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ADMISSIBLE WEAK SOLUTION FOR NONLINEAR SYSTEM OF CONSERVATION LAWS IN MIXED TYPE

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1. Introduction

For a strictly hyperbolic system of conservation laws, it is well-known that the classical solution of initial value problem exists only locally in time, in general, and one has to extend the concept of classical solution to weak solution or discontinuous solution in order to obtain a globally defined solution. Since weak solutions are not unique, one has to use admissibility condition or sometimes called entropy condition to pick out an admissible weak solution which is physically reasonable. There has been a general theory about the existence, uniqueness, asymptotic behavior of the admissible weak solution of Cauchy problem for the one-space dimensional strictly hyperbolic system of conservation laws. Moreover, there are different kinds of admissibility criteria proposed from either physical point of view or mathematical consideration and there are certain results about the equivalence among these different forms of entropy conditions.

What will occur if the strict hyperbolicity fails? Parabolic degeneracy will arise which can be found in the literature in connection with various models in applied sciences ([8], [2]). Furthermore, elliptic domain may occur in the phase space, in other words, the system of conservation laws is of mixed type. The following quasilinear system is the simplest model of mixed type which can be used as the equations of motion for dynamic elastic bar theory ([4]) or used as the equations governing isothermal motion of a Van der Waals fluid ([7]).

$$\begin{cases} u_t + p(v)_x = 0 \\ v_t - u_x = 0 \end{cases} \tag{1. 1}$$

where $p(v)$ is given by a nonmonotone function and the elliptic domain is a strip $\{v_\alpha < v < v_\beta\}$ on the (u, v) plane since the eigenvalue is defined by $\lambda^2 = -p'(v)$ and $p'(v) > 0$ when $v_\alpha < v < v_\beta$ and $p'(v) \leq 0$ when $v \leq v_\alpha$ or $v \geq v_\beta$.

It is an open problem to determine the extent to which the Cauchy problem is meaningful for such kind of nonlinear system of mixed type. For a first step, we study the simplest Cauchy problem—Riemann problem, namely

$$(u,\ v)\ (\lambda,\ 0) = \begin{cases} (u_-,\ v_-), & x < 0 \\ (u_+,\ v_+), & x > 0 \end{cases} \tag{1. 2}$$

where $(u_\mp,\ v_\mp)$ are arbitrary constant states.

An essential feature of mixed type nonlinear systems is the possibility of shocks between values one of which is in the elliptic domain and the other in the hyperbolic region. Such discontinuities are routinely observed in transonic flow, but are not described by linear system of mixed type or by purely hyperbolic nonlinear systems. It is obvious that in order to determine which shocks are admissible on physical grounds the classical entropy condition is not appropriate for shocks connecting states in elliptic domain with states in hyperbolic domain ([4]).

For handling the elliptic domain, people did various efforts ([1], [4], [5], [7], [6]). We introduce a different approach in this paper. We give a new definition of generalized entropy condition and a different definition of admissible weak solution of (1. 1), (1. 2) first in section 2 and prove the existence and uniqueness of the admissible weak solution then in section 3. This approach can be used for much more general system of mixed type for which the elliptic domain is of the following property: there is at least one direction on the $(u,\ v)$ plane such that for any given straight line, parallel to this direction, the intersection of the elliptic domain with the straight line is finite in length. The result about this kind of more general system of mixed type can be found in a coming paper ([3]).

2. Preliminary Remarks

Since both the system (1. 1) and the initial data (1. 2) are invariant under the transformation $x \to ax,\ t \to at,\ a > 0$, we look for similarity solutions $u = u(\xi),\ v = v(\xi),\ \xi = x/t$ for which the condition (1. 2) becomes into the boundary condition $(u(\xi),\ v(\xi)) \to (u_\mp,\ v_\mp)$ as $\xi \to \mp\infty$.

Substitute $u(\xi),\ v(\xi)$ into (1. 1), we obtain

$$\begin{pmatrix} \xi & -p'(v) \\ 1 & \xi \end{pmatrix} \begin{pmatrix} \dfrac{du}{d\xi} \\ \dfrac{dv}{d\xi} \end{pmatrix} = 0 \tag{2. 1}$$

which supplies the solution wherever it is smooth. Namely, either

$$\begin{cases} u = \text{constant} \\ v = \text{constant} \end{cases}$$

this is called constant state, or $\xi = \lambda_i(v)$ and the vector $\left(\dfrac{du}{d\xi},\ \dfrac{dv}{d\xi}\right)^T$ is parallel to the right eigenvector r_i, corresponding to λ_i. This defines the i-th rarefaction wave solution if $\lambda_i(v)$ is defined as a real valued function and $\lambda_i(v)$ is monotone along the integral curve of the vector field r_i, i. e. so called rarefaction wave curve. More precisely, for $v \leq v_\alpha$ or $v \geq v_\beta$

$$\lambda_1 = -\sqrt{-p'(v)} \qquad \left(\text{or} \quad \lambda_2 = \sqrt{-p'(v)}\right).$$

R_1 (or R_2) is the integral curve of $\dfrac{du}{dv} = \sqrt{-p'(v)}\left(\text{or} \quad \dfrac{du}{dv} = -\sqrt{-p'(v)}\right)$.

Suppose that $p(v)$ has the type of graph as in Figure 2. 1, namely, $p(v)$ satisfies the following hypotheses (H).

(H)

(i) $p(v)$ is a smooth function defined on (b, ∞), where b is a given positive constant;

(ii) $p'(v) < 0$ for $0 < b < v < v_\alpha$ or $v > v_\beta$ and $p'(v_\alpha) = p'(v_\beta) = 0$, $p'(v) \to 0$ as $v \to \infty$;

(iii) $p'(v) > 0$ for $v_\alpha < v < v_\beta$;

For simplicity, we make further assumptions

(iv) $p''(v) \geq 0$ if $v \leq v_\alpha$, and $p''(v)$ changes sign only once for $v \geq v_\beta$ where $v = v_*$ and $p(v_*) > p(v_\alpha)$ for definiteness.

(v) $\displaystyle\int_v^{v_0} \sqrt{-p'(\eta)}\,d\eta \to +\infty$ as $v \to b$ for any given $v_0 \leq v_\alpha$,

$\displaystyle\int_{v_0}^{v} \sqrt{-p'(\eta)}\,d\eta \to +\infty$ as $v \to +\infty$ for any given $v_0 \geq v_\beta$

Proposition 2. 1 *For any given (u_0, v_0) with $b < v_0 < v_\alpha$, the state (u, v) which can be joined to (u_0, v_0) on the right hand side by a 1st (or 2nd) rarefaction wave is defined by*

$$\begin{cases} v_0 \leq v \leq v_\alpha \\ u - u_0 = \displaystyle\int_{v_0}^{v} \sqrt{-p'(\eta)}\,d\eta \end{cases} \qquad \left(\text{or} \quad \begin{cases} b < v \leq v_0 \\ u - u_0 = -\displaystyle\int_{v_0}^{v} \sqrt{-p'(\eta)}\,d\eta \end{cases}\right) \tag{2. 2$_1$}$$

denoted still by R_1 (or R_2) and for any given (u_0, v_0) with $v_* \geq v_0 > v_\beta$, the state (u, v) which can be joined to (u_0, v_0) on the right hand side by a 1st (or 2nd) rarefaction wave is defined by

$$\begin{cases} v_\beta \leq v \leq v_0 \\ u - u_0 = \displaystyle\int_{v_0}^{v} \sqrt{-p'(\eta)}\,d\eta \end{cases} \qquad \left(\text{or} \quad \begin{cases} v_0 \leq v \leq v_* \\ u - u_0 = -\displaystyle\int_{v_0}^{v} \sqrt{-p'(\eta)}\,d\eta \end{cases}\right) \tag{2. 2$_2$}$$

while for any given (u_0, v_0) with $\infty > v_0 > v_*$, the above kind of state (u, v) is defined by

$$\begin{cases} v \geq v_0 \\ u - u_0 = \int_{v_0}^{v} \sqrt{-p'(\eta)}\, d\eta \end{cases} \quad \left(\text{or} \quad \begin{cases} v_* \leq v \leq v_0 \\ u - u_0 = -\int_{v_0}^{v} \sqrt{-p'(\eta)}\, d\eta \end{cases} \right) \tag{2. 2}$$

(see Figure 2. 2)

Turn to discontinuity now. A discontinuity is defined by Rankine-Hugoniot Condition which takes the form

$$\begin{aligned} \sigma(u - u_0) &= p(v) - p(v_0) \\ \sigma(v - v_0) &= -(u - u_0) \end{aligned} \tag{2. 3}$$

For any given (u_0, v_0), the state which can be joined to (u_0, v_0) by a discontinuity defines the shock curve S_1 with σ_1 and S_2 with σ_2 from (2. 3). Namely

$$\sigma_1 = -\sqrt{-\frac{p(v) - p(v_0)}{v - v_0}}$$

$$\frac{u - u_0}{v - v_0} = \sqrt{-\frac{p(v) - p(v_0)}{v - v_0}} \tag{S_1}$$

and

$$\sigma_2 = \sqrt{-\frac{p(v) - p(v_0)}{v - v_0}}$$

$$\frac{u - u_0}{v - v_0} = -\sqrt{-\frac{p(v) - p(v_0)}{v - v_0}} \tag{S_2}$$

It is easy to show that S_i is a single-valued function of v but is not necessary to be connected. According to different location of (u_0, v_0): $b < v_0 \leq \bar{v}$, $\bar{v} < v_0 < v_\alpha$, $v_\alpha \leq v_0 \leq v_\beta$, $v_\beta < v_0 < \hat{v}$, $v \geq \hat{v}$, where $\hat{v}$ is defined by $p(\hat{v}) = p(v_\alpha)$, $\bar{v}$ is defined by $p(\bar{v}) = p(v_\beta)$, S_i is shown in Figure 2. 3 respectively.

In order to determine which shocks are admissible on physical grounds we introduce the following generalized entropy condition (G. E. C.) for handling the elliptic domain.

