J. Partial Differential Equations
Vol. 2, *No.* 1 (1989), 31—39

THE SPIN WAVE SYSTEM IN FERROMAGNETIC LATTICS

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(Received July 7, 1987; revised December 17, 1987)

Recently, the Heisenberg system:

$$S_t = S \times S_{xx} + S \times h$$

has aroused the more and more interest. There are many works upon it in various places, such as soliton solutions, infinit conservation laws, etc. [1-7]. But this system is only the primary approximation of its original mathematical-physical model.

In ferromagnetic lattics[1, 2], spin vectors satisfies the relation:

$$\frac{\overline{h}}{2}\frac{\partial S_i}{\partial t} = \sum_k \frac{A}{2}[S_i, \ S_k] + [S_i, \ h]$$

where i points to the i th atom. The summing indices k's point to these atoms which near to the i th. A is the exchange integral. $\overline{h}$ is the Planck constant. h is a constant vector. For α-Fe with ferromagnetic property is a simple cubic lattic with lattic constant $\overline{a}$, to which we have expansion:

$$\frac{A}{2}\sum_k S_k = \sum_{m=0}^{\infty} \widetilde{\Delta}^m S_i$$

where $\widetilde{\Delta}^m = \sum_{|\alpha|=m} a_\alpha D_x^{2\alpha}$, $(m = 1, 2, \ldots)$ are elliptic operators. α's are N- tuple indices. a_α's are positive constants depending on $\overline{a}$. The unequal relation between α's and the meaning of $\begin{pmatrix} \alpha' \\ \alpha'' \end{pmatrix}$ are understood as usual.

In this paper, we will discuss the existence of the global weak solution of the system which is much nearer to the original model than before, only neglect the terms whose orders are higher than $2M$:

$$Z_t = Z \times \sum_{m=1}^{M} \widetilde{\Delta}^m Z + f(x, \ t) Z + B(x, \ t) \qquad \text{in} \quad Q_T = \Omega \times [0, \ T) \tag{0. 1}$$

with the boundary-initial conditions:

$$\left.\frac{d^l Z}{d\gamma^l}\right|_{\partial\Omega} = 0 \qquad l = 0, \ 1, \ \ldots, \ M-1 \qquad \text{on} \quad S_T = \partial\Omega \times [0, \ T) \tag{0. 2}$$

$$Z(x, \ 0) = Z_0(x) \qquad \text{on} \quad \Omega \tag{0. 3}$$

where $Z(x,t)$, $Z_0(x)$, and $B(x,t)$ are all 3-dimensional vector functions. $f(x,t)$ is a 3×3 matrix function. γ is the out-normal vector. $x\in R^N$, $N<2M$. Ω is a C^M bound domain in N-dimensional space. The Z_0 also satisfies the condition of compatibility: $\left.\frac{d^l Z_0}{d\gamma^l}\right|_{\partial\Omega}=0,\ l=0,1,\ldots,M-1$.

In our model, N is a finite number (generally it is 3). But M can be arbitrary great. so the condition $N<2M$ can be satisfied easily, and it is quite natural.

The coefficient matrix of the highest order elliptic operator of the system (0. 1) is:

$$A_{2M}(Z)=\begin{pmatrix} 0 & -Z_3 & Z_2 \\ Z_3 & 0 & -Z_1 \\ -Z_2 & Z_1 & 0 \end{pmatrix}$$

It is a null definite matrix [3]. So (0. 1) is a high order degenerate quasilinear parabolic system.

Being based on the work of the second order ferromagnetic chain system by Zhou Yulin and Guo Boling [3, 4], we first discuss the solution of a system with a small parameter $\varepsilon\ (>0)$ (a_0 below will be determined later):

$$Z_t=\varepsilon(-1)^{M+1}\left(\sum_{m=1}^{M}\tilde{\Delta}^m Z+a_0 Z\right)+Z\times\sum_{m=1}^{M}\tilde{\Delta}^m Z+f(x,t)Z+B(x,t) \qquad (0.\ 4)$$

under the boundary-initial conditions (0. 2), (0. 3). And at last, we study the weak convergence of it when ε tends to zero.

