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REFLECTION OF SINGULARITIES AT BOUNDARY FOR PIECEWISE SMOOTH SOLUTIONS TO SEMILINEAR HYPERBOLIC SYSTEMS

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1. Introduction

During the last decade, much has been done in the study of singularity propagation for nonlinear partial differential equations, which has already been introduced in [1] and [6]. However, one has not seen much work on the reflection of singularities at boundary before. In 1979, M. Reed & J. Berning [8] proved that the singularities still propagate along the characteristic curves after reflection at the boundary for semilinear wave equations in one-dimensional case. For the multi-dimensional case, there have been a lot of work lately, such as that done by G. Métivier [6], M. Beals & G. Métivier [2] and M. Sablé-Tougeron [9], who use the tools of pseudodifferential and paradifferential operators in conormal distributions. Nevertheless, one has not seen any work done by classical methods in piecewise smooth solutions up till now, while to solve this problem is of equal importance.

On the other hand, J. Rauch & M. Reed [7] proved the following result about the singularity propagation of the solution to the Cauchy problem for semilinear hyperbolic systems by classical methods:

Given a symmetric strictly hyperbolic system

$$\begin{aligned} &\partial_t u(t, x) + \sum_{j=1}^{n} A_j(t, x)\,\partial_{x_j} u(t, x) + B(t, x)\,u(t, x) \\ &= (Pu)(t, x) \\ &= f(u; t, x), \qquad x \in \Omega \end{aligned} \tag{1.1}$$

where $x = (x_1, x_2, \dots, x_n) = (x_1, x')$; P is the partial differential operator representing the first line of the equations; $A_j's$ are 2×2 symmetric matrices, B is a 2×2 matrix, $u = (u_1, u_2)^T$; $A_j(t, x)'s$ and $B(t, x)$ are smooth enough, $f(u; t, x)$ is smooth enough with respect to its arguments; $A_j's$, B and f are all constants outside a compact set with respect to (t, x) . If the initial data are piecewise smooth with a jump discontinuity across an $(n$

— 1) -dimensional hypersurface σ, then there is a unique local piecewise smooth solution to the Cauchy problem of (1.1), whose singularities propagate along the two characteristic hypersurfaces issuing from σ. After that, Chen Shuxing proved the same result in the 3×3 case in [5].

In this paper, one has proved the following main theorem:

Theorem. *For (1.1) and a boundary condition*

$$Mu|_{\partial\Omega}=0 \qquad (1.2)$$

where $M\in C(\partial\Omega:\ Hom\ R^2)$, the boundary $\partial\Omega$ of the region Ω is regular, (1.2) is a stably admissible boundary condition (cf. [11], [12]), $[0, T]\times\partial\Omega$ is noncharacteristic for small T. If in $t\leq 0$, there is a piecewise smooth solution to (1.1) and (1.2) whose singularities propagate along a characteristic hypersurface $\sum_2$ where $\sum_2\cap\{t\geq 0\}$ intersects with the boundary $\partial\Omega\times[0, T]$ at an $(n-1)$ -dimensional hypersurface σ and is reflected into another characteristic hypersurface $\sum_1$; then in $t\geq 0$, there is still a local piecewise smooth solution whose singularities propagate along $\sum_1\cup\sum_2$. Furthermore, the order of the singularities is maintained after the reflection.

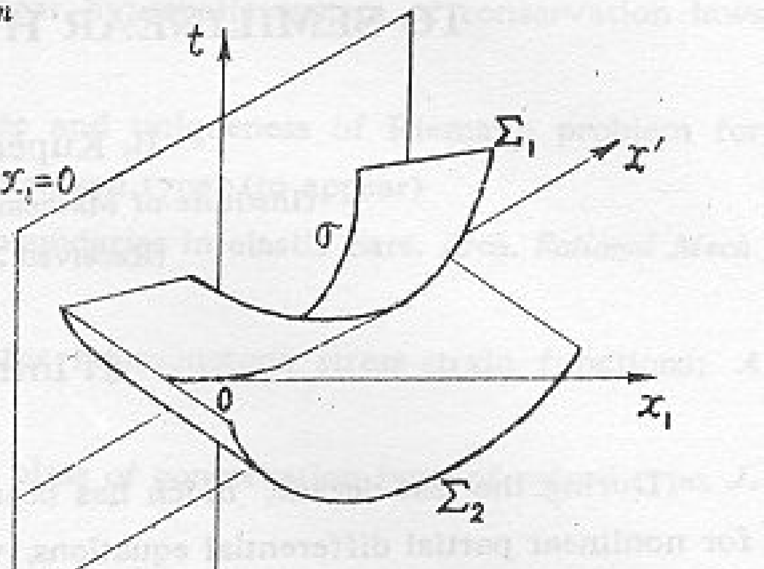

Fig. 1

In our paper, we will first simplify the problem in section 2, and then we will estimate the jumps across the characteristic hypersurface $\sum_1$. The proof of the main theorem will be given in section 4 and the order of the singularities will be discussed in section 5.