Definition *For any given (u_-, v_-), $(u_+, v_+) \in S_i(u_-, v_-)$ is said to satisfy the G. E. C. if either*

I. *when v varies from v_- to v_+, excluding v_+ itself, the corresponding σ_i is decreasing wherever it is defined, or*

II. *for any v between v_- and v_+ where σ_i is defined, it holds that*

$$\sigma_i(v;\, u_-,\, v_-) \geq \sigma_i(v_+;\, u_-,\, v_-) \qquad (i = 1,\ 2)$$

This kind of discontinuity is called admissible discontinuity.

For any given (u_-, v_-), the state $(u_+, v_+) \in S_i(u_-, v_-)$ which, with (u_-, v_-) together, supplies an admissible discontinuity can be determined. According to different location of (u_-, v_-), the following proposition shows the set of state (u, v) which belongs to $S_i(u_-, v_-)$ and satisfies the G. E. C., is still denoted by $S_i(u_-, v_-)$.

Proposition 2. 2 *Corresponding to different location of* (u_-, v_-) *with* $b < v_- \leq \bar{v}$, $\bar{v} < v_- < v_\alpha$, $v_\alpha \leq v_- \leq v_\beta$, $v_\beta < v_- < v_*$, $v_* \leq v_- \leq \hat{v}$, $\hat{v} < v_- < \infty$, $S_i(u_-, v_-)$ *can be defined as follows respectively (see Figure 2. 4). For the case when* $b < v_- \leq \bar{v}$

$$S_1: \begin{cases} b < v < v_- \\ \dfrac{u - u_-}{v - v_-} = \sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases} \qquad S_2: \begin{cases} v_- \leq v \leq v_{R_\beta(-)} \quad \text{or} \quad v \geq v_{D_\beta(-)} \\ \dfrac{u - u_-}{v - v_-} = -\sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases}$$

where $v_{R_\beta(-)}$ *is defined by* $\dfrac{p(v_{R_\beta}) - p(v_-)}{v_{R_\beta} - v_-} = p'(v_{R_\beta})$, $v_{R_\beta} \geq v_\beta$ *and* $v_{D_\beta(-)}$ *is defined by*

$$\frac{p(v_{D_\beta}) - p(v_-)}{v_{D_\beta} - v_-} = \frac{p(v_{R_\beta}) - p(v_-)}{v_{R_\beta} - v_-} = p'(v_{R_\beta}), \quad v_{D_\beta} > v_\beta.$$

For the case when $\bar{v} < v_- < v_\alpha$, there are two subcases: $v_- < \tilde{v}$ or $v_- > \tilde{v}$, where $\tilde{v}$ is defined by $p'(\tilde{v}) = p'(\tilde{\tilde{v}}) = \dfrac{p(\tilde{v}) - p(\tilde{\tilde{v}})}{\tilde{v} - \tilde{\tilde{v}}}$, $\tilde{v} < v_\alpha$, $\tilde{\tilde{v}} > v_\beta$. When $\bar{v} < v_- < \tilde{v}$

$$S_1: \begin{cases} b < v \leq v_- \\ \dfrac{u - u_-}{v - v_-} = \sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases} \qquad S_2: \begin{cases} v_- \leq v \leq v_{B_i(-)} \quad \text{or} \quad v = v_{B_\beta(-)} \\ \dfrac{u - u_-}{v - v_-} = -\sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases}$$

where $p(v_{B_i}) = p(v_-) = p(v_{B_\beta})$ and $p'(v_{B_i}) > 0$, $p'(v_{B_\beta}) < 0$, $v_{B_\beta} \geq v_\beta$.

When $\tilde{v} < v_- < v_\alpha$

$$S_1: \begin{cases} b < v \leq v_- \quad \text{or} \quad v_{L_\beta(-)} \leq v \leq v_{R_\beta(-)} \\ \dfrac{u - u_-}{v - v_-} = \sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases} \qquad S_2: \begin{cases} v_- \leq v \leq v_{B_i(-)} \quad \text{or} \quad v = v_{B_\beta(-)} \\ \dfrac{u - u_-}{v - v_-} = -\sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases}$$

where $v_{L_\beta(-)}$ is defined by $p'(v_-) = \dfrac{p(v_{L_\beta}) - p(v_-)}{v_{L_\beta} - v_-}$, $v_{L_\beta} \geq v_\beta$.

For the case when $v_\alpha \leq v_- \leq v_\beta$

$$S_1: \begin{cases} b < v \leq v_{B_\alpha} \quad \text{or} \quad v_{B_\beta} \leq v \leq v_{R_\beta} \\ \dfrac{u - u_-}{v - v_-}\sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases} \qquad S_2: \begin{cases} v = v_{B_\alpha} \quad \text{or} \quad v = v_{B_\beta} \\ \dfrac{u - u_-}{v - v_-} = -\sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases}$$

where v_{B_α} is defined by $p(v_{B_\alpha}) = p(v_-)$ and $v_{B_\alpha} \leq v_\alpha$.

For the case when $v_\beta < v_- < v_*$

$$S_1: \begin{cases} b < v \leq v_{L_\alpha} \quad \text{or} \quad v_- \leq v \leq v_{R_\beta(-)} \\ \dfrac{u - u_-}{v - v_-} = \sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases} \qquad S_2: \begin{cases} v_{B_i} \leq v \leq v_- \quad \text{or} \quad v = v_{B_\alpha} \\ \quad \text{or} \quad v_{L_\beta} \leq v < \infty \\ \dfrac{u - u_-}{v - v_-} = -\sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases}$$

where $v_{L_a(-)}$ is defined by $\frac{p(v_{L_a}) - p(v_-)}{v_{L_a} - v_-} = p'(v_-)$, $v_{L_a} \leq v_a$.

For the case when $v_* \leq v_- \leq \hat{v}$

$$S_1: \begin{cases} v_{R_\beta} \leq v \leq v_- \quad \text{or} \quad b < v \leq v_{D_a} \\ \dfrac{u - u_-}{v - v_-} = \sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases} \qquad S_2: \begin{cases} v \geq v_- \quad \text{or} \quad v_{B_1} \leq v \leq v_{L_1} \\ \quad \text{or} \quad v = v_{B_a} \\ \dfrac{u - u_-}{v - v_-} = -\sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases}$$

where v_{D_a} is defined by

$$\frac{p(v_{D_a}) - p(v_-)}{v_{D_a} - v_-} = \frac{p(v_{R_\beta}) - p(v_-)}{v_{R_\beta} - v_-} = p'(v_{R_\beta}), \qquad v_{D_a} < v_a,$$

v_{L_1} is defined by $\frac{p(v_{L_1}) - p(v_-)}{v_{L_1} - v_-} = p'(v_-)$, $\quad v_a \leq v_{L_1} \leq v_*$.

For the case when $v_- > \hat{v}$, there are two subcases: $v_- < \tilde{v}$ or $v_- \geq \tilde{v}$. When $\hat{v} < v_- < \tilde{v}$,

$$S_1: \begin{cases} v_{R_\beta} \leq v \leq v_- \quad \text{or} \quad b < v \leq v_{D_a} \\ \dfrac{u - u_-}{v - v_-} = \sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases} \qquad S_2: \begin{cases} \infty > v \geq v_- \quad \text{or} \quad v_{R_a} \leq v \leq v_{L_1} \\ \dfrac{u - u_-}{v - v_-} = -\sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases}$$

When $v_- \geq \tilde{v}$

$$S_1: \begin{cases} v_{R_\beta} \leq v \leq v_- \quad \text{or} \quad b < v \leq v_{D_a} \\ \dfrac{u - u_-}{v - v_-} = \sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases} \qquad S_2: \begin{cases} v \geq v_- \\ \dfrac{u - u_-}{v - v_-} = -\sqrt{-\dfrac{p(v) - p(v_-)}{v - v_-}} \end{cases}$$

Now it is ready to introduce the definition of admissible weak solution.

Definition 2. 3 *A single-valued function* $(u(\xi), v(\xi))$ *is called an admissible weak solution of* (1. 1), (1. 2) *if*

I. *It satisfies the boundary condition* $(u, v) \to (u_\mp, v_\mp)$ *as* $\xi \to \mp\infty$;

II. *It is either a rarefaction wave or a constant state wherever it is smooth;*

III. *Any discontinuity satisfies the Rankine-Hugoniot condition and the above generalized entropy condition;*

IV. *The image in the phase plane takes the minimum variation among all possible single-valued function* $(u(\xi), v(\xi))$ *satisfying* (I) — (III).

3. Existence and Uniqueness of an Admissible Weak Solution

For any given state (u_-, v_-), we consider the set of states which, as a state (u_+, v_+), can be joined to (u_-, v_-) by a single-valued function $(u(\xi), v(\xi))$ satisfying the definition 2. 3 and consisting of the first kind of waves. Namely, it contains either a first kind of admissible discontinuity of rarefaction wave or a fan of first kind of waves, consisting of first kinds of rarefaction waves and admissible discontinuities. We call the whole set the first kind of wave curve, denoted by $W_1(u_-, v_-)$. For each point (u_1, v_1) on the curve $W_1(u_-, v_-)$, we determine the set of states which, as a state (u_+, v_+) , can be joined to (u_1, v_1) , as a state (u_-, v_-) , by a single-valued function $(u(\xi), v(\xi))$ satisfying the definition 2. 3 and consisting of the second kind of waves. Namely, it contains either a second kind of admissible discontinuity or rarefaction wave or a fan of second kind of waves, consisting of second kind of rarefaction waves and admissible discontinuities. We call the whole set the second kind of wave curve, denoted by $W_2(u_1, v_1)$.

In order to prove the existence and uniqueness of the admissible weak solution for the Riemann problem (1. 1), (1. 2), it suffices to show that for any given (u_-, v_-) with $b < v_- < \infty$ the family of curves $\{W_2(u_1, v_1) ; (u_1, v_1) \in W_1(u_-, v_-)\}$ covers the whole domain $D: \{-\infty < u < \infty, b < v < \infty\}$ univaluedly. We give the constructive proof next which shows the structure of the solution simultaneously.