1

As preparing work, in this section, we will discuss a linear high order parabolic system:

$$u_t=(-1)^{M+1}A_{2M}(x,t)\tilde{\Delta}^m u+\sum_{|\alpha|\leq 2M-1}A_\alpha D_x^\alpha u+B(x,t) \quad \text{in } Q_T \qquad (1.\ 1)$$

with the first boundary-initial conditions:

$$\left.\frac{d^l u}{d\gamma^l}\right|_{\partial\Omega}=0 \qquad l=0,1,\ldots,M-1 \quad \text{in } S_T \qquad (1.\ 2)$$

$$u(x,0)=u_0(x) \quad \text{in } \Omega \qquad (1.\ 3)$$

where u, u_0 and B are all J-dimensional vector functions. $x\in R^N$, $A_\alpha(x,t)=(A_\alpha^{ij}(x,t))_{i,j=1}^{J}$, $|\alpha|=0,1,\ldots,2M$. (When $|\alpha|=2M$, let $(A_\alpha)=A_{2M}$. It is same to a_M in the following.) And $\left.\frac{d^l u_0}{d\gamma^l}\right|_{\partial\Omega}=0 \quad l=0,1,\ldots,M-1$.

Denote by $\mathscr{B}$ a function space. Say $u=(u_1,\ldots,u_J)\in\mathscr{B}$, if $u_i\in\mathscr{B}$, $i=1,2,\ldots,J$. And $\|u\|_{\mathscr{B}}=\{\sum_{i=1}^{J}\|u_i\|_{\mathscr{B}}^2\}^{\frac{1}{2}}$.

Lemma 1 *Let the problem (1. 1)—(1. 3) satisfy the following conditions:*

(H1) Matrix $A_{2M}(x, t)$ is a positive definite matrix in Q_T, that is, there exists $\delta_0 (>0)$, such that for any $\xi \in \mathbf{R}^J$, $\xi' A_{2M}\xi \geqslant \delta_0 |\xi|^2$; $A_\alpha (|\alpha| \leqslant 2M)$ belong to $L_\infty(Q_T)$;

(H2) $B(x, t) \in L_2(Q_T)$, $u_0(x) \in H_0^M(\Omega)$.

Then, it has a unique solution $u \in G = L_\infty([0, T]; H_0^M(\Omega)) \cap W_2^{2M, 1}(Q_T)$ and there is a priori estimation:

$$\sup_{0 \leqslant t \leqslant T} \| u(\cdot, t) \|_{H^M(\Omega)} + \| u_t \|_{L_2(Q_T)} + \| \tilde{\Delta}^M u \|_{L_2(Q_T)}$$
$$\leqslant K (\| u_0 \|_{H^M(\Omega)} + \| B \|_{L_2(Q_T)}) \tag{1. 4}$$

where K is depending on δ_0 and T.

Proof First, let us proceed to a priori estimates of the solution. Take (1. 1) get the scalar product by the vector u and $\tilde{\Delta}^M u$ respectively, then integrate over Ω respecting to x to obtain:

$$\int_\Omega u \cdot u_t dx = (-1)^{M+1} \int_\Omega u \cdot A_{2M} \tilde{\Delta}^M u dx$$
$$+ \sum_{|\alpha| \leqslant 2M-1} \int_\Omega u \cdot A_\alpha D_x^\alpha u dx + \int_\Omega u \cdot B dx \tag{1. 5}$$

$$\int_\Omega \tilde{\Delta}^M u \cdot u_t dx = (-1)^{(M+1)} \int_\Omega \tilde{\Delta}^M u \cdot A_{2M} \tilde{\Delta}^M u dx$$
$$+ \sum_{|\alpha| \leqslant 2M-1} \int_\Omega \tilde{\Delta}^M u \cdot A_\alpha D_x^\alpha u dx + \int_\Omega \tilde{\Delta}^M u \cdot B dx \tag{1. 6}$$

Note that by integrating by parts:

$$\int_\Omega \tilde{\Delta}^M u \cdot u_t dx = (-1)^M \frac{1}{2} \frac{d}{dt} \sum_{|\alpha| = M} a_\alpha \| D_x^\alpha u \|_{L_2(\Omega)}^2 \tag{1. 7}$$

