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GLOP OLUTIONS FOR A COUPLED KDV SYSTEM

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1. Introduction

The coupled KdV system

$$\begin{cases} u_t - a(u_{xxx} + 6uu_x) - 2bvv_x = 0, \\ v_t + v_{xxx} + cuv_x = 0 \end{cases} \quad (1.\ 1)$$

arises in physics[1, 2], which describes the interaction of two long waves with different dispersion relations. It has been proved that system (1. 1) has two- and three-soliton solutions if there is a special relation between the dispersion relations of the two long waves.

In the present work we shall show existence and uniqueness of global solutions satisfying the periodic initial-value conditions

$$\begin{cases} U(x+2D,\ t) = U(x,\ t) \\ U(x,\ 0) = U_0(x) \end{cases} \quad (1.\ 2)$$

or the initial value condition

$$U(x,\ 0) = U_0(x) \quad (1.\ 3)$$

for the coupled system (1. 1) in the domain $Q_T^* = \{|x| < \infty, 0 \leq t \leq T\}$, where $T > 0$, $U(x, t) = (u(x, t), v(x, t))$, $U_0(x) = (u_0(x), v_0(x))$.

We shall obtain the solution to the periodic problem (1. 1), (1. 2) as a limit of solutions to the perturbed system

$$\begin{cases} u_t = -\varepsilon u_{xxxx} + a(u_{xxx} + 6uu_x) + 2bvv_x \\ v_t = -\varepsilon v_{xxxx} - v_{xxx} - cuv_x \end{cases} \quad (1.\ 4)$$

with periodic condition (1. 2). The difficult part of our development, as in all previous work on the KdV and its generalizations, is in obtaining a priori estimates for the norms of solutions to the perturbed problem. In the final section we will also state theorems for the initial-value problem (1. 1), (1. 3) analogous to the periodic initial-value problem (1. 1), (1. 2).

2. The Existence Theorem for the Perturbed Problem

Let us consider the periodic initial-value problem (1. 4), (1. 2). To solve the problem we linearize system (1. 4) and obtain

Lemma 1 *Let $U_0 \in H^2(-D, D)$ and $f \in L_2(Q_T)$ be periodic with respect to x with period $2D$, where $f=(f_1, f_2)$, $Q_T=\{(x, t): -D<x<D, 0 \leq t \leq T\}$, then the linear parabolic system*

$$\begin{cases} u_t = -\varepsilon u_{xxxx} + a u_{xxx} + f_1 \\ v_t = -\varepsilon v_{xxxx} - v_{xxx} + f_2 \end{cases} \tag{2. 1}$$

with the periodic initial-value condition (1. 2) has one and only one solution $U(x, t)$ and

$$\| U \|_{L_\infty(0, T; H^2(-D, D))} + \| U \|_{W_2^{(4, 1)}(Q_T)} \leq C_1 (\| U_0 \|_{H^2(-D, D)} + \| f \|_{L_2(Q_T)}) \tag{2. 2}$$

where C_1 is a constant.

Proof From the theory on parabolic partial differential equation, we can obtain the existence of solutions to the periodic initial-value problem (2. 1), (1. 2).

In order to get the estimation, we take the inner product of (2. 1) and U, then integrate the resultant relation over rectangular domain Q_t, we have

$$\| U(\cdot, t) \|^2_{L_2(-D, D)} + 2 \| U_{xx} \|^2_{L_2(Q_t)} \leq \| U \|^2_{L_2(Q_t)} + \| f \|^2_{L_2(Q_t)} + \| U_0 \|^2_{L_2(-D, D)}$$

By the Gronwall inequality, there is

$$\| U \|^2_{L_\infty(0, T; L_2(-D, D))} \leq e^T (\| U_0 \|^2_{L_2(-D, D)} + \| f \|^2_{L_2(Q_T)})$$