Case 1 $\quad b < v_- \leq \tilde{v}$

It can be shown, by (H), Propositions 2. 1 and 2. 2, that $W_1(u_-, v_-)$ is defined as

$$W_1(u_-, v_-) = \begin{cases} S_1(u_-, v_-) & \text{for} \quad b < v < v_- \\ R_1(u_-, v_-) & \text{for} \quad v_- \leq v \leq v_a \\ C_1(\tilde{v}, v_a; R_1(\text{-})) & \text{for} \quad \hat{v} \leq v \leq \tilde{\tilde{v}} \\ R_1(\tilde{\tilde{u}}, \tilde{\tilde{v}}) & \text{for} \quad v > \tilde{\tilde{v}} \end{cases} \qquad (3.\ 1)$$

where $C_1(\tilde{v}, v_a; R_1(\text{-}))$ consists of states $(u_{L_\beta(1)}, v_{L_\beta(1)})$ such that corresponding to each state $(u_1, v_1) \in R_1(u_-, v_-)$ with $\tilde{v} \leq v_1 \leq v_a$, it holds that

$$\begin{cases} \dfrac{p(v_{L_\beta(1)}) - p(v_1)}{v_{L_\beta(1)} - v_1} = p'(v_1), & v_{L_\beta(1)} > v_\beta \\ (u_{L_\beta(1)}, v_{L_\beta(1)}) \in S_1(u_1, v_1) & \end{cases}$$

and $v_{L_\beta(1)}$ varies from $\hat{v}$ to $\tilde{\tilde{v}}$ as v_1 varies from v_a to $\tilde{v}$. $\tilde{\tilde{u}} = u_{L_\beta(\tilde{u}, \tilde{v})}$, $(\tilde{u}, \tilde{v}) \in R_1(u_-, v_-)$, $(u_{L_\beta(1)}, v_{L_\beta(1)})$ and (u_1, v_1) supply an admissible first kind of discontinuity. Moreover, $u \to -\infty$ along the curve S_1 monotonically as $v \to b$, the curve $C(\tilde{v}, v_a; R_1(u_-, v_-))$ is smooth on $\hat{v} \leq v \leq \tilde{\tilde{v}}$ which contacts with $R_1(\tilde{\tilde{u}}, \tilde{\tilde{v}})$ at $v = \tilde{\tilde{v}}$ and finally, $u \to +\infty$ along $R_1(\tilde{\tilde{u}}, \tilde{\tilde{v}})$ monotonically as $v \to +\infty$. (see Figure 3. 1)

Consider the curve $W_2(u_1, v_1)$ for any $(u_1, v_1) \in W_1(u_-, v_-)$. It can be shown that there exists $(u_1^*, v_1^*) \in S_1(u_-, v_-)$ such that $\dfrac{p(v_1^*) - p(v_*)}{v_1^* - v_*} = p'(v_*)$. For any

$(u_1, v_1) \in W_1(u_-, v_-)$ with $b < v_- \leq v_1^*$.

$$W_2(u_1, v_1) = \begin{cases} R_2(u_1, v_1) & \text{for } b < v \leq v_1 \\ S_2(u_1, v_1) & \text{for } v_1 < v < \infty \end{cases} \tag{3. 2}$$

For any $(u_1, v_1) \in W_1(u_-, v_-)$ with $v_1^* < v_1 < \bar{v}$, there are $v_{R_\beta(1)}$ and $v_{D_\beta(1)}$ determined in Proposition 2. 2, $W_2(u_1, v_1)$ is defined as

$$W_2(u_1, v_1) = \begin{cases} R_2(u_1, v_1) & \text{for } b < v \leq v_1 \\ S_2(u_1, v_1) & \text{for } v_1 < v \leq v_{R_\beta(1)} \\ R_2(u_{R_\beta(1)}, v_{R_\beta(1)}) & \text{for } v_{R_\beta(1)} < v \leq v_* \\ C_2(v_{R_\beta(1)}, v_*; R_2(u_{R_\beta(1)}, v_{R_\beta(1)})) & \text{for } v_* < v < v_{D_\beta(1)} \\ S_2(u_1, v_1) & \text{for } v \geq v_{D_\beta(1)} \end{cases} \tag{3. 3}$$

where $(u_{R_\beta(1)}, v_{R_\beta(1)}) \in S_2(u_1, v_1)$, $C_2(v_{R_\beta(1)}, v_*; R(u_{R_\beta(1)}, v_{R_\beta(1)}))$ consists of states $(u_{L_\beta(2)}, v_{L_\beta(2)})$ such that corresponding to each state $(u_2, v_2) \in R_2(u_{R_\beta(1)}, v_{R_\beta(1)})$ with $v_{R_\beta(1)} \leq v_2 \leq v_*$ it holds that

$$\begin{cases} \dfrac{p(v_{L_\beta(2)}) - p(v_2)}{v_{L_\beta(2)} - v_2} = p'(v_2), & v_{L_\beta(2)} > v_\beta \\ (u_{L_\beta(2)}, v_{L_\beta(2)}) \in S_2(u_2, v_2) \end{cases}$$

as in Proposition 2. 2 and $v_{L_\beta(2)}$ varies from v_* to $v_{D_\beta(1)}$ as v_2 varies from v_* to $v_{R_\beta(1)}$, $(u_{L_\beta(2)}, v_{L_\beta(2)})$ and (u_2, v_2) supply an admissible second kind of discontinuity. Moreover, $(u_{L_\beta(2)}, v_{L_\beta(2)}) \in S_2(u_1, v_1)$ when $v_2 = v_{R_\beta(1)}$ and $W_2(u_1, v_1)$ is a smooth curve defined for $b < v < \infty$ on which $u \to +\infty$ as $v \to b$ and $u \to -\infty$ as $v \to +\infty$.

It is easy to see that $v_{D_\beta(1)} \to +\infty$ when $v_1 \to \bar{v}$, therefore when $v_1 = \bar{v}$,

$$W_2(u_1, v_1) = \begin{cases} R_2(u_1, v_1) & \text{for } b < v \leq v_1 = \bar{v} \\ S_2(u_1, v_1) & \text{for } \bar{v} < v \leq v_{R_\beta(1)} = v_\beta \\ R_2(u_\beta, v_\beta) & \text{for } v_\beta < v \leq v_* \\ C_2(v_\beta, v_*; R_2(u_\beta, v_\beta)) & \text{for } v > v_* \end{cases} \tag{3. 4}$$

For any $(u_1, v_1) \in W_1(u_-, v_-)$ with $\bar{v} < v_1 < v_a$, there are $v_{B_1(1)}$ and $v_{B_\beta(1)}$ detemined in Proposition 2. 2, $W_2(u_1, v_1)$ is defined differently corresponding to $v_1 < v_a^*$ or $v_1 > v_a^*$, where $p(v_a^*) = p(v_*)$, $v_a^* < v_a$. For the case when $\bar{v} < v_1 < v_a^*$,

$$W_2(u_1, v_1) = \begin{cases} R_2(u_1, v_1) & \text{for} \quad b < v \leq v_1 \\ S_2(u_1, v_1) & \text{for} \quad v_1 < v \leq v_{B_1(1)} \\ S_2(u_{B_\beta(1)}, v_{B_\beta(1)}) & \text{for} \quad v_{B_1(1)} < v \leq v_{B_\beta(1)} \\ R_2(u_{B_\beta(1)}, v_{B_\beta(1)}) & \text{for} \quad v_{B_\beta(1)} < v \leq v_* \\ C_2(v_{B_\beta(1)}, v_*; R_2(u_{B_\beta(1)}, v_{B_\beta(1)})) & \text{for} \quad v_* < v < v_{L_\beta(B_\beta(1))} \\ S_2(u_{B_\beta(1)}, v_{B_\beta(1)}) & \text{for} \quad v \geq v_{L_\beta(B_\beta(1))} \end{cases} \tag{3. 5}$$

where $C_2(v_{B_\beta(1)}, v_*; R_2(u_{B_\beta(1)}, v_{B_\beta(1)}))$ consists of states $(u_{L_\beta(2)}, v_{L_\beta(2)}) \in S_2(u_2, v_2)$ as before and $v_{L_\beta(2)}$ varies from v_* to $v_{L_\beta(B_\beta(1))}$ as v_2 varies from v_* to $v_{B_\beta(1)}$. Moreover, $(u_{L_\beta(B_\beta(1))}, v_{L_\beta(B_\beta(1))}) \in S_2(u_{B_\beta(1)}, v_{B_\beta(1)})$ and $W_2(u_1, v_1)$ is a smooth curve defined for $b < v < \infty$ on which $u \to +\infty$ as $v \to b$ and $u \to -\infty$ as $v \to +\infty$.

It is clear that $v_{B_\beta(1)} \to v_*$ and $v_{L_\beta(B_\beta(1))} \to v_*$ when $v_1 \to v_\alpha^*$, therefore when $v_1 = v_\alpha^*$,

$$W_2(u_1, v_1) = \begin{cases} R_2(u_1, v_1) & \text{for} \quad b < v \leq v_1 \\ S_2(u_1, v_1) & \text{for} \quad v_1 < v \leq v_{B_1(1)} \\ S_2(u_{B_\beta(1)}, v_{B_\beta(1)}) & \text{for} \quad v_{B_1(1)} < v \leq v_{B_\beta(1)} = v_* \quad \text{or} \quad v > v_* \end{cases} \tag{3. 6}$$

For the case when $v_\alpha^* < v_1 < v_\alpha$,

$$W_2(u_1, v_1) = \begin{cases} R_2(u_1, v_1) & \text{for} \quad b < v \leq v_1 \\ S_2(u_1, v_1) & \text{for} \quad v_1 < v \leq v_{B_1(1)} \\ S_2(u_{B_\beta(1)}, v_{B_\beta(1)}) & \text{for} \quad v_{B_1(1)} < v \leq v_{L_1(B_\beta(1))} \\ C_2(v_*, v_{B_\beta(1)}; R_2(u_{B_\beta(1)}, v_{B_\beta(1)})) & \text{for} \quad v_{L_1(B_\beta(1))} < v < v_* \\ R_2(u_{B_\beta(1)}, v_{B_\beta(1)}) & \text{for} \quad v_* \leq v < v_{B_\beta(1)} \\ S_2(u_{B_\beta(1)}, v_{B_\beta(1)}) & \text{for} \quad v \geq v_{B_\beta(1)} \end{cases} \tag{3. 7}$$

where $v_{L_1(\cdot)}$ is defined by $\dfrac{p(v_{L_1(\cdot)}) - p(v.)}{v_{L_1(\cdot)} - v.} = p'(v.)$ and $v_* \leq v. \leq v$; $C_2(v_*, v_{B_\beta(1)}; R_2(u_{B_\beta(1)}, v_{B_\beta(1)}))$ consists of states $(u_{L_\beta(2)}, v_{L_\beta(2)}) \in S_2(u_2, v_2)$ as before and $v_{L_\beta(2)}$ varies from $v_{L_1(B_\beta(1))}$ to v_* as v_2 varies from $v_{B_\beta(1)}$ to v_*. Moreover, $S_2(u_{B_\beta(1)}, v_{B_\beta(1)})$ tends to the same state as $S_2(u_1, v_1)$ does at $v = v_{B_1(1)}$ with the same slope there and $W_2(u_1, v_1)$ is a smooth curve defined for $b < v < +\infty$ on which $u \to +\infty$ as $v \to b$ and $u \to -\infty$ as $v \to +\infty$.