Let (1. 6) times $(-1)^M$, then add to (1. 5), we obtain:

$$\int_\Omega u \cdot u_t dx + \frac{1}{2} \frac{d}{dt} \sum_{|\alpha| = M} a_\alpha \| D_x^\alpha u \|_{L_2(\Omega)}^2 + \int_\Omega \tilde{\Delta}^M u \cdot A_{2M} \tilde{\Delta}^M u dx$$
$$= (-1)^{M+1} \int_\Omega u \cdot A_{2M} \tilde{\Delta}^M u dx + \int_\Omega (u + (-1)^M \tilde{\Delta}^M u) \cdot (\sum_{|\alpha| \leqslant 2M-1} A_\alpha D_x^\alpha u + B) dx \tag{1. 8}$$

By conditions (H1), (H2) and using Hölder inequality, (1. 8) can be changed into:

$$\frac{1}{2} \frac{d}{dt} \{ \| u \|_{L_2(\Omega)}^2 + \sum_{|\alpha| = M} a_\alpha \| D_x^\alpha u \|_{L_2(\Omega)}^2 + \delta_0 \| \tilde{\Delta}^M u \|_{L_2(\Omega)}^2 \}$$
$$\leqslant \frac{\delta_0}{2} \| \tilde{\Delta}^M u \|_{L_2(\Omega)}^2 + \frac{C}{\delta_0^{2M-1}} \{ \| u \|_{L_2(\Omega)}^2 + \sum_{|\alpha| = M} a_\alpha \| D_x^\alpha u \|_{L_2(\Omega)}^2 \} + C_{\delta_0} \| B \|_{L_2(\Omega)}^2 \tag{1. 9}$$

where the constant C depend only on L_∞ norms of $A_\alpha (|\alpha| \leqslant 2M)$, and B. In the process above, we have used following two inequalities [8, 9]:

$$\sum_{|\alpha| = m} \| D_x^\alpha u \|_{L_2(\Omega)}^2 \leqslant \varepsilon K \| u \|_{H^{2M}(\Omega)}^2 + K \varepsilon^{\frac{-m}{2M-m}} \| u \|_{L_2(\Omega)}^2, \quad (m < 2M)$$

$$\| u \|^2_{H^{2M}(\Omega)} \leqslant K (\| \tilde{\Delta}^M u \|^2_{L_2(\Omega)} + \| u \|^2_{L_2(\Omega)}) \tag{1. 10}$$

Let (1. 9) integrate respecting to t over $[0, \tau]$, use Gronwall inequality, get:

$$\sup_{0 \leqslant t \leqslant T} \{ \| u(\cdot, t) \|^2_{L_2(\Omega)} + \sum_{|\alpha| = M} a_\alpha \| D^\alpha_x u(\cdot, t) \|^2_{L_2(\Omega)} \} + \| \tilde{\Delta}^M u \|^2_{L_2(Q_T)}$$
$$\leqslant K (\| u_0 \|^2_{H^M(\Omega)} + \| B \|^2_{L_2(\Omega)}) \tag{1. 11}$$

Observe the system (1. 1), and note (1. 11), we have:

$$\| u_t \|^2_{L_2(Q_T)} \leqslant K (\| u_0 \|^2_{H^M(\Omega)} + \| B \|^2_{L_2(Q_T)}) \tag{1. 12}$$

From (1. 11), (1. 12), we can get the estimate (1. 4), then $u \in G$ evidently.

The existence of the solution of the problem (1. 1) — (1. 3) can be proved by the method of the continuity in a parameter with the estimate (1. 4) without difficulty. The uniqueness is also from (1. 4).

2

Now we can consider the first boundary-initial value problem of (0. 4) with a small parameter diffusion term. Using Leray-Schauder Fixed Point Theorem, we discuss the existence of the solution $Z(x, t)$ on Q_T.