Then taking the inner product of system (2. 1) and vector U and integrating the resultant relation over Q_t, we obtain the expression

$$\| U_{xx}(\cdot, t) \|^2_{L_2(-D, D)} + \varepsilon \| U_{xxxx} \|^2_{L_2(Q_t)} \leq \| U_{0xx} \|^2_{L_2(-D, D)} + \frac{1}{\varepsilon} \| f \|^2_{L_2(Q_t)}$$

from which we have

$$\| U_{xx} \|_{L_\infty(0, T; L_2(-D, D))} + \varepsilon \| U_{xxxx} \|^2_{L_2(Q_t)} \leq \| U_{0xx} \|^2_{L_2(-D, D)} + \frac{1}{\varepsilon} \| f \|^2_{L_2(Q_t)}$$

Besides, using system (2. 1) and the above results, we can also get the estimation for $\| U_t \|_{L_2(Q_t)}$. So the inequality (2. 2) holds, which ensures the uniqueness of solution.

Corollary *Let $D_x^k D_t^h f(x, t) \in L_2(Q_t)$, $U_0 \in H^{(k+4h+2)}(-D, D)$ for $h \geq 0$ and $k \geq 0$, then for the solution U to the problem (2. 1), (1. 2), we have*

$$D_x^k D_t^h U \in L_\infty(0, T; H^2(-D, D)) \cap W_2^{(4, 1)}(Q_T)$$

and the inequality analogous to the inequality (2. 2) holds.

Using lemma 1, we can show the following result:

Lemma 2 *Let $a+1>0$, $bc>0$, $U_0(x) \in H^2(-D, D)$ be periodic with period $2D$. Then* *the periodic initial-value problem (1. 4), (1. 2) has at least one generalized global solution.*

Proof We use the Leray-Schauder fixed point theorem to prove. Let $B = L_\infty(0, T; H^1(-D, D))$ be the base space. Denote $Z = L_\infty(0, T; H^2(-D, D)) \cap W_2^{(4,1)}(Q_T)$. For any function $\bar{U} = (\bar{u}, \bar{v})$, we define a two-dimensional vector-valued function $U = (u, v)$ to be the generalized solution to the linear parabolic system

$$\begin{cases} u_t = -\varepsilon u_{xxxx} + a u_{xxx} + 6ra\bar{u}\bar{u}_x + 2rb\bar{v}\bar{v}_x \\ v_t = -\varepsilon v_{xxxx} - v_{xxx} - cr\bar{u}\bar{v}_x \end{cases} \tag{2. 3}$$

with the periodic initial-value problem (1. 2), where $0 \leq r \leq 1$ is a parameter. From

$$\| rb\bar{v}\bar{v}_x \|^2_{L_2(Q_T)} \leq C_0 \sup_{Q_T} |\bar{v}|^2 \| \bar{v}_x \|^2_{L_2(Q_T)}$$
$$\leq C_1 \| \bar{v}_x \|_{L_\infty(0,T;L_2(-D,D))} \| \bar{v} \|_{L_\infty(0,T;L_2(-D,D))} \| \bar{v}_x \|^2_{L_2(Q_T)}$$
$$\leq C_1 \| \bar{U} \|^4_B$$
$$\| ra\bar{u}\bar{u}_x \|_{L_2(Q_T)} + \| rc\bar{u}\bar{v}_x \|_{L_2(Q_T)} \leq C_1 \| \bar{U} \|^2_B$$

where C_1 is a constant, we can see that the right hand side of (2. 3) is quadratically integrable over Q_T. Hence from Lemma 1 the periodic initial-value problem (2. 3), (1. 2) has a unique generalized solution $U(x, t) \in Z$ for every $0 \leq r \leq 1$. This defines an operator $T_r: B \to Z \subset B$ for any $0 \leq r \leq 1$.

It is easy to show that the operator T_r is continuous.