Since $v_{B_1(1)} \to v_\alpha$ and $v_{B_\beta(1)} \to v$ when $v_1 \to v_\alpha$, one obtains $W_2(u_1, v_1)$ for $v_1 = v_\alpha$ as follows

$$W_2(u_1, v_1) = \begin{cases} R_2(u_1, v_1) & \text{for } b < v \leq v_1 = v_a \\ S_2(u, v) & \text{for } v_a < v \leq v_{L_1(u,v)} \\ C_2(v_*, v; R_2(u, v)) & \text{for } v_{L_1(u,v)} < v < v_* \\ R_2(u, v) & \text{for } v_* \leq v < v \\ S_2(u, v) & \text{for } v \geq v \end{cases} \quad (3.8)$$

where $(u, v) \in S_2(u_a, v_a)$, $(u_a, v_a) \in R_1(u_-, v_-)$. Obviously, (u, v) is the same state on $C_1(\tilde{v}, v_a; R_1(u_-, v_-))$ as the starting point for $v = v$.

Now we turn to (u_1, v_1) on $W_1(u_-, v_-)$ for $v_1 \geq v$. For the case when $v \leq v_1 \leq \tilde{\tilde{v}}$

$$W_2(u_1, v_1) = \begin{cases} S_2(u_1, v_1) & \text{for } v > v_1 \\ R_2(u_1, v_1) & \text{for } v_* \leq v \leq v_1 \\ C_2(v_*, v_1; R_2(u_1, v_1)) & \text{for } v_{L_1(1)} < v < v_* \\ S_2(u_1, v_1) & \text{for } v_{R_a(1)} \leq v \leq v_{L_1(1)} \\ R_2(u_{R_a(1)}, v_{R_a(1)}) & \text{for } b < v < v_{R_a(1)} \end{cases} \quad (3.9)$$

It is easy to find that the $W_2(u_1, v_1)$ defind in (3.9) gives the same curve as in (3.8) when $v_1 = v$. On the other hand, $v_{L_1(1)} \to \tilde{v}$ and $v_{R_a(1)} \to \tilde{v}$ as $v_1 \to \tilde{\tilde{v}}$, this implies the following $W_2(u_1, v_1)$ for $v_1 = \tilde{\tilde{v}}$

$$W_2(u_1, v_1) = \begin{cases} S_2(u_1, v_1) & \text{for } v > v_1 = \tilde{\tilde{v}} \\ R_2(u_1, v_1) & \text{for } v_* \leq v \leq v_1 \\ C_2(v_*, v_1; R_2(u_1, v_1)) & \text{for } \tilde{v} \leq v < v_* \\ R_2(\tilde{u}, \tilde{v}) & \text{for } b < v < \tilde{v} \end{cases} \quad (3.10)$$

For the case when $v_1 > \tilde{\tilde{v}}$

$$W_2(u_1, v_1) = \begin{cases} S_2(u_1, v_1) & \text{for } v > v_1 \\ R_2(u_1, v_1) & \text{for } v_* \leq v \leq v_1 \\ C_2(v_*, \tilde{\tilde{v}}; R_2(u_1, v_1)) & \text{for } \tilde{v} \leq v < v_* \\ R_2(\tilde{u}, \tilde{v}) & \text{for } b < v < \tilde{v} \end{cases} \quad (3.11)$$

In summary, (3.2) — (3.11) show that the family $\{W_2(u_1, v_1) : (u_1, v_1) \in W_1(u_-, v_-),\ b < v_- \leq \bar{v}\}$ is divided into 6 groups according to $b < v_1 \leq v_1^*$, $v_1^* < v_1 \leq \bar{v}$, $\bar{v} < v_1 \leq v_a^*$, $v_a^* < v_1 \leq v_a$, $v < v_1 \leq \tilde{\tilde{v}}$, $v_1 > \tilde{\tilde{v}}$ which are expressed in (3.2), (3.3), (3.5), (3.7), (3.9), (3.11) respectively. The boundary between each couple of neighboring groups, corresponding to $v_1 = v_1^*$, $v_1 = \bar{v}$, $v_1 = v_a^*$, $v_1 = v_a (v_1 = v)$, $v_1 = \tilde{\tilde{v}}$, is expressed in (3.2) for $v_1 = v_1^*$ and (3.4), (3.6), (3.8), (3.10) respectively.

It can be shown, by (H), Proposition (2.1), (2.2) and the formular in (3.1) — (3.11) that the family of curves $\{W_2(u_1, v_1)\ ;\ (u_1, v_1) \in W_1(u_-, v_-)\}$ for given (u_-, v_-) with $-\infty < u < \infty$ (see Figure 3.1) $b < v_- \leq \bar{v}$ covers the whole domain

$D:\{-\infty<u<+\infty, b<v<+\infty\}$ univaluedly. Therefore, the existence and uniqueness of an admissibly weak solution for the problem (1. 1) (1. 2) can be obtained for any given $(u_+, v_+)\in D$ and $(u_-, v_-)\in D$ with $b<v_-\leq\bar{v}$. Each group of curves in the above 6 groups covers a subdomain $G_i\subset D$, $i=1, ..., 6$, shown in Figure 3. 1, where we assume $v_->v_1^*$ for definiteness. We use dotted line for S, flack line for R and dotted-flack line for C.

For the case when $(u_+, v_+)\in G_1$ (corresponding to $b<v_1\leq v_1^*$ in Figure 3. 1), the solution contains two waves: either a 1-shock and a 2-shock or a 1-shock and a 2-rarefaction wave, denoted by S_1-S_2 or S_1-R_2 respectively.

For the case when $(u_+, v_+)\in G_2$ (corresponding to $v_1^*<v_1\leq\bar{v}$ in Figure 3. 1) there are 8 different kinds of wave patterns for the solution which are S_1-R_2, S_1-S_2, $S_1-S_2^{CR}-R_2$, $S_1-S_2^{CR}-R_2-S_2^{CL}$; R_1-R_2, R_1-S_2, $R_1-S_2^{CR}-R_2$, $R_1-S_2^{CR}-R_2-S_2^{CL}$, where S_2^{CR} denotes the second kind of right hand side contact discontinuity, namely, $\sigma_2(v_r, v_l)=\lambda_2(v_r)$, S_2^{CL} denotes the second kind of left hand side contact discontinuity, namely, $\sigma_2(v_r, v_l)=\lambda_2(v_l)$.

For the case when $(u_+, v_+)\in G_3$ (corresponding to $\bar{v}<v_1\leq v_a^*$ in Figure 3. 1) there are 5 different kinds of wave patterns for the solution which are R_1-R_2, R_1-S_2, $R_1-S_2^{o\beta}-S_2(B_\beta)$, $R_1-S_2^{o\beta}-R_2(B_\beta)$, $R_1-S_2^{o\beta}-R_2-S_2^{CL}$ respectively where $S_2^{o\beta}$ denotes the second kind of admissible discontinuity with $\sigma_2=0$ and the right hand side of value v is greater than v_β. $S_2(B_\beta)$ denotes the second kind of admissible discontinuity which has the state B_β as the left hand side of value. $B_\beta=(u_{B_\beta}, v_{B_\beta})$, $B_\beta\in S_2(u_1, v_1)$, $\sigma_2(B_\beta)=0$.

For the case when $(u_+, v_+)\in G_4$ (corresponding to $v_a^*<v_1\leq v_a$) the 5 different wave patterns are the same as above.

For the case when $(u_+, v_+)\in G_5$ (corresponding to $v<v_1\leq\tilde{\tilde{v}}$) the different wave patterns are $R_1-S_1^{CL}$, $R_1-S_1^{CL}-S_2$, $R_1-S_1^{CL}-R_2$, $R_1-S_1^{CL}-R_2-S_2^{CL}$, $R_1-S_1^{CL}-S_2^{CR}-R_2$, where S_1^{CL} denotes the first kind of left hand side contact discontinuity with $\sigma_1(v_r, v_l)=\lambda_1(v_l)$.

For the case when $(u_+, v_+)\in G_6$ (corresponding to $\tilde{\tilde{v}}<v_1<\infty$) the different wave patterns are $R_1-S_1^{CD}-R_1$, $R_1-S_1^{CD}-R_1-S_2$, $R_1-S_1^{CD}-R_1-R_2$, $R_1-S_1^{CD}-R_1-R_2-S_2^{CL}$, $R_1-S_1^{CD}-R_1-R_2-S_2^{CD}-R_2$, where S_1^{CD} denotes the first kind of admissible double contact discontinuity, namely, $\sigma_1(v_r, v_l)=\lambda_1(v_l)=\lambda_1(v_r)$, the same for S_2^{CD}.

Case 2 $\tilde{v}<v_-<v_a$

Similarly as in Case 1, $W_1(u_-, v_-)$ is defined as

$$W_1(u_-, v_-) = \begin{cases} S_1(u_-, v_-) & \text{for} \quad b < v < v_- \\ R_1(u_-, v_-) & \text{for} \quad v_- \leq v \leq v_\alpha \\ C_1(v_-, v_\alpha; R_1(u_-, v_-)) & \text{for} \quad v_\alpha \leq v \leq v_{L_\beta(u_-, v_-)} \\ S_1(u_-, v_-) & \text{for} \quad v_{L_\beta(-)} < v \leq v_{R_\beta(-)} \\ R_1(R_\beta(-)) & \text{for} \quad v > v_{R_\beta(-)} \end{cases} \tag{3.12}$$

For any $(u_1, v_1) \in W_1(u_-, v_-)$, $W_2(u_1, v_1)$ can be defined similarly as in (3.2) — (3.11) respectively according to different v_1 and the family of curves $\{W_2(u_1, v_1); (u_1, v_1) \in W_1(u_-, v_-)\}$ covers the whole domain D univaluedly (see Figure 3.2).