Take the space $\mathscr{B} = L_\infty(Q_T)$, on which define a functional mapping with a parameter λ $(0 \leqslant \lambda \leqslant 1)$ $T_\lambda: \mathscr{B} \to \mathscr{B}$. For any $u \in \mathscr{B}$, $Z = T_\lambda(u)$ is the solution of the problem:

$$Z_t = \varepsilon (-1)^{M+1} \left(\sum_{m=1}^{M} \tilde{\Delta}^m Z + a_0 Z \right) + \lambda u \times \sum_{m=1}^{M} \tilde{\Delta}^m Z + \lambda f Z + \lambda B \tag{2. 1}$$

$$\left. \frac{d^l Z}{d \gamma^l} \right|_{\partial \Omega} = 0 \qquad l = 0, 1, \ldots, M-1. \qquad Z(x, 0) = Z_0(x) \tag{2. 2}$$

Because of $u \in L_\infty(Q_T)$, the main coefficient matrix is positive definite and bounds in L_∞. We suppose that:

(A1) $f(x, t) \in L_\infty([0, T]; C^M(\Omega))$, $B(x, t) \in L_\infty([0, T]; H^M_0(\Omega))$.

(A2) $Z_0(x) \in H^M_0(\Omega)$.

From last section, we know that $Z(x, t)$ is uniquely determinated from (2. 1), (2. 2), and it belongs to the space G.

It is not difficult to examine that:

1 T_λ is completely continuous.

2 T_λ is uniformly continuous respecting to λ on any bound set $\tilde{M} \subset \mathscr{B}$.

3 When $\lambda = 0$, $T_0(\mathscr{B})$ is a zero vector.

Now only one thing we need to do is to give the uniform estimation with respect to λ $(0 \leqslant \lambda \leqslant 1)$ of the solution of the quasilinear parabolic system:

$$Z_t = (-1)^{M+1}\varepsilon\Big(\sum_{m=1}^{M}\tilde{\Delta}^m Z + a_0 Z\Big) + \lambda Z \times \sum_{m=1}^{M}\tilde{\Delta}^m Z + \lambda f Z + \lambda B \tag{2. 3}$$

with the boundary-initial condition (2. 2) on the space $L_\infty(Q_T)$.

Take (2. 3) get the scalar product by $(-1)^M a_0 Z$ and $(-1)^M \tilde{\Delta}^n Z$ respectively, then integrate over Ω respecting to x, $(n=1, \ldots, M)$, and obtain:

$$(-1)^M a_0 \int_\Omega Z\cdot Z_t dx = (-1)^{(2M+1)} a_0\varepsilon \int_\Omega Z\cdot\Big(\sum_{m=1}^{M}\tilde{\Delta}^m Z + a_0 Z\Big)dx$$

$$+\lambda(-1)^M a_0\int_\Omega Z\cdot\Big(Z\times\sum_{m=1}^{M}\tilde{\Delta}^m Z\Big)dx + \lambda(-1)^M a_0\int_\Omega Z\cdot(fZ+B)\,dx \tag{2. 4$_0$}$$

$$(-1)^M\int_\Omega \tilde{\Delta}^n Z\cdot Z_t dx = (-1)^{(2M+1)}\varepsilon\int_\Omega \tilde{\Delta}^n Z\Big(\sum_{m=1}^{M}\tilde{\Delta}^m Z + a_0 Z\Big)dx$$

$$+\lambda(-1)^M\int_\Omega \tilde{\Delta}^n Z\cdot\Big(Z\times\sum_{m=1}^{M}\tilde{\Delta}^m Z\Big)dx + \lambda(-1)^M\int_\Omega \tilde{\Delta}^n Z\cdot(fZ+B)\,dx \tag{2. 4$_n$}$$