For any $\bar{U} \in B$, $U = T_r\bar{U} \in L_\infty(0, T; H^2(-D, D)) \cap W_2^1(0, T; L^2(-D, D))$. For any $t_1, t_2 \in [0, T]$, using the interpolation formulas we have

$$\sup_{-D \leq x \leq D} | u(x, t_2) - u(x, t_1) | \leq C_1 \| u(\cdot, t_2) - u(\cdot, t_1) \|^{3/4}_{L_2(-D,D)} \cdot$$
$$\cdot \| u(\cdot, t_2) - u(\cdot, t_1) \|^{1/4}_{H^2(-D,D)}$$

$$\sup_{-D \leq x \leq D} | u_x(x, t_2) - u_x(x, t_1) | \leq C_1 \| u(\cdot, t_2) - u(\cdot, t_1) \|^{1/4}_{L_2(-D,D)} \cdot$$
$$\cdot \| u(\cdot, t_2) - u(\cdot, t_1) \|^{3/4}_{H^2(-D,D)}$$

From the following integral relation

$$\int_{-D}^{+D} (u(x, t_2) - u(x, t_1))^2 dx = \int_{-D}^{D} \left(\int_{t_1}^{t} u_h(x, h)\, dh \right)^2 dx$$
$$\leq |t_2 - t_1| \cdot \| u \|^2_{W_2^1(0,T;L_2(-D,D))}$$

we get

$$\sup_{-D \leq x \leq D} | u(x, t_2) - u(x, t_1) | \leq C_1 |t_2 - t_1|^{3/8} \cdot$$
$$\cdot \| u \|^{1/4}_{W_2^1(0,T;L_2(-D,D))} \| u \|^{3/4}_{L_\infty(0,T;H^2(-D,D))}$$

$$\sup_{(-D,D)} | u_x(x, t_2) - u(x, t_1) | \leq c_1 |t_2 - t_1|^{1/8} \| u \|^{1/4}_{W_2^1(0,T;L_2(-D,D))} \cdot$$
$$\cdot \| u \|^{3/4}_{L(0,T;H^2(-D,D))}$$

For function v, there are same estimates. Hence $U(x, t)$ and $U_x(x, t)$ are Hölder continuous with respect to t with the order $3/8$ and $1/8$ respectively. Besides, the imbedding mapping from $H^2(-D, D)$ to $H^1(-D, D)$ is compact, so the imbedding mapping from space Z to B is also compact. Therefore the operators T_r are completely continuous for every $0 \leq r \leq 1$. It can be seen that for any bounded set $M \subset B$, the continuity of T_r with respect to parameter $0 \leq r \leq 1$ is uniform for $M \subset B$. It is obvious that $T_0 B$ is a fixed function.

In order to obtain the generalized solution to the periodic initial-value problem (1. 4), (1. 2), it is sufficient to verify the uniform boundedness of all possible solutions to the periodic initial-value condition (1. 2) of the nonlinear parabolic system

$$\begin{cases} u_t = -\varepsilon u_{xxxx} + a u_{xxx} + 6rauu_x + 2brvv_x \\ v_t = -\varepsilon v_{xxxx} - v_{xxx} - cruv_x \end{cases} \tag{2. 4}$$

with respect to parameter $0 \leq r \leq 1$. Taking the scalar product of system (2. 4) and vector $(cu, 2bv)$ and integrating the resultant relation over domain Q_t $(0 \leq t \leq T)$, we have

$$\int_{-D}^{D} (cu^2(x, t) + 2bv^2(x, t))\,dx - \int_{-D}^{D} (cu_0^2 + 2bv_0^2)\,dx$$
$$+ 2\int_{-D}^{D} (cu_{xx}^2 + 2bv_{xx}^2)\,dx = 0$$

from which we obtain

$$\| U \|^2_{L_\infty(0,T:L_2(-D,D))} + 2\| U_{xx} \|^2_{L_2(Q_T)}$$
$$\leq C_1 \| U_0 \|^2_{L_2(-D,D)}$$