For the case when $(u_+, v_+) \in G_1, G_2$ and G_6 the wave patterns are the same as in Case 1 respectively.

For the case when $(u_+, v_+) \in G_3$, the wave patterns are $S_1 - R_2$, $S_1 - S_2$, $S_1 - S_2^{o\beta} - S_2(B_\beta)$, $S_1 - S_2^{o\beta} - R_2(B_\beta)$, $S_1 - S_2^{o\beta} - R_2 - S_2^{CL}$ which have the same form as in Case 1 if we replace R_1 by S_1 there.

For the case when $(u_+, v_+) \in G_4$, there are 10 different wave patterns, 5 of which are the same as above and the others are $R_1 - R_2$, $R_1 - S_2$, $R_1 - S_2^{o\beta} - S_2(B_\beta)$, $R_1 - S_2^{o\beta} - R_2(B_\beta)$, $R_1 - S_2^{o\beta} - R_2 - S_2^{CL}$.

For the case when $(u_+, v_+) \in G_5$, in addition to the five kinds of wave patterns shown in the corresponding place in Case 1, there are more patterns as $S_1 - R_2$, $S_1 - S_2$, $S_1 - R_2 - S_2^{CL}$, $S_1 - R_2 - S_2^{CD} - R_2$, $S_1^{CR} - R_1 - R_2$, $S_1^{CR} - R_1 - S_2$, $S_1^{CR} - R_1 - R_2 - S_2^{CL}$, $S_1^{CR} - R_1 - R_2 - S_2^{CD} - R_2$.

Case 3 $\quad v_\alpha \leq v_- \leq v_\beta$

Similarly, $W_1(u_-, v_-)$ is defined as

$$W_1(u_-, v_-) = \begin{cases} S_1(u_-, v_-) & \text{for} \quad b < v \leq v_{B_\alpha(-)} \\ S_1(u_-, v_-) & \text{for} \quad v_{B_\beta(-)} \leq v \leq v_{R_\beta(-)} \\ R_1(u_{R_\beta(-)}, v_{R_\beta(-)}) & \text{for} \quad v > v_{R_\beta(-)} \end{cases}$$

For any $(u_1, v_1) \in W_1(u_-, v_-)$, $W_2(u_1, v_1)$ can be defined as in (3.2) — (3.11) respectively$^{(*)}$ and the family of curves $\{W_2(u_1, v_1); (u_1, v_1) \in W_1(u_-, v_-)\}$ covers the whole domain D univaluedly (see Figure 3.3).

For the case when $(u_+, v_+) \in G_1, G_2, G_3$, the wave patterns are the same as in Case 2 respectively.

For the case when $(u_+, v_+) \in G_4$, the wave patterns are $S_1 - R_2$, $S_1 - S_2$, $S_1 - S_2^{o\beta} - S_2(B_\beta)$, $S_1 - S_2^{o\beta} - R_2$, $S_1 - S_2^{o\beta} - R_2 - S_2^{CL}$, $S_1 - R_2 - S_2^{CL}$, $S_1 - S_2^{o\alpha} - S_2(B_\alpha)$, $S_1 - S_2^{o\alpha} - R_2$, where $S_2^{o\alpha}$ denotes the second kind of admissible discontinuity with $\sigma_2 = 0$ and $v_r < v_\alpha$.

For the case when $(u_+, v_+) \in G_5$, the wave patterns are in two groups, one of which are $S_1 - S_2$, $S_1 - R_2$, $S_1 - R_2 - S_2^{CL}$, $S_1 - S_2^{CR} - R_2$ and another one has $S_1^{CR} - R_1$ as the first wave instead of S_1.

For the case when $(u_+, v_+) \in G_6$, the wave patterns are the same as is the second group above.

Case 4 $v_\beta < v_- \leq v_*$

$W_1(u_-, v_-)$ is defined as

$$W_1(u_-, v_-) = \begin{cases} R_1(u_{R_\beta(-)}, v_{R_\beta(-)}) & \text{for} \quad v > v_{R_\beta(-)} \\ S_1(u_-, v_-) & \text{for} \quad v_- < v \leq v_{R_\beta(-)} \\ R_1(u_-, v_-) & \text{for} \quad v_\beta \leq v \leq v_- \\ C_1(v_\beta, v_-; R_1(-)) & \text{for} \quad \bar{v} \geq v \geq v_{L_a(-)} \\ S_1(u_-, v_-) & \text{for} \quad b < v < v_{L_a(-)} \end{cases}$$

similarly, $W_2(u_1, v_1)$ can be defined and the family of curves $\{W_2(u_1, v_1) : (u_1, v_1) \in W_1(u_-, v_-)\}$ covers the whole domain D univaluedly. $(v_{R_\beta(-)} = v_*$ when $v_- = v_*)$ (see Figure 3. 4, where the family $\{W_2(u_1, v_1) : (u_1, v_1) \in W_1(u_-, v_-)\}$ is divided into 6 groups according to $b < v_1 \leq v_1^*$, $v_1^* < v_1 \leq \bar{v}$, $v_\beta < v_1 \leq v_*$, $v_* < v_1 \leq \hat{v}$, $\hat{v} < v_1 \leq \tilde{v}$, $v_1 > \tilde{v}$ and covers $G_1, G_2, \ldots, G_6$ respectively.)

For $(u_+, v_+) \in G_1$, the wave patterns are the same as in Case 3. For $(u_+, v_+) \in G_2$, the wave patterns are $R_1 - S_1^{CL}$, $R_1 - S_1^{CL} - R_2$, $R_1 - S_1^{CL} - S_2$, $R_1 - S_1^{CL} - S_2^{CR} - R_2$, $R_1 - S_1^{CL} - S_2^{CR} - R_2 - S_2^{CL}$ and another group is with S_1 as the first wave instead of $R_1 - S_1^{CL}$.

For $(u_+, v_+) \in G_3$, the wave patterns are in two groups, one of which contains $S_1 - S_2$, $S_1 - S_2^{oa} - S_2(B_a)$, $S_1 - S_2^{oa} - R_2$, $S_1 - R_2$, $S_1 - R_2 - S_2^{CL}$, another group has R_1 as the first wave instead of S_1.

For $(u_+, v_+) \in G_4$, the wave patterns can be also divided into two groups. One group is $S_1 - S_2$, $S_1 - R_2$, $S_1 - R_2 - S_2^{CL}$, $S_1 - S_2^{oa} - S_2(B_a)$, $S_1 - S_2^{oa} - R_2$. Another has $S_1^{CR} - R_1$ as the first two waves instead of S_1.

For $(u_+, v_+) \in G_5$, the wave patterns are $S_1^{CR} - R_1 - S_2$, $S_1^{CR} - R_1 - R_2$, $S_1^{CR} - R_1 - R_2 - S_2^{CL}$, $S_1^{CR} - R_1 - S_2^{CR} - R_2$.

For $(u_+, v_+) \in G_6$, the wave patterns are the same as in G_5 except the last one is $S_1^{CR} - R_1 - R_2 - S_2^{CD} - R_2$.

Case 5 $v_* < v_- \leq \hat{v}$

$W_1(u_-, v_-)$ is defined as

$$W_1(u_-, v_-) = \begin{cases} R_1(u_-, v_-) & \text{for } v \geq v_- \\ S_1(u_-, v_-) & \text{for } v_{R_\beta(-)} \leq v < v_- \\ R_1(u_{R_\beta(-)}, v_{R_\beta(-)}) & \text{for } v_\beta \leq v < v_{R_\beta(-)} \\ C_1(v_\beta, v_{R_\beta(-)}; R_1(R_{\beta(-)})) & \text{for } v_{L_\alpha(R_\beta(-))} \leq v \leq \bar{v} \\ S_1(u_-, v_-) & \text{for } b < v < v_{L_\alpha(R_\beta(-))} \end{cases}$$

Similarly, $W_2(u_1, v_1)$ can be defined and the family of curves $\{W_2(u_1, v_1); (u_1, v_1) \in W_1(u_-, v_-)\}$ covers the whole domain D univaluedly (see Figure 3. 5).

For $(u_+, v_+) \in G_1$, and G_2, the wave patterns are the same as in Case 4 respectively except $S_1^{CR} - R_1 - S_1^{CL}$ is the head instead of $R_1 - S_1^{CL}$ in one of the groups for G_2. For $(u_+, v_+) \in G_3$, the wave patterns are the same as in the corresponding Case 4 but the first wave is $S_1^{CR} - R_1$ or S_1 in the two groups instead of R_1 or S_1 respectively. For $(u_+, v_+) \in G_4$, the wave patterns are the same as in Case 4 too, but the first wave is S_1 or R_1 in the two groups instead of S_1 or $S_1^{CR} - R_1$. For $(u_+, v_+) \in G_5$ or G_6, the wave patterns are the same as in Case 4 respectively, except the first wave is R_1 instead of $S_1^{CR} - R_1$.