2. Simplification

Thanks to the regularity of the boundary, we can flatten it locally by a transformation of the independent variables. So we suppose the boundary to be $\{x_1=0\}$ and O to be the origin, as shown in Fig. 1. We suppose further that $\sum_1$ is above $\sum_2$, where $\sum_i (i=1, 2)$ is the characteristic hypersurfaces corresponding to the eigenvalues $\lambda_i (i=1, 2)$, that is to say, $\lambda_2<0<\lambda_1$. First, we must solve the problem in $I=\{(t, x)\ |t\geq 0,\ (t, x)$ is below $\sum_2\}$.

Denote σ_2 to be the $(n-1)$ -dimensional hypersurface at which $\sum_2$ intersects with the initial plane. σ_2 is tangent to the boundary at O. The solution is known in the region

$\widetilde{\Omega}$ between σ_2 and the boundary on the initial plane thanks to the given condition, as shown in Fig. 3. We continue the solution in $\widetilde{\Omega}$ into a neighbourhood B of O smoothly and then get a local solution u near O by solving an initial-boundary value problem in $[0, +\infty] \times B$ (cf. [4]). As I is in fact in the determinate region of $\widetilde{\Omega}$ for the initial-boundary value problem (1. 1) and (1. 2), the value of u in I is the very solution in I to the problem we consider, that is to say $\tilde{u}|_I = u|_I$. While using the results in [4], we must see that we do not need the restriction on f for $t=0$, because we have already known $f(u; t, x)$ for $t \leq 0$.

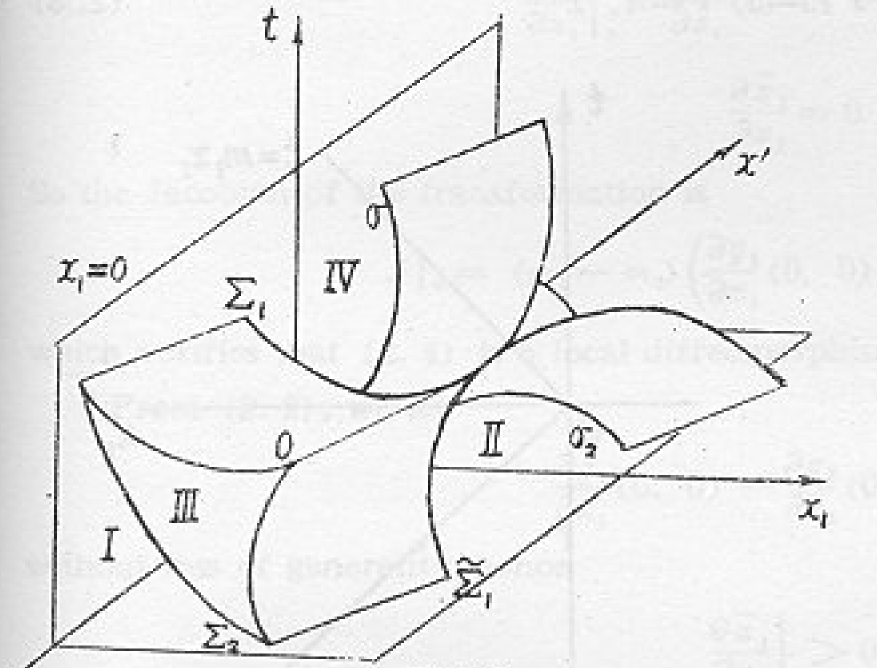

Fig. 2

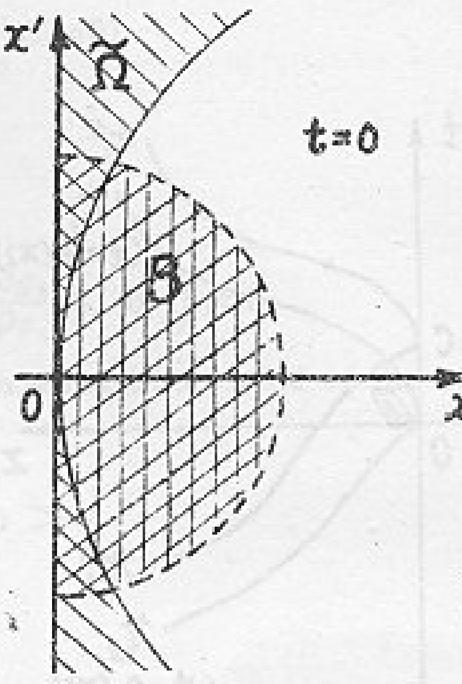

Fig. 3

Let $\widetilde{\sum}_1$ to be the characteristic hypersurface issuing upward from σ_2 corresponding to λ_1, we can get u in the region II $= \{(t, x) \mid t \geq 0,\ (t, x)$ is below $\widetilde{\sum}_1\}$ by solving a Cauchy problem.