Sum up (2. 4)$_n$, $n=0, 1, \ldots, M$. Note that $\Big(Z\times\sum_{m=1}^{M}\tilde{\Delta}^m Z\Big)\cdot Z = 0$, then:

$$\frac{1}{2}\frac{d}{dt}\Big\{\sum_{0<|\alpha|\le M}(-1)^{M+|\alpha|}a_\alpha \|D_x^\alpha Z\|^2_{L_2(\Omega)} + (-1)^M a_0\|Z\|^2_{L_2(\Omega)}\Big\}$$

$$\le -\varepsilon\int_\Omega\Big(\sum_{n=1}^{M}\tilde{\Delta}^n Z + a_0 Z\Big)\Big(\sum_{m=1}^{M}\tilde{\Delta}^m Z + a_0 Z\Big)dx$$

$$+(-1)^M\int_\Omega\Big(\sum_{n=1}^{M}\tilde{\Delta}^n Z + a_0 Z\Big)\Big(Z\times\sum_{m=1}^{M}\tilde{\Delta}^m Z\Big)dx +$$

$$+(-1)^M\int_\Omega\Big(\sum_{n=1}^{M}\tilde{\Delta}^n Z + a_0 Z\Big)(fZ+B)\,dx$$

$$\le C\Big(\sum_{|\alpha|=M}\|D_x^\alpha Z\|^2_{L_2(\Omega)} + \|Z\|^2_{L_2(\Omega)} + C\Big) \tag{2. 5}$$

in which C is depending only on the bounds of f and B in their space. For:

$$\sum_{0<|\alpha|\le M}(-1)^{M+|\alpha|}a_\alpha\|D_x^\alpha Z(\cdot,\tau)\|^2_{L_2(\Omega)}$$

$$=\Big(\sum_{|\alpha|=M}+\sum_{0<|\alpha|<M}\Big)(-1)^{M+|\alpha|}a_\alpha\|D_x^\alpha Z(\cdot,\tau)\|^2_{L_2(\Omega)}$$

$$\ge\sum_{|\alpha|=M}\frac{a_\alpha}{2}\|D_x^\alpha Z(\cdot,\tau)\|^2_{L_2(\Omega)} - C(M, N, a_M)\|Z(\cdot,\tau)\|^2_{L_2(\Omega)} \tag{2. 6}$$

let us choose a_0, such that $(-1)^M a_0 > C(M, N, a_M) > 0$, then integrate (2. 5) with respect to t over $[0, \tau]$, and get:

$$\sum_{|\alpha|=M}\|D_x^\alpha Z(\cdot,\tau)\|^2_{L_2(\Omega)} + \|Z(\cdot,\tau)\|^2_{L_2(\Omega)}$$

$$\le C\Big[\|Z_0\|^2_{H^M(\Omega)} + \int_0^\tau\Big(\sum_{|\alpha|=M}\|D_x^\alpha Z(\cdot,t)\|^2_{L_2(\Omega)} + \|Z(\cdot,t)\|^2_{L_2(\Omega)}\Big)dt\Big] + C \tag{2. 7}$$

Use Gronwall inequality, then:

$$\sup_{0\leqslant t\leqslant T}\| Z(\cdot,\ t)\|_{H^M(\Omega)}\leqslant K \tag{2. 8}$$

By Imbedding Theorem $H^M(\Omega)\hookrightarrow L_\infty(\Omega)$, $(N<2M)$, [9], we have:

$$\| Z\|_{L_\infty(Q_T)}\leqslant K \tag{2. 9}$$

where K is independent of λ, ε; and depending only on M, N, T, and the bounds of the coefficients. Namely, $Z_\lambda(0\leqslant\lambda\leqslant 1)$ are uniformly bounded in $L_\infty(Q_T)$. By Leray-Schauder Fixed Point Theorem, we know that the problem (0. 4), (0, 2), (0. 3) has at least one solution $Z(x,t)\in G$.

In the following, we will show the uniqueness of $Z(x,t)$ in the space G.

Let $Z(x,t)$, $\widehat{Z}(x,t)$ are two solutions of the problem. Set $W=Z-\widehat{Z}$ then

$$W_t=(-1)^{M+1}\varepsilon\Big(\sum_{m=1}^{M}\widetilde{\Delta}^m W+a_0W\Big)+Z\times\sum_{m=1}^{M}\widetilde{\Delta}^m W+W\times\sum_{m=1}^{M}\widetilde{\Delta}^m\widehat{Z}+fW \tag{2. 10}$$

$$\frac{d^lW}{d\gamma^l}\Big|_{\partial\Omega}=0 \qquad l=0,1,\ldots,M-1 \qquad W(x,0)=0 \tag{2. 11}$$