Case 6 $v_- > \hat{v}$

$W_1(u_-, v_-)$ is defined as

$$W_1(u_-, v_-) = \begin{cases} R_1(u_-, v_-) & \text{for } v \geq v_- \\ S_1(u_-, v_-) & \text{for } v_{R_\beta(-)} \leq v < v_- \\ R_1(u_{R_\beta(-)}, v_{R_\beta(-)}) & \text{for } v_\beta \leq v < v_{R_\beta(-)} \\ C_1(v_\beta, v_{R_\beta(-)}; R_1(R_\beta(-))) & \text{for } v_{D_\alpha(-)} \leq v \leq \bar{v} \\ S_1(u_-, v_-) & \text{for } b < v < v_{D_\alpha(-)} \end{cases}$$

The others can be discussed similarly as before. We end up with

Theorem 3. 1 *For any given $(u_-, v_-) \in D$, $(u_+, v_+) \in D$, there exists a unique admissible similarity weak solution of the Riemann problem (1. 1) and (1. 2).*

(*) $G_4 = G_4^{(1)} \cup G_4^{(2)}$ corresponding to $v_\alpha^* < v_1 \leq v_{B_\beta(-)}$ and $v_{B_\beta(-)} \leq v_1 \leq \hat{v}$ respectively. $W_2(u_1, v_1)$ can be defined as in (3. 7) for $v_\alpha^* < v_1 \leq v_{B_\alpha(-)}$. However, $W_2(u_1, v_1)$ is defined as follows for $v_{B_\beta(-)} \leq v_1 \leq \hat{v}$.

$$W_2(u_1, v_1) = \begin{cases} S_2(u_1, v_1) & \text{for } v > v_1 \\ R_2(u_1, v_1) & \text{for } v_* \leq v \leq v_1 \\ C_2(v_*, v_1; R_2(1)) & \text{for } v_{L_i(-)} < v < v_* \\ S_2(u_1, v_1) & \text{for } v_{B_i(1)} \leq v \leq v_{L_i(1)} \\ S_2(u_{B_\alpha(1)}, v_{B_\alpha(1)}) & \text{for } v_{B_\alpha(1)} \leq v < v_{B_i(1)} \\ R_2(u_{B_\alpha(1)}, v_{B_\alpha(1)}) & \text{for } b < v < v_{B_\alpha(1)} \end{cases} \qquad (3.\ 7)'$$

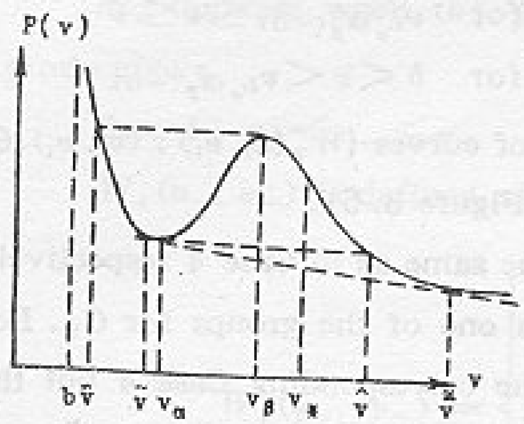

Fig.2.1

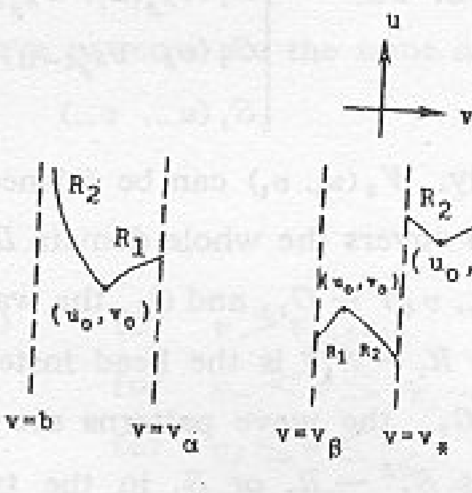

Fig.2.2

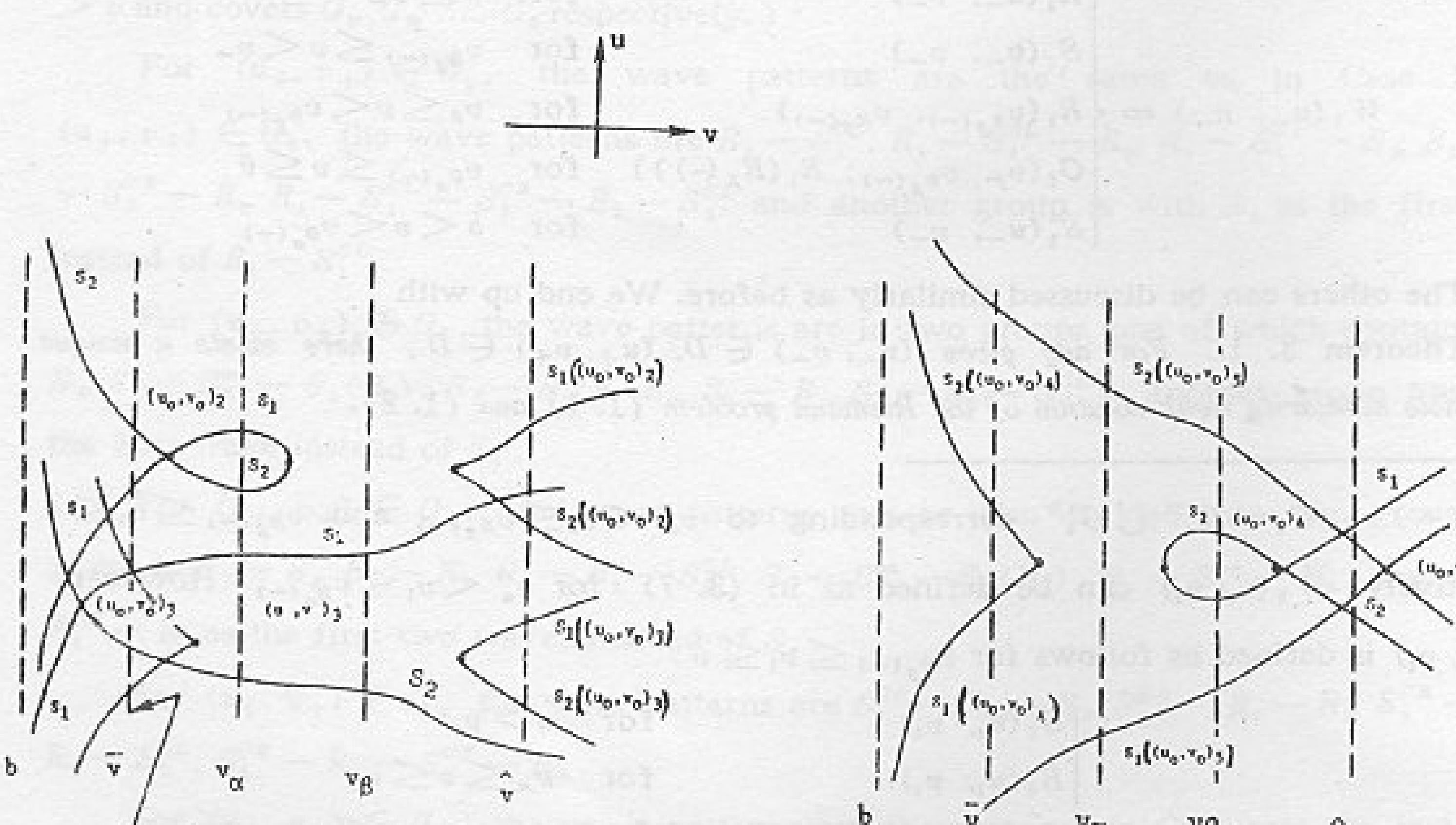

Fig.2.3

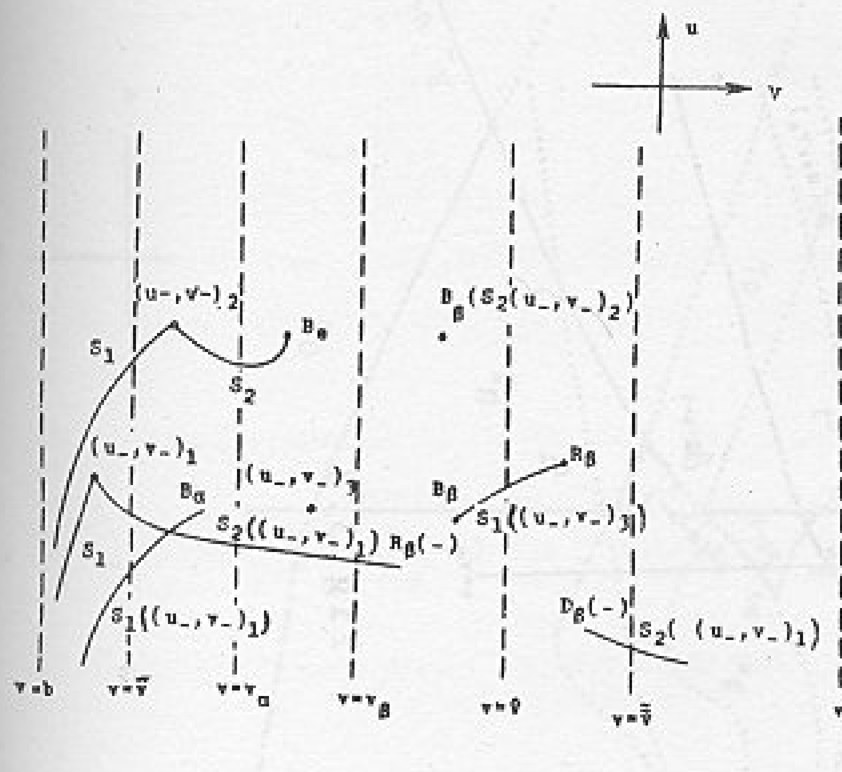

Fig.2.4a

the cases: $b<\underline{v}\leq\bar{v}$, $\bar{v}<v_-<\bar{\bar{v}}$

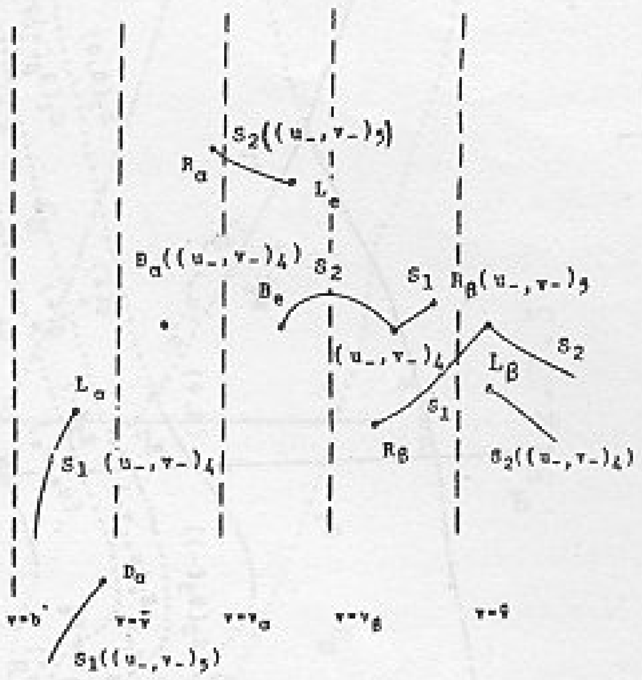

Fig.2.4b

the case: $\bar{\bar{v}}<v_-<v_\alpha$, $v_\alpha\leq v_-\leq v_\beta$, $v_\beta<v_-<v_*$, $v_*\leq v_-\leq\hat{v}$, $v_-<\hat{v}$