Up till now, we have got the solution in the region $\{(t, x) \mid t \geq 0,\ (t, x)$ is below $\sum_1\}$ except for a part region III $= \{(t, x)$ is above $\sum_2$ and $\widetilde{\sum}_1$, below $\sum_1\}$. For further discussion, we first introduce a transformation of the independent variables.

Lemma 2. 1 *For two hypersurfaces π_1 and π_2 in the half space $\{x_1 \geq 0\}$, if π_1 and π_2 intersect with the boundary $\beta = \{x_1 = 0\}$ at the same $(n-1)$-dimensional hypersurface σ and π_1, π_2, β are transversal to each other; then there exists a local diffeomorphism which transforms the half space into itself and transforms π_1, π_2 into $\{t = m_1 x_1\}$, $\{t = m_2 x_1\}$ respectively, the origin being fixed, where m_1 and m_2 are two different real numbers.*

Proof First, we transform σ into an $(n-1)$-dimensional plane $\{(t, x) \mid t = 0,\ x_1 = 0\}$.

For the transversality of π_1 and π_2 with the boundary, we can express them as

$\{(t, x) \mid t = g_i(x)\}$ $(i = 1, 2)$ respectively. It is obvious that

$$g_i(0, x') = 0, \ i = 1, \ 2 \tag{2.1}$$

$$\left(\frac{\partial}{\partial x_1} g_1\right)(0, \ x') \neq \left(\frac{\partial}{\partial x_1} g_2\right)(0, \ x') \tag{2.2}$$

For any point P near O, there exists two real numbers c_1, c_2 such that $P \in \{(t, x) \mid t = g_1(x_1, x') + c_1\} \cap \{(t, x) \mid t = g_2(x_1, x') + c_2\}$. So we can define two functions φ_1 and φ_2 such that $\varphi_i(P) = c_i$ $(i = 1, 2)$. We can see that φ_i are C^∞-smooth functions and that

$$\varphi_i(t, \ 0, \ x') = t, \ i = 1, \ 2 \tag{2.3}$$

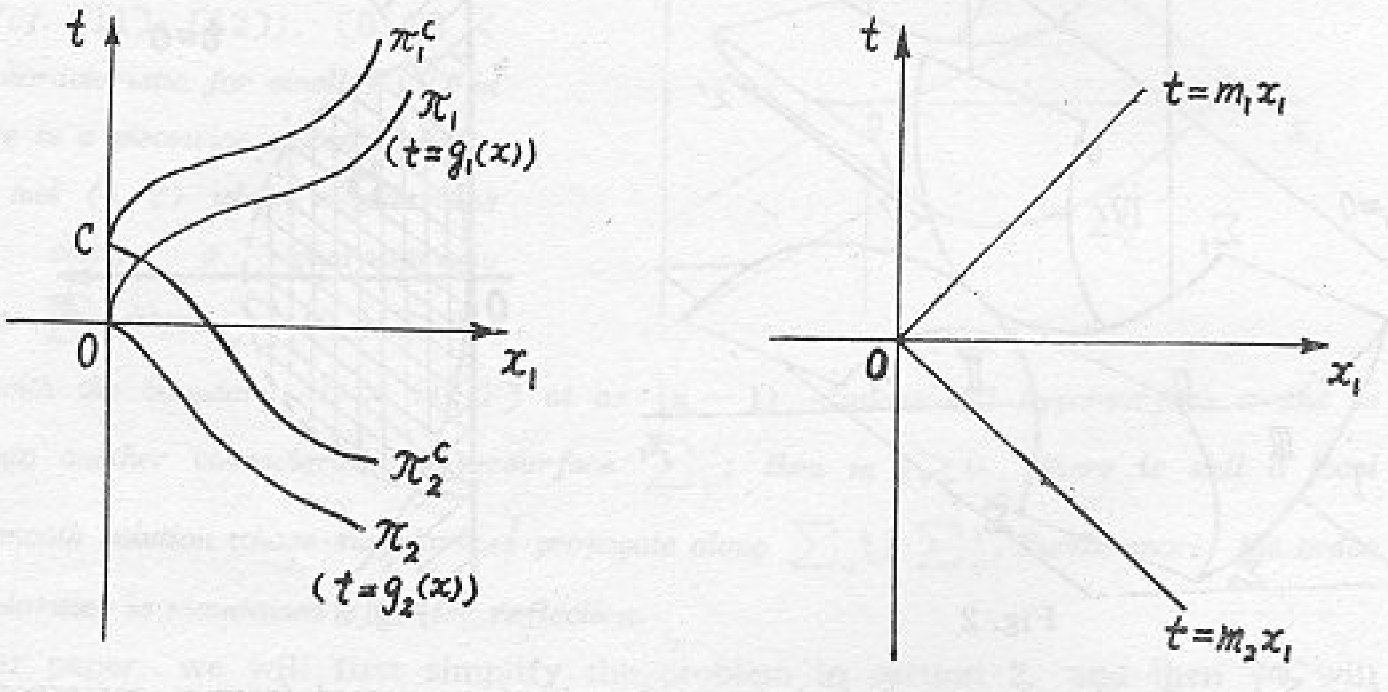

Fig. 4

Therefore, we define a transformation T:

$$\begin{cases} \tilde{t} = m_1\varphi_2 - m_2\varphi_1 \\ \tilde{x}_1 = \varphi_2 - \varphi_1 \\ \tilde{x}' = x' \end{cases} \tag{2.4}$$

From (2.3), $\frac{\partial}{\partial t}\varphi_i|_0 = 1$, $\frac{\partial}{\partial x_j}\varphi_i|_0 = 0$ $(i = 1, 2; j = 2, 3, \dots, n)$.

