these estimates of dJ_i and K_i into (3.28), we have an inequality

$$|d\dot{\mu}(t)| \leq C\Big\{\|\delta f\|^2 + \|\delta g\|^2 + \int_0^t d\tau \int_0^1 \sqrt{\frac{\rho}{1-\rho}}|d\dot{\mu}(\tau + \rho(t-\tau))|d\tau + \int_0^t \Big(1 + \frac{1}{\sqrt{t-\tau}}\Big)|d\dot{\mu}(\tau)|d\tau\Big\} \tag{3.49}$$

it implies the desired result.

Theorem 3.4 shows that the free boundary operator $S(f, g)$ is Fréchet differentiable from $\sum$ to $C^2[0, T^*]$, and the Fréchet differential is

$$dS(f, g)(t) = ds(t) = \int_0^t d\tau \int_0^\tau \delta\dot{\mu}(\xi)\,d\xi \tag{3.50}$$

By the previous results we are now able to investigate the Lipschitz continuous dependence of $\delta\dot{\mu}(t)$, which is stated in the next theorem. Let $\delta\dot{\mu}(t)$ and $\delta\dot{\bar{\mu}}(t)$ be the Fréchet differential corresponding to (f, g) and $(\bar{f}, \bar{g}) \in \sum$ respectively.

Theorem 3.5 *There exists a constant $N_3 > 0$, depending only on the given data and δ, T^*, N, N_1, N_2, such that*

$$|\delta\dot{\bar{\mu}}(t) - \delta\dot{\mu}(t)| \leq N_3\{\|\delta f\| + \|\delta g\|\}\{\|\bar{f} - f\| + \|\bar{g} - g\|\}$$

for $0 \leq t \leq T^*$. (3.51)

Proof Since it is similar to the proof of Theorem 2.5, the details are omitted here.

Thus we have proved that the free boundary operator $S(f, g)$ is Lipschitz continuous Fréchet differentiable in its domain of definition. This result can be used to construct an effective procedure for solving the inverse Muskat's problems of this type.

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