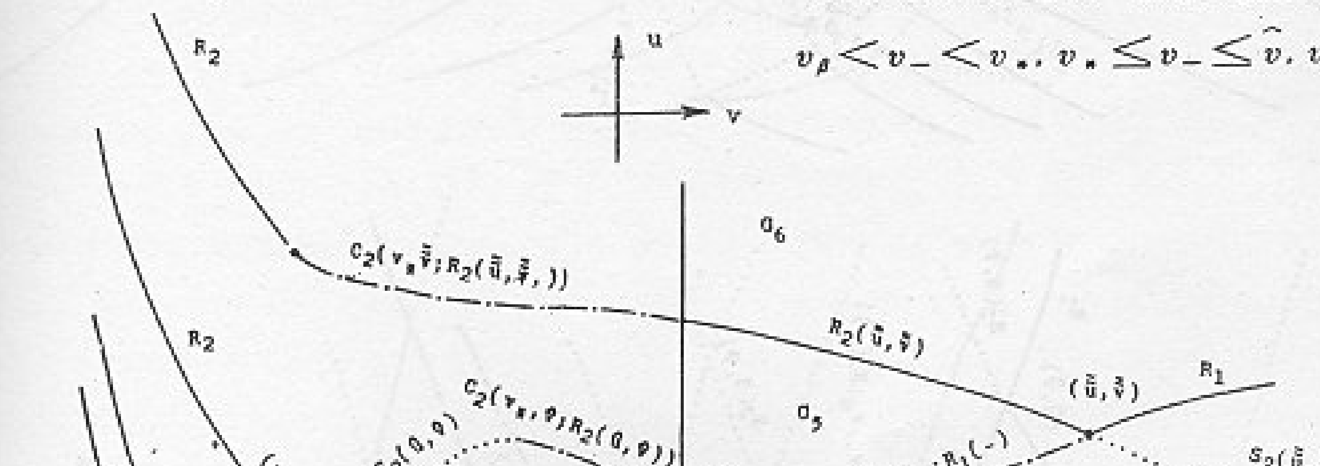

Fig.3.1

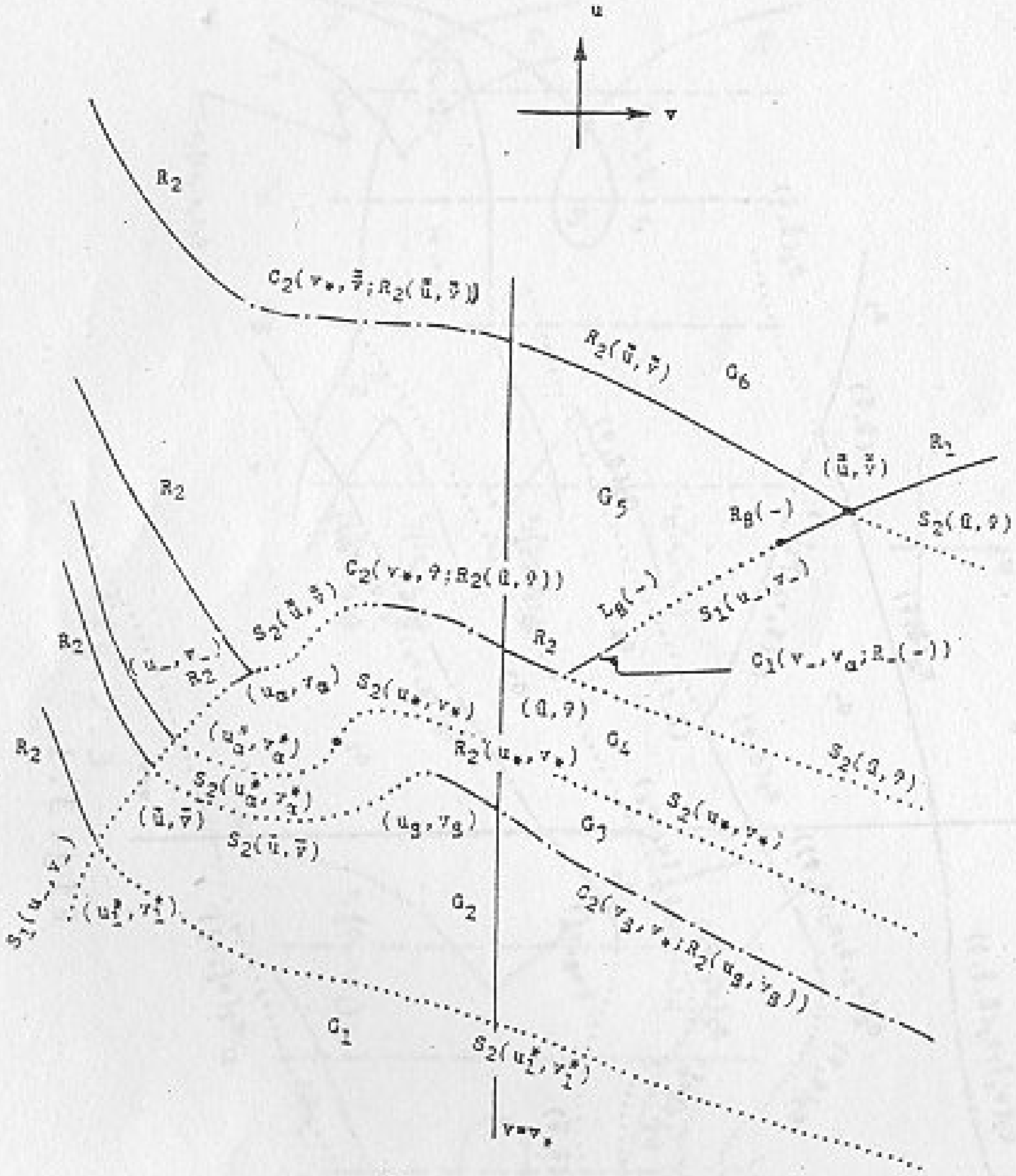
u
v
R2
C2(v*, v̄; R2(ū, v̄))
R2(ū, v̄)
G6
G5
R1
(ū, v̄)
R8(−)
S2(û, v̂)
S1(u−, v−)
C1(v−, vα; R−(−))
R2
(û, v̂)
G4
S2(û, v̂)
R2(u*, v*)
S2(u*, v*)
G3
G2
C2(vg, v*; R2(ug, vg))
G1
S2(u1*, v1*)
v=v*
S1(u−, v−)
(u1*, v1*)
(ū, v̄)
S2(ū, v̄)
(ug, vg)
(uα, vα)
(u−, v−)

Fig.3.2

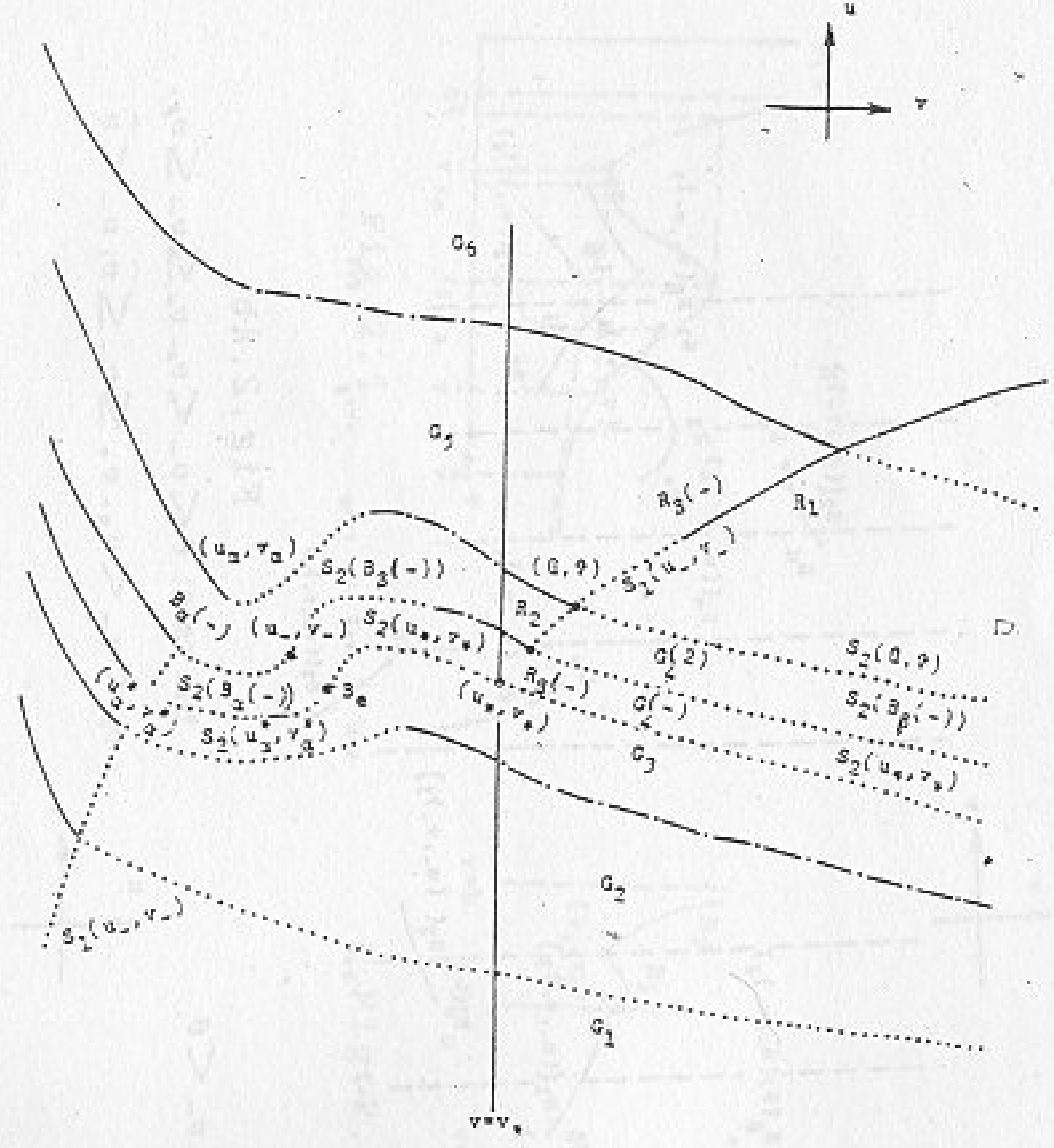
u
v
G6
G5
R1
R3(−)
S1(u−, v−)
(û, v̂)
S2(û, v̂)
S2(B3(−))
(uα, vα)
S2(u*, v*)
(u−, v−)
G(2)
G(−)
G3
G2
G1
D
S2(u*, v*)
v=v*
S1(u−, v−)