Take (2. 10) get the scalar product by W, and integrate over Ω to obtain

$$\int_\Omega W\cdot W_t dx=(-1)^{M+1}\varepsilon\sum_{m=1}^{M}\int_\Omega(W\cdot\widetilde{\Delta}^m W+W\cdot a_0W)\,dx$$
$$+\sum_{m=1}^{M}\int_\Omega W\cdot(Z\times\widetilde{\Delta}^m W)\,dx+\int_\Omega W\cdot\Big(W\times\sum_{m=1}^{M}\widetilde{\Delta}^m\widehat{Z}\Big)\,dx$$
$$+\int_\Omega fW\cdot W dx \tag{2. 12}$$

where

$$(-1)^{M+1}\varepsilon\Big(\sum_{m=1}^{M}\int_\Omega(W\cdot\widetilde{\Delta}^m W+a_0W\cdot W)\,dx\Big)$$
$$\leq-\varepsilon\Big(\sum_{|\alpha|=M}\frac{a_\alpha}{2}\| D_x^\alpha W\|^2_{L_2(\Omega)}+[(-1)^M a_0-C(M,\ N,\ a_M)]\| W\|^2_{L_2(\Omega)}\Big)$$
$$\leq-\varepsilon\sum_{|\alpha|=M}\frac{a_\alpha}{2}\| D_x^\alpha W\|^2_{L_2(\Omega)}$$

$$\sum_{m=1}^{M}\int_\Omega W\cdot(Z\times\widetilde{\Delta}^m W)\,dx\leq C_\varepsilon\| W\|^2_{L_2(\Omega)}+\varepsilon\sum_{|\alpha|=M}\frac{a_\alpha}{2}\| D_x^\alpha W\|^2_{L_2(\Omega)}$$

so

$$\frac{1}{2}\frac{d}{dt}\| W\|^2_{L_2(\Omega)}(t)\leq C_\varepsilon\| W\|^2_{L_2(\Omega)}(t) \tag{2. 13}$$

Integrate (2. 13) respecting to t over $[0,\tau]$, then

$$\| W(\cdot,\ \tau)\|_2^2\leq C_\varepsilon\int_0^\tau\| W(\cdot,\ t)\|^2_{L_2(\Omega)}dt \tag{2. 14}$$

By Gronwall Inequality, (2. 14) imply $\| W(\cdot,\tau)\|_2^2\leq 0$, $\forall\ \tau\in[0,T]$. Namely, $W=0$ a. e. Now we have

Theorem 1 *The problem (0. 4), (0. 2), (0. 3) has a unique solution in the space G, if*

(A1), (A2) are satisfied.

3

Now we can discuss the existence of the problem (0. 1), (0. 2), (0. 3).

Definition *A function vector $Z(x,t) \in L_\infty([0,T]; H^M(\Omega))$ is called a weak solution of the problem (0. 1), (0. 2), (0. 3), if for any $\varphi(x,t) \in C_0^M(Q_T)$ holds the integral relation:*

$$\iint_{Q_T} \Big[\varphi_t Z + \sum_{0<|\alpha|\le M} (-1)^{|\alpha|} a_\alpha D_x^\alpha Z \cdot \Big(\sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} (D_x^\beta \varphi \times D_x^{\alpha-\beta} Z) \Big) + \varphi(fZ+B) \Big] dx dt + \int_\Omega \varphi(x,0) Z_0(x)\, dx = 0 \tag{3. 1}$$

We manage to let the unique solution $Z_\varepsilon(x,t)$ of the problem (0. 4), (0. 2), (0. 3) approximate the solution $Z(x,t)$ of the problem (0. 1), (0. 2), (0. 3). At first, from last section, we have

Lemma 2 *For all $\varepsilon > 0$, the solutions $\{Z_\varepsilon(x,t)\}$ of the problem (0. 4), (0. 2), (0. 3) have a uniform bound respecting to $\varepsilon(>0)$ in the space $L_\infty([0,T]; H_0^M(\Omega))$. That is:*

$$\sup_{0\le t\le T} \| Z_\varepsilon(\cdot,t) \|_{H_0^M(\Omega)} \le K \tag{3. 2}$$

where K is depending only on M, N, T, the bounds of the coefficients and the boundary-initial functions.