Besides, from the nonlinear parabolic system (2. 4) we derive the following relations

$$\frac{d}{dt}\int_{-D}^{D} \frac{1}{2}u_x^2 dx + \varepsilon \int_{-D}^{D} u_{xx}^2 dx = 3ar \int_{-D}^{D} u^2 u_{xxxx} dx$$
$$- 2br \int_{-D}^{D} u\,(vv_{xxx} + 3vv_x)\,dx \tag{1*}$$

$$\frac{d}{dt}\int_{-D}^{D} \frac{1}{2}v_x^2 dx + \varepsilon \int_{-D}^{D} v_{xx}^2 dx = -cr \int_{-D}^{D} uv_x v_{xx} dx \tag{2*}$$

$$\frac{d}{dt}\int_{-D}^{D} uv^2 dx = -\varepsilon \int_{-D}^{D} u_{xxxx} v^2 dx - 2\varepsilon \int_{-D}^{D} uvv_{xxxx} dx$$
$$-2(1+a)\int_{-D}^{D} uvv_{xxx} dx - 6a \int_{-D}^{D} uv_x v_{xx} dx$$
$$+ 6ar \int_{-D}^{D} uu_x v^2 dx$$
$$- 2cr \int_{-D}^{D} u^2 vv_x dx, \tag{3*}$$

$$\frac{d}{dt}\int_{-D}^{D} u^3 dx = -3\varepsilon \int_{-D}^{D} u^2 u_{xxxx} dx + 3a \int_{-D}^{D} u^2 u_{xxx} dx$$
$$+ 6br \int_{-D}^{D} u^2 vv_x dx. \tag{4*}$$

Making a linear combination of the four relation formulas to eliminate three terms $\int_{-D}^{D} u^2 u_{xxx} dx$, $\int_{-D}^{D} uvv_{xxx} dx$, $\int_{-D}^{D} uv_x v_{xx} dx$, we obtain

$$\frac{d}{dt}\int_{-D}^{D}\left(\frac{a+1}{2}u_x^2+\frac{3b}{c}v_x^2-bruv^2-(a+1)ru^3\right)dx=-(a+1)\varepsilon\int_{-D}^{D}u_{xxx}^2dx$$

$$-6c\frac{1}{b}\varepsilon\int_{-D}^{D}v_{xxx}^2dx+br\varepsilon\int_{-D}^{D}u_{xxxx}v^2dx+2br\varepsilon\int_{-D}^{D}uvv_{xxxx}dx$$

$$+3(a+1)r\varepsilon\int_{-D}^{D}u^2u_{xxxx}dx-6abr^2\int_{-D}^{D}uu_xv^2dx$$

$$+2br^2(c-3a-3)\int_{-D}^{D}u^2vv_xdx \tag{2. 5}$$

For the terms which contain ε, using the interpolation formulas we obtain

$$\left|br\int_{-D}^{D}u_{xxxx}v^2dx\right|\leq\frac{1}{6}\left((a+1)\int_{-D}^{D}u_{xxx}^2dx+6\frac{b}{c}\int_{-D}^{D}v_{xxxx}^2dx\right)+C_1$$

$$\left|2br\int_{-D}^{D}uvv_{xxxx}dx\right|\leq\frac{1}{6}\left((a+1)\int_{-D}^{D}u_{xxx}^2dx+6\frac{b}{c}\int_{-D}^{D}v_{xxx}^2dx\right)+C_1$$

$$\left|3(a+1)r\int_{-D}^{D}u^2u_{xxxx}dx\right|\leq\frac{1}{6}\left((a+1)\int_{-D}^{D}u_{xxx}^2dx+6\frac{b}{c}\int_{-D}^{D}v_{xxx}^2dx\right)+C_1$$

For the last two terms in the equality (2. 5), we have

$$\int_{-D}^{D}u^2vv_xdx\leq C_1\int_{-D}^{D}U_x^2dx,\qquad\int_{-D}^{D}uu_xv^2dx\leq C_1\int_{-D}^{D}U_x^2dx$$