For φ_i = const. in π_i $(i = 1, 2)$, we have

$$\frac{\partial \varphi_i}{\partial x_1} + \frac{\partial \varphi_i}{\partial t}\frac{\partial t}{\partial x_1} + \sum_{j=2}^{n} \frac{\partial \varphi_i}{\partial x_j}\frac{\partial x_j}{\partial x_1} = 0, \quad (t, x) \in \varphi_i$$

Hence,

$$\frac{\partial \varphi_i}{\partial x_1} + \frac{\partial t}{\partial x_1} = 0, \ \text{for } (t, \ x) = 0$$

$$\left.\frac{\partial \varphi_i}{\partial x_1}\right|_0 = -\left.\frac{\partial t}{\partial x_1}\right|_0 = -\frac{\partial g_i}{\partial x_1}(0, \ 0); \ i = 1, \ 2 \tag{2.5}$$

Therefore,

$$\left.\frac{\partial \tilde{t}}{\partial t}\right|_0 = m_1 \left.\frac{\partial \varphi_2}{\partial t}\right|_0 - m_2 \frac{\partial \varphi_1}{\partial t} = m_1 - m_2 \tag{2.6}$$

$$\left.\frac{\partial \tilde{t}}{\partial x_1}\right|_0 = -m_1 \frac{\partial g_2}{\partial x_1}(0,\ 0) + m_2 \frac{\partial g_1}{\partial x_1}(0,\ 0) \tag{2.7}$$

$$\left.\frac{\partial \tilde{t}}{\partial x_j}\right|_0 = 0, \quad (j = 2,\ 3,\ \ldots,\ n) \tag{2.8}$$

$$\frac{\partial \tilde{x}_1}{\partial t} = 0 \tag{2.9}$$

$$\left.\frac{\partial \tilde{x}_1}{\partial x_1}\right|_0 = \frac{\partial g_1}{\partial x_1}(0,\ 0) - \frac{\partial g_2}{\partial x_1}(0,\ 0) \tag{2.10}$$

$$\frac{\partial \tilde{x}_1}{\partial x_j} = 0 \tag{2.11}$$

So the Jacobian of the transformation is

$$J|_0 = (m_1 - m_2)\left(\frac{\partial g_1}{\partial x_1}(0,\ 0) - \frac{\partial g_2}{\partial x_1}(0,\ 0)\right) \neq 0 \tag{2.12}$$

which verifies that (2. 4) is a local diffeomorphism in a neighbourhood of the origin.

From (2. 2), we let

$$\frac{\partial g_1}{\partial x_1}(0,\ 0) - \frac{\partial g_2}{\partial x_1}(0,\ 0) > 0 \tag{2.13}$$

without loss of generality, hence

$$\left.\frac{\partial \tilde{x}_1}{\partial x_1}\right|_0 > 0 \tag{2.14}$$

From the above reasoning, we have proved that the half space $\{x_1 \geq 0\}$ is still changed into the half space $\{\tilde{x}_1 \geq 0\}$ and π_i' s $(i = 1, 2)$ are changed into $\{\tilde{t} = m_i \tilde{x}_1\}$ $(i = 1, 2)$.

Using Lemma 2. 1, we change $\sum_1$ and $\sum_2$ into $\{x_1 = t\}$ and $\{x_1 = -t\}$ respectively. As the problem (1. 1) and

$$\begin{cases} u|_{\Gamma_2} \text{ and } u|_{t=-\tau} \quad (\tau > 0) \\ \text{are known} \end{cases} \tag{2.15}$$

the solution can be obtained; the solution in III can be obtained, for the existence region of the former is dependent only upon the equation itself, the value of u on $\sum_2$ and τ. When τ is sufficiently small, the existence region is independent of τ and so contains the point O. While the existence of the solution to the problem (1. 1) & (2. 15) can be deduced from the results in [7].