Take $\psi(x) \in H_0^{2M}(\Omega)$, and calculate that:

$$\begin{aligned} &\int_\Omega \psi(x) Z_{\varepsilon t}(x,t)\, dx \\ &= \int_\Omega (-1)\varepsilon\psi \Big(\sum_{m=1}^M \Delta^m Z_\varepsilon + a_0 Z_\varepsilon\Big) dx + \int_\Omega \psi \Big(Z_\varepsilon \times \sum_{m=1}^M \Delta^m Z_\varepsilon\Big) dx + \int_\Omega \psi(fZ_\varepsilon + B)\, dx \\ &\le \varepsilon C \|\psi\|_{H_0^M(\Omega)} \|Z_\varepsilon\|_{H_0^M(\Omega)} + C \sum_{0<|\beta|\le M} \|D_x^\beta \psi\|_{L_\infty(\Omega)} \Big(\sum_{|\alpha|\le M} \|D_x^\alpha Z_\varepsilon\|^2_{L_2(\Omega)} \Big) \\ &\quad + (C\|Z_\varepsilon\|_{L_2(\Omega)} + C) \|\psi\|_{L_2(\Omega)} \\ &\le K \|\psi\|_{H_0^{2M}(\Omega)} \end{aligned} \tag{3. 3}$$

in which we have used that $H_0^{2M}(\Omega) \hookrightarrow C_0^M(\Omega)$, $(N < 2M)$ [6]*, and in which K is independent of $\varepsilon(>0)$.*

If we call $T_f(\psi) = \int_\Omega f\psi dx$ as a linear functional on $\psi \in \mathscr{D}$, (3. 3) implies that the norm of $T_{Z_{\varepsilon t}}$, which is a linear functional on $H_0^{2M}(\Omega)$, is bound. So $Z_{\varepsilon t}(x,t) \in H^{-2M}(\Omega)$, $\forall t \in [0,T]$. Then we have

Lemma 3 *For any $t \in [0,T]$, $Z_{\varepsilon t}(x,t)$ is uniformly bounded in space $H^{-2M}(\Omega)$. That is:*

$$\sup_{0\le t\le T} \| Z_{\varepsilon t}(\cdot,t) \|_{H^{-2M}(\Omega)} \le K \not\sim \varepsilon \tag{3. 4}$$

At last, let us consider the limit process when ε tends to 0. Since for any $\varphi \in C_0^M(Q_T)$, the solution $Z_\varepsilon(x, t)$ of the approximating problem (0. 4), (0. 2), (0. 3) satisfies that:

$$\iint_{Q_T} \Big[\varphi_t Z_\varepsilon + \varepsilon \Big(\sum_{|\alpha| \leq M} (-1)^{M+|\alpha|+1} a_\alpha D_x^\alpha Z_\varepsilon \cdot D_x^\alpha \varphi \Big)$$

$$+ \sum_{0<|\alpha|\leq M} (-1)^{|\alpha|} a_\alpha D_x^\alpha Z_\varepsilon \cdot \Big(\sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} D_x^\beta \varphi \times D_x^{\alpha-\beta} Z_\varepsilon + \varphi(fZ_\varepsilon + B) \Big)\Big] dxdt$$

$$+ \int_\Omega \varphi(x, 0) Z_0(x)\, dx = 0 \tag{3. 5}$$

By lemma 2. 3, we know that the set $\{Z_\varepsilon(x, t) \mid \varepsilon > 0\}$ is uniformly bounded respecting to $\varepsilon (> 0)$ in space $\mathscr{Z} = L_\infty([0, T]; H^M(\Omega)) \cap W_\infty^1([0, T]; H^{-2M}(\Omega))$. From Imbedding Theorems[9], it holds that:

$$H_0^{2M}(\Omega) \subset\!\!\to H_0^M(\Omega) \subset\!\!\to L_p(\Omega) \subset\!\!\to H^{-2M}(\Omega)$$