Fig.3.3

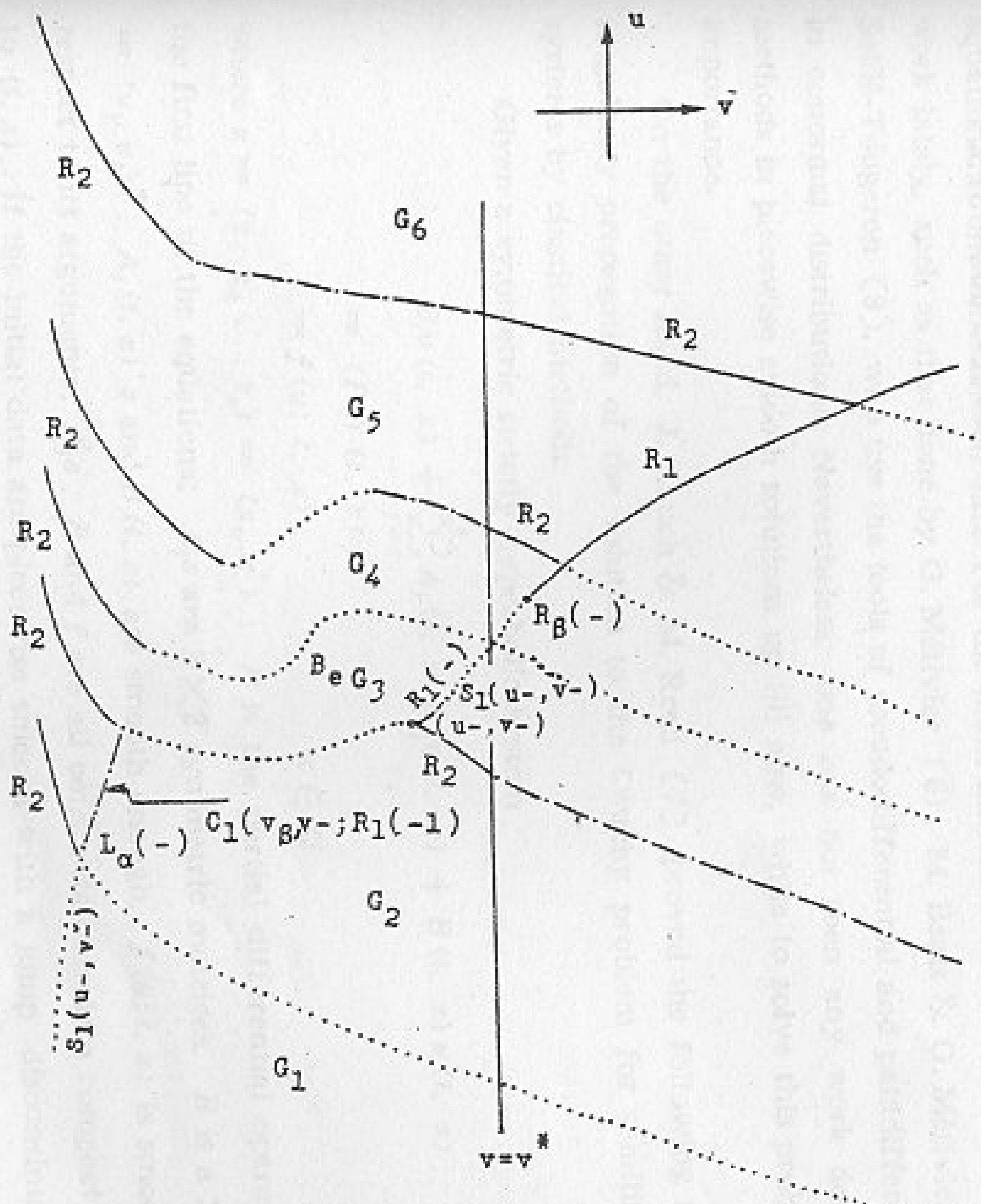
u
v
R2
G6
R2
G5
R1
R2
G4
R2
Rβ(-)
R2
Be G3
R1(-)
S1(u-,v-)
(u-,v-)
R2
R2
C1(vβ,v-;R1(-1)
Lα(-)
G2
S1(u-A'-n)I
G1
v=v*

Fig.3.4

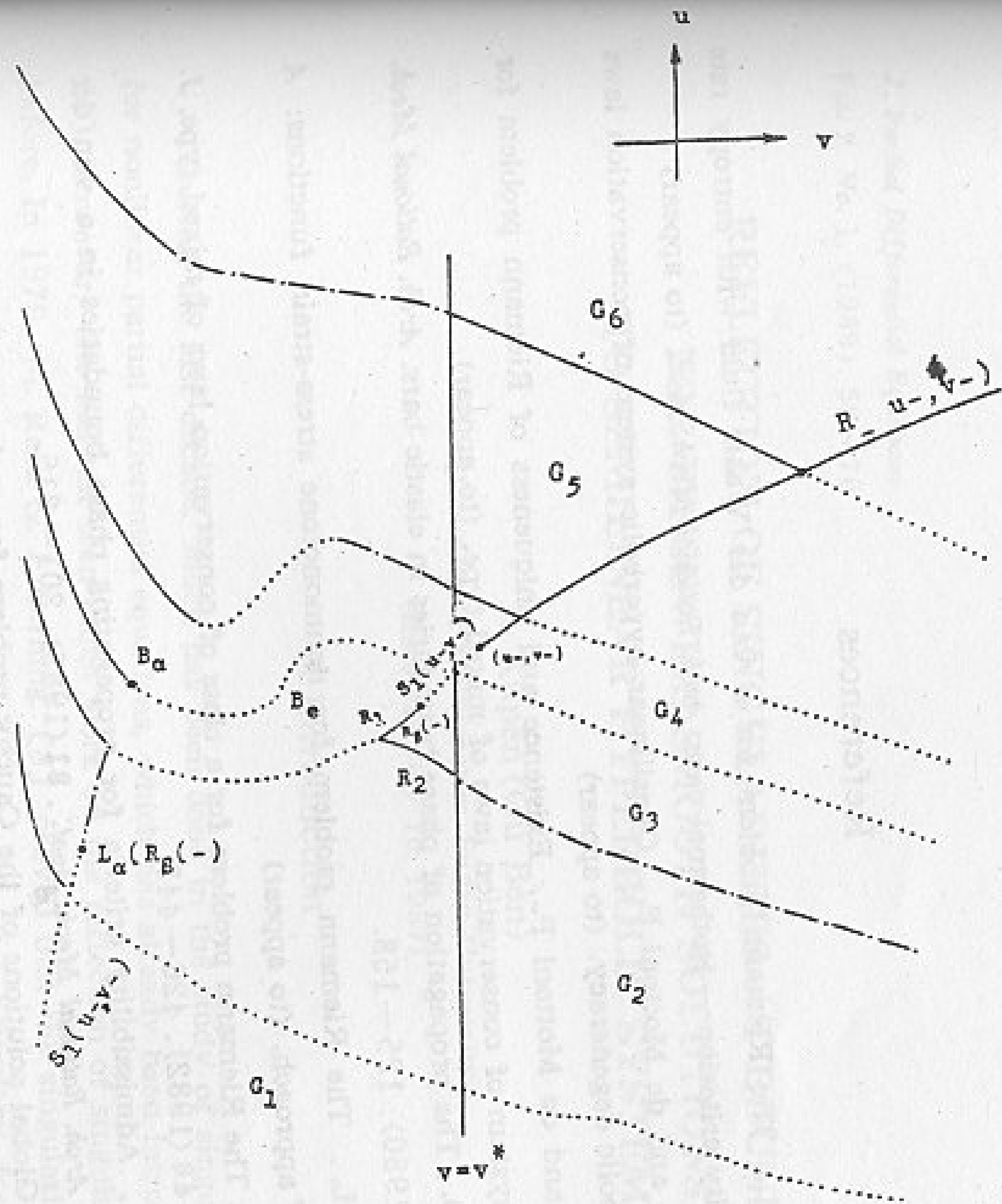
u
v
G6
R_ u-,v-)
G5
Bα
Be
S1(u-,v-)
(u-,v-)
G4
R1
R2
G3
Lα(Rβ(-)
G2
S1(u-A'-n)
G1
v=v*

Fig.3.5

References

〔1〕 Hattori H., The Riemann problem for a van der Waals fluid with entropy rate admissibility criterion— Isothermal case, *Arch. Rational Mech. Anal.* (to appear)

〔2〕 Hsiao L. and de Mottoni P., Quasilinear hyperbolic system of conservation laws with parabolic degeneracy. (to appear)

〔3〕 Hsiao L. and de Mottoni P., Existence and uniqueness of Riemann problem for nonlinear system of conservation laws of mixed type. (to appear)

〔4〕 James R. D., The propagation of phase boundaries in elastic bars. *Arch. Rational Mech. Anal.*, 73 (1980), 125—158.

〔5〕 Keyfitz B. L., The Riemann problem for nonmonotone stress-strain functions: *A* "Hysteresis" approach. (to appear)

〔6〕 Shearer M., The Riemann problem for a class of conservation laws of mixed type, *J. Diff. Eqns.*, 46 (1982), 426—443.

〔7〕 Slemrod M., Admissibility criteria for propagating phase boundaries in a van der Waals fluid. *Arch. Rational Mech. Anal.*, 81 (1983), 301—315.

〔8〕 Temple B., Global solutions of the Cauchy problem for a class of 2×2 nonstrictly hyperbolic conservation laws, *Advances in Applied Math.*, 3 (1982), 335—375.