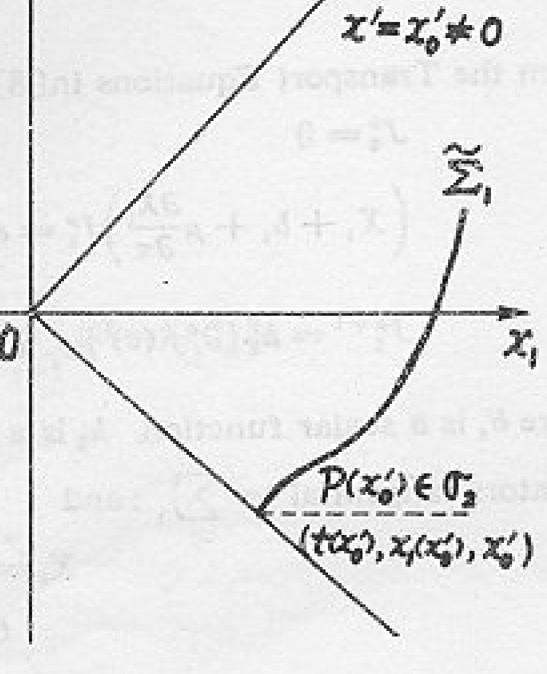

Fig. 5

For further simplification, we change A_1 in (1. 1) into a diagonal matrix through a linear transformation of the dependent variables

$$A_1 = A = \mathrm{diag}\,(\lambda_1(t,\ x),\ \lambda_2(t,\ x)) \tag{2. 16}$$

According to the discussion in section 2 of Chap. Ⅳ in [10], the admissible boundary condition (1. 2) is in fact

$$u_1(t,\ 0,\ x') = a_1(t,\ x')\,u_2(t,\ 0,\ x') \tag{2. 17}$$

while the stable admissibility require

$$\lambda_1 a_1^2 + \lambda_2 < 0 \tag{2. 18}$$

Finally, we suppose the function f to be bounded. In fact, there is a sufficiently small neighbourhood of O, in which u is sufficiently small and so f is bounded, as we only consider the local behaviour of the solution. We suppose that $f|_{u=0} = 0$.

3. Estimate of Jumps

To get the a priori estimate of the solution, we must estimate the jumps of the solution across the characteristic hypersurface $\sum_1$. After that, we can get the estimates of all the derivatives of the solution u on $\sum_1^-$ and then get the estimates of those on $\sum_1^+$.

Lemma 3. 1 *All the jumps $\{\partial_t^\mu u\}_{\Sigma_1}$ of the piecewise smooth solution to the problem (1. 1) & (1. 2) across $\sum_1$ can be estimated by the jumps of itself across $\sum_2$ and the derivatives of u on $\sum_1^-$, where $\mu = 0,\ 1,\ \ldots$..*

Proof Denote

$$J_i^\mu = \{\partial_t^\mu u_i\}_{\Sigma_1} \tag{3. 1}$$

$$J^\mu = \{\partial_t^\mu u\}_{\Sigma_1} \tag{3. 2}$$

From the Transport Equations in[8], we have

$$J_2^0 = 0 \tag{3. 3}$$

$$\left(X_1 + b_1 + \mu\frac{\partial\lambda_1}{\partial x}\right)J_1^\mu = e\{\partial_t^\mu f(u)\}_{\Sigma_1} + e\sum_{\kappa=0}^{\mu-1} T_\mu^{(\kappa)} J^\kappa + eT_\mu(\mathrm{I}-e)J^\mu \tag{3. 4}$$

$$J_2^{\mu+1} = h_2^T\{\partial_t^\mu f(u)\}_{\Sigma_1} + \sum_{\kappa=0}^{\mu} h_2^T T_\mu^{(\kappa)} J^\kappa \tag{3. 5}$$

where b_1 is a scalar function, h_2 is a vector function, $T_\mu^{(\kappa)}$ and T_μ are first-order differential operators tangential to $\sum_1$; and

$$X_i = \partial_t + \lambda_i\partial_x \quad (i = 1,\ 2) \tag{3. 6}$$

$$(x,\ y) = (x_1,\ x')$$

$$e(x) = \mathrm{diag}\,(1,\ 0) \tag{3. 7}$$

If we know the value of J_1^0 at $\{t=0,\ x=0\}$, we can solve the Cauchy problem of the ordinary differential equation (3. 4) to obtain J_1^0, while the former can be obtained as follows:

For the same reason as (3. 3), we have also

$$\{u_1\}_{\Sigma_2}=0$$

Because u is piecewise smooth,

$$\begin{aligned}&\sum_{i=1}^{2}\{u_1\}_{\Sigma_i}|_{x=0}\\&=\{u_1|_{x=0}\}_{t=0}\\&=a_1(0,\ y)\{u_2|_{x=0}\}_{t=0}\\&=a_1(0,\ y)\sum_{i=1}^{2}\{u_1\}_{\Sigma_i}|_{x=0}\end{aligned}$$

So, at $\{t=0,\ x=0\}$

$$\begin{aligned}J_1^0&=\{u_1\}_{\Sigma_1}|_{x=0}\\&=a_1(0,\ y)\{u_2\}_{\Sigma_2}|_{x=0}\end{aligned}$$

is known. Hence we have estimated J^0 on $\sum_1$.

From (3. 5) we get J_2^1.