and the imbedding mapping $\mathscr{Z} \subset\!\!\to L_p(Q_T)$, $(1 < p < +\infty)$ is compact. So $\{Z_\varepsilon(x, t)\}$ has a subsequence $\{Z_{\varepsilon_i}(x, t)\}$, such that $Z_{\varepsilon_i}(x, t)$ converge to a vector function $Z(x, t) \in L_p(Q_T)$ with ε_i going to zero. And $D_x^\alpha Z_{\varepsilon_i}$ weakly converge to $D_x^\alpha Z$ in $L_p([0, T]; L_2(\Omega))$, where $|\alpha| \leq M$. When $|\gamma| < M$, because of $\mathscr{Z} \subset\!\!\to\subset\!\!\to L_p([0, T]; H^{M-1}(\Omega))$, we can find a subsequence of $D_x^\gamma Z_{\varepsilon_i}$ to converge to $D_x^\gamma Z$ by norm in the space $L_p([0, T]; L_2(\Omega))$. We still denote by $D_x^\gamma Z_{\varepsilon_i}$ this subsequence. When $\varepsilon_i \to 0$, a term of the difference between (3. 5) and (3. 1) is that:

$$\iint_{Q_T} \sum_{0<|\alpha|\leq M} (-1)^{|\alpha|} a_\alpha D_x^\alpha Z_{\varepsilon_i} \cdot \Big(\sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} D_x^\beta \varphi \times D_x^{\alpha-\beta} Z_{\varepsilon_i} \Big) dxdt$$

$$- \iint_{Q_T} \sum_{0<|\alpha|\leq M} (-1)^{|\alpha|} a_\alpha D_x^\alpha Z \cdot \Big(\sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} D_x^\beta \varphi \times D_x^{\alpha-\beta} Z \Big) dxdt$$

$$= \iint_{Q_T} \sum_{0<|\alpha|\leq M} (-1)^{|\alpha|} a_\alpha \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} [D_x^\alpha (Z_{\varepsilon_i} - Z) \cdot (D_x^\beta \varphi \times D_x^{\alpha-\beta} Z)$$

$$+ D_x^\alpha Z_{\varepsilon_i} \cdot (D_x^\beta \varphi \times D_x^{\alpha-\beta}(Z_{\varepsilon_i} - Z))] dxdt \tag{3. 6}$$

On the right, the first part goes to zero because of $D_x^\alpha Z_{\varepsilon_i} \to D_x^\alpha Z$ (weakly), and the second part, since $\beta > 0$, $|\alpha - \beta| \leq M - 1$, can be estimated like that:

$$\iint_{Q_T} \sum_{0<|\alpha|\leq M} (-1)^{|\alpha|} a_\alpha \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} [D_x^\alpha Z_{\varepsilon_i} \cdot (D_x^\beta \varphi \times D_x^{\alpha-\beta}(Z_{\varepsilon_i} - Z))] dxdt$$

$$\leq C \| Z_{\varepsilon_i} \|_{L_p([0, T]; H^M(\Omega))} \cdot \| \varphi \|_{C^M(Q_T)} \| Z_{\varepsilon_i} - Z \|_{L_p([0, T]; H^{M-1}(\Omega))} \to 0$$

So (3. 6) tends to zero with ε. The other terms of the difference between (3. 5) and (3. 1) go to zero, too, because of the weak convergence of $Z_{\varepsilon_i}(x, t)$. Thus, when $i \to \infty$ $(\varepsilon_i \to 0)$, there exists $Z(x, t) \in L_p(Q_T)$ to be the limit of the $Z_{\varepsilon_i}(x, t)$, and it satisfies the integral relation (3. 1). Since $p\ (1 < p < \infty)$ is arbitrary in our discussion, the solution $Z(x, t)$ belongs to $L_\infty([0, T]; H_0^M(\Omega))$. Now we have

Theorem 2 *If the conditions (A1), (A2) are satisfied, then there exists at least a solution of the problem* (0. 1), (0. 2), (0. 3) *belonging to the space* $L_\infty([0, T]; H_0^M(\Omega))$.

Acknowledgement: The authoress is grateful for the helpful suggestion of Prof. Chen Guowang and Prof. Zhou Yulin.

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