For the piecewise-smoothness of u,

$$\begin{aligned}&\sum_{j=1}^{2}\{\partial_t u_1\}_{\Sigma_j}|_{x=0}\\&=\{\partial_t u_1|_{x=0}\}_{t=0}\\&=(\partial_t a_1)(0,\ y)\{u_2|_{x=0}\}_{t=0}+a_1(0,\ y)\{\partial_t u_2|_{x=0}\}_{t=0}\\&=(\partial_t a_1)(0,\ y)\sum_{j=1}^{2}\{u_2\}_{\Sigma_j}|_{x=0}\\&\quad+a_1(0,\ y)\sum_{j=1}^{2}\{\partial_t u_2\}_{\Sigma_j}|_{x=0}\end{aligned}$$

Hence,

$$\begin{aligned}&J_1^1|_{x=0}\\&=\{\partial_t u_1\}_{\Sigma_1}|_{x=0}\\&=(\partial_t a_1)(0,\ y)\sum_{j=1}^{2}\{u_2\}_{\Sigma_j}|_{x=0}+a_1(0,\ y)\sum_{j=1}^{2}\{\partial_t u_2\}_{\Sigma_j}|_{x=0}\\&\quad-\{\partial_t u_1\}_{\Sigma_2}|_{x=0}\end{aligned}$$

which is known. So we can solve J^1 by (3. 4) and (3. 5).

Repeating the above steps, we can get all the jumps J^n.

4. Proof of the Main Theorem

For convenience, we define a norm for piecewise smooth functions:

$$\begin{cases} |||u|||_k = \sum\limits_{|\alpha|\le k} \sup\limits_{(t,x,y)\in R^{n+1}\cap(x\ge 0)\setminus(\Sigma_1\cup\Sigma_2)} |\partial^\alpha u| \\ |||u^\circ|||_k = \sum\limits_{|\alpha|\le k} \sup\limits_{(x,y)\in R^n\cap(x\ge 0)\setminus\sigma_2} |\partial^\alpha u| \end{cases} \tag{4. 1}$$

where $u^\circ(x, y) = u(t, x, y)\big|_{t=0}$, $(x, y) = (x_1, x')$.

Lemma 4. 1 *If u is a piecewise smooth solution to the problem* (1. 1) & (1. 2), *then the value of u in the region* IV *can be controlled by the initial data*

$$|||u|||_k \le g_k(|||u^\circ|||_{\nu_k}) \tag{4. 2}$$

where ν_k is a positive integer related to k.

Proof Seeing (2. 16) we write the system (1. 1) as

$$X_1u_1 + \sum_{j=2}^{n}(A_j)_{1,1}\partial_{y_j}u_1 + \sum_{j=2}^{n}(A_j)_{1,2}\partial_{y_j}u_2 = f_1 \tag{4. 3}$$

$$X_2u_2 + \sum_{j=2}^{n}(A_j)_{2,1}\partial_{y_j}u_1 + \sum_{j=2}^{n}(A_j)_{2,2}\partial_{y_j}u_2 = f_2 \tag{4. 4}$$

Then we make an energy integral over IV_{t_0}:

$$\int_{\mathrm{IV}_{t_0}} (4.\,3)\,u_1 + (4.\,4)\,u_2$$

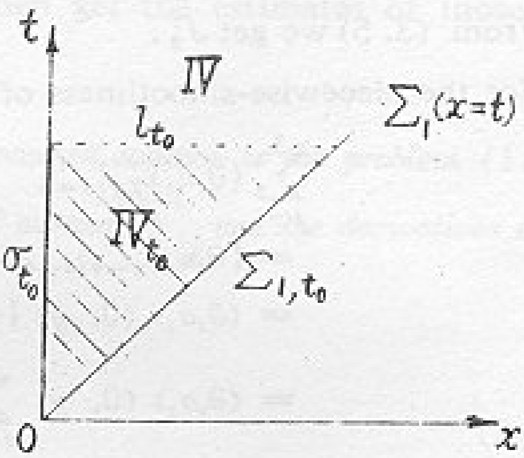

Fig. 6

and by Green's formula we have

$$\int_{l_{t_0}} \frac{1}{2}(u_1^2 + u_2^2) - \int_{\sigma_{t_0}} \frac{1}{2}(\lambda_1u_1^2 + \lambda_2u_2^2)$$

$$+ \int_{\Sigma^+_{1,t_0}} \frac{1}{2}\left(-\frac{\sqrt{2}}{2}(u_1^2 + u_2^2) + \frac{\sqrt{2}}{2}(\lambda_1u_1^2 + \lambda_2u_2^2)\right)$$

$$+ \left(\int_{y=Y} - \int_{y=-Y}\right)\sum_{j=2}^{n}\left((A_j)_{1,1}\frac{1}{2}u_1^2 + (A_j)_{2,2}\frac{1}{2}u_2^2 + (A_j)_{1,2}u_1u_2\right)$$

$$= \int_{\mathrm{IV}_{t_0}}\Big(u_1f_1 + u_2f_2 + (\partial_x\lambda_1)\frac{1}{2}u_1^2 + (\partial_x\lambda_2)\frac{1}{2}u_2^2$$

$$+ \sum_{j=2}^{n}(\partial_{y_j}(A_j)_{1,1}\frac{1}{2}u_1^2 + \partial_{y_j}(A_j)_{2,2}\frac{1}{2}u_2^2 + \partial_{y_j}(A_j)_{1,2}u_1u_2)\Big) \tag{4. 5}$$

As we only consider local solutions, we suppose that u has compact support with respect to y. So $\left(\int_{y=Y} - \int_{y=-Y}\right) = 0$, when Y is sufficiently large. From the admissible boundary condition,

$$-\int_{\sigma_{t_0}} \frac{1}{2}(\lambda_1u_1^2 + \lambda_2u_2^2) = -\int_{\sigma_{t_0}} \frac{1}{2}(\lambda_1a_1^2 + \lambda_2)\,u_2^2 \ge 0$$

According to the results of J. Rauch & M. Reed in [7], the values of u on $\sum_{1,t_0}^{-}$ can be controlled by the initial data and by Lemma 3.1, the jumps of u across $\sum_{1,t_0}$ can also be controlled by the initial data. So, the values of u on $\sum_{1,t_0}^{+}$ can be controlled by the initial data. Then, from (4.5), we have

$$\int_{l_{t_0}} (u_1^2+u_2^2) \leq C \int_{W_{t_0}} (u_1^2+u_2^2) + \tilde{g}_0(\| u^0 \|_{\dot{v}_0}) \tag{4.6}$$

By Gronwall's Inequality we get

$$\int_{l_{t_0}} (u_1^2+u_2^2) \leq \hat{g}_0(\| u^0 \|_{\dot{v}_0}) \tag{4.7}$$

We will then estimate the derivatives of u. Because the boundary $\{x=0\}$ is noncharacteristic, the operator P can be continued (cf. [12]). So, there exists a complete system of tangential operators $\{D_\rho\}$ ($\rho = 0, 1, \ldots, M$), such that

$$\begin{aligned}[P, D_\alpha] &= PD_\alpha - D_\alpha P \\ &= \sum_{\rho=1}^{M} p_\alpha^\rho(t, x, y) D_\rho + q_\alpha(t, x, y) P\end{aligned} \tag{4.8}$$

where $D_0 = \mathrm{Id}$.

Denoting $\underset{1}{u} = (D_0 u^T, D_1 u^T, \ldots, D_M u^T)^T$, we get an enlarged system of $\underset{1}{u}$:

$$p(D_\rho u) = -\sum_{\alpha=0}^{M} p_\rho^\alpha (D_\alpha u) + (D_\rho - q_\rho) f \quad (\rho = 0, 1, \ldots, M) \tag{4.9}$$

The continued boundary condition corresponding to it is

$$\underset{1}{M}\underset{1}{u} = 0, \quad (x_1, x') \in \partial\Omega \tag{4.10}$$

i.e.

$$MD_0 u = 0, \quad \text{on} \quad \partial\Omega \tag{4.11}$$

and

$$M(D_\rho u) + (D_\rho M)(D_0 u) = 0, \qquad \text{on} \qquad \partial\Omega \tag{4.12}$$

Using the notation in section 2, this becomes

$$u_1(t, 0, y) = a_1(t, y) u_2(t, 0, y) \tag{4.11'}$$

and

$$\begin{aligned}(D_\rho u_1)(t, 0, y) &= a_1(t, y)(D_\rho u_2)(t, 0, y) \\ &\quad + (D_\rho a_1)(t, y) u_2(t, 0, y)\end{aligned} \tag{4.12'}$$

Replacing D_ρ with $\varepsilon \cdot D_\rho$ and seeing (4.11') & (4.12'), we find the summation $\sum_{\rho=0}^{M}\sum_{j=1}^{2} \lambda_j (u_j^\rho)^2$ (where $u^\rho = D_\rho u$) to be

$$(\lambda_1 a_1^2 + \lambda_2) u_2^2 + \varepsilon^2 \sum_{\rho=1}^{M} (\lambda_1 (a_1 u_2 + (D_\rho a_1) u_2)^2 + \lambda_2 (u_2)^2) \tag{4. 13}$$

which is positively definite when ε is sufficiently small. Therefore, (4. 10) is still an admissible boundary condition.

As we have done in (4. 5), we make an energy integral with respect to the enlarged system (4. 9) and then get an estimate of the tangential derivatives of u. Because X_1 and X_2 are all transversal to the boundary, we can get the estimate of the normal derivative by (4. 3) and (4. 4). Hence, we have

$$\sum_{|\alpha|\le 1} \int_{l_{t_0}} |\partial^\alpha u|^2 \le \hat{g}_1 (\| u^0 \|_{\tilde{\nu}_1}) \tag{4. 14}$$

Applying the above steps inductively we have in general

$$\sum_{|\alpha|\le m} \int_{l_{t_0}} |\partial^\alpha u|^2 \le \hat{g}_m (\| u^0 \|_{\tilde{\nu}_m}) \tag{4. 15}$$

According to Sobolev's Imbedding Theorem, we verify (4. 2).

Define two difference operators $L_h^{(1)}$, $L_h^{(2)}$:

$$(L_h^{(1)} u)(t, x, y) = \sum_{j=2}^{n} (A_j)_{1,2} \frac{u(t, x, y+he_j) - u(t, x, y-he_j)}{2h} \tag{4. 16}$$

$$(L_h^{(2)} u)(t, x, y) = \sum_{j=2}^{n} (A_j)_{2,1} \frac{u(t, x, y+he_j) - u(t, x, y-he_j)}{2h} \tag{4. 17}$$

Lemma 4. 2 *If $u^h = (u_1^h, u_2^h)$ satisfies*

$$(X_1 + \sum_{j=2}^{n} (A_j)_{1,1} \partial_{y_j}) u_1^h + L_h^{(1)} u_2^h = f_1(u^h; t, x, y) \tag{4. 18}$$

$$(X_2 + \sum_{j=2}^{n} (A_j)_{2,2} \partial_{y_j}) u_2^h + L_h^{(2)} u_1^h = f_2(u^h; t, x, y) \tag{4. 19}$$

$$u^h|_{t=0} = u|_{t=0} = u^0 \tag{4. 20}$$

and the boundary condition (2. 17); then u^h can be controlled by the initial data uniformly, i. e.,

$$\| u^h \|_k \le g_k (\| u^0 \|_{\nu_k}) \tag{4. 21}$$

where g_k is independent of h.

Proof All the steps of the proof of this lemma repeat that of Lemma 4. 1 except for two points:

1° For $i = 1, 2$; $L_h^{(i)} + (L_h^{(i)})^*$ is bounded linear operators.

2° For the estimate of the jumps, the tangential differential operators in the Transport Equations will be replaced by the corresponding difference operators, while, in fact, the latter can be controlled by the former uniformly.

The justification of these two points is not difficult.

Proof of the Main Theorem Considering the initial-boundary value problem (4.

18), (4.19), (4.20) and (2.17), we can prove the existence of the solution u^h by integration over characteristic curves and by the contraction mapping principle. By Lemma 4.2, u^h is uniformly bounded with respect to h. According to Arzela-Ascoli's theorem, there exists a subsequence $\{u^{h_\nu}\}_{\nu=1}^{\infty}$ which converges to a function u in C^k. That is the very solution.

5. Orders of the Singularities

Finally, we give the proof of the later part of the main theorem.

For $k=0$, we have already in section 3 that

$$\{u_i\}_{\Sigma_j}=0, \quad i\neq j \tag{5.1}$$

As u is piecewise smooth,

$$\begin{aligned}\{u_1\}_{\Sigma_1}\big|_{x=0} &= \{u_1|_{x=0}\}_{t=0}-\{u_1\}_{\Sigma_2}\big|_{x=0}\\ &= \{a_1u_2|_{x=0}\}_{t=0}\\ &= a_1(\{u_2\}_{\Sigma_1}+\{u_2\}_{\Sigma_2})\big|_{x=0}=0\end{aligned} \tag{5.2}$$

Hence, $\{u\}_{\Sigma_1}=0$, i. e., $u\in C^0$.

For $k>0$, we prove it by contradiction. Suppose that there existed an index α_0 ($|\alpha_0|=m\leq k$) such that

$$\{\partial^{\alpha_0}u\}_{\Sigma_1}\neq 0 \tag{5.3}$$

while for any β ($|\beta|<m$)

$$\{\partial^{\beta}u\}_{\Sigma_1}\equiv 0 \tag{5.4}$$

We make any one of the tangential derivatives δ of $\sum_1$ act on the two sides of (5.4), getting

$$\{\delta\partial^{\beta}u\}_{\Sigma_1}\equiv 0 \tag{5.5}$$

and make ∂^β act on the two sides of (4.4), getting

$$\{\partial^{\alpha}u_2\}_{\Sigma_1}\equiv 0, \ \forall\ \alpha\ (|\alpha|\leq m) \tag{5.6}$$

So,

$$\begin{aligned}&\{\partial^{\alpha_0}u_1\}_{\Sigma_1}\big|_{x=0}\\ &=\{\partial^{\alpha_0}u_1|_{x=0}\}_{t=0}-\{\partial^{\alpha_0}u_1\}_{\Sigma_2}\big|_{x=0}\\ &=\{a_1\partial^{\alpha_0}u_2|_{x=0}+\cdots\}_{t=0}\\ &=a_1(\{\partial^{\alpha_0}u_2\}_{\Sigma_1}\big|_{x=0}+\{\partial^{\alpha_0}u_2\}_{\Sigma_2}\big|_{x=0})+\cdots\\ &=0\end{aligned} \tag{5.7}$$

By the Transport Equations in [7],

$$\{\partial^{\alpha_0}u_1\}_{\Sigma_1}\equiv 0 \tag{5.8}$$

Summing (5. 6) and (5. 8) up, we have

$$\{\partial^{\alpha_0} u\}_{\Gamma_1} \equiv 0$$

which is a contradiction.

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