By all the above estimates, the formula (2. 5) can be turned into

$$\frac{d}{dt}\int_{-D}^{D}\left(\frac{a+1}{2}u_x^2+\frac{3b}{c}v_x^2-bruv^2-(a+1)ru^3\right)dx$$

$$\leq C_1\int_{-D}^{D}U_x^2dx+C_1$$

where C_1 is a constant independent of r and ε. Integrating the inequality over $[0, T]$ with respect to t we have

$$\int_{-D}^{D}\left(\frac{a+1}{2}u_x^2+\frac{3b}{c}v_x^2-bruv^2-(a+1)ru^3\right)dx\leq C_1\int_{-D}^{D}U_x^2dx+C_1$$

using the interpolation formula and Gronwall inequality to calculate, we obtain

$$\| U_x \|_{L_\infty(0,T:L_2(-D,D))}\leq K_1$$

where K_1 is a constant depending only on a, b, c and initial value U_0. This shows that the solutions to the problem (2. 4), (1. 2) is uniformly bounded in B for $0\leq r\leq 1$. Therefore the periodic initial-value problem (1. 4), (1. 2) has at least one solution in B and hence in Z.

The lemma is proved.

Corollary *Let $U_0(x)$ belong to $H^{2(2h+1)+k}(-D, D)$ for $k\geq 0$ and $h\geq 0$. Under the conditions in Lemma 2, for the solutions $U(x, t)$ to the periodic initial-value problem (1. 4), (1.*

2), there are $D_x^k D_t^h U(x,t) \in Z$.

3. The a Priori Estimates

In order to obtain a solution to the problem (1. 4), (1. 2) by taking the limit of solutions $U_\varepsilon(x,t)$ with ε tending to zero, it is necessary to get the estimates for the norms of solutions to the problem (1. 4), (1. 2) and for the estimates to be independent of ε.

As a consequence of the proof of Lemma 2, we have the following lemma:

Lemma 3 *Under the conditions in Lemma 2, the generalized solutions U_ε to the periodic initial-value problem* (1. 4), (1. 2) *admit the uniform estimates*

$$\| U_\varepsilon \|_{L_\infty(0,T;H^1(-D,D))} + \sqrt{\varepsilon}\, \| U_{\varepsilon xx} \|_{L_2(Q_T)} \leq K_1$$
$$\| U_\varepsilon \|_{L_\infty(Q_T)} \leq K_1$$

where K_1 is a constant independent of ε, D and T.

Lemma 4 *Let $-1 < a < 0$, $bc > 0$, $U_0 \in H^2(-D,D)$ be periodic with period $2D$. For the solutions U_ε to the problem* (1. 4), (1. 2) *there is the following inequality*

$$\| U_{\varepsilon xx} \|_{L_\infty(0,T;L_2(-D,D))} + \sqrt{\varepsilon}\, \| U_{\varepsilon xxxx} \|_{L_2(Q_T)} \leq K_2$$

where K_2 is a constant, independent of ε, D and T.

Proof Using system (1. 4), by lengthy computation we obtain

$$\frac{d}{dt}\int_{-D}^{D} \frac{1}{2} u_{xx}^2 dx + \varepsilon \int_{-D}^{D} u_{xxxx}^2 dx = 15a \int_{-D}^{D} u_x u_{xx}^2 dx$$
$$+ 2b \int_{-D}^{D} u\,(v v_{x^5} + 5 v_x v_{xxxx} + 10 v_{xx} v_{xxx})\, dx \qquad (A_1)$$

$$\frac{d}{dt}\int_{-D}^{D} \frac{1}{2} v_{xx}^2 dx + \varepsilon \int_{-D}^{D} v_{xxxx}^2 dx = -C \int_{-D}^{D} u v_x v_{xxxx} dx \qquad (A_2)$$

$$\frac{d}{dt}\int_{-D}^{D} u v_x^2 dx = -\varepsilon \int_{-D}^{D} u_{xxxx} v_x^2 dx - 2\varepsilon \int_{-D}^{D} u v_x v_{xxxxx} dx$$
$$-2(a+1) \int_{-D}^{D} u v_x v_{xxxx} dx - 6a \int_{-D}^{D} u v_{xx} v_{xxx} dx + 6a \int_{-D}^{D} u u_x v_x^2 dx$$
$$+ 2b \int_{-D}^{D} v v_x^3 dx \qquad (A_3)$$

$$\frac{d}{dt}\int_{-D}^{D} u_{xx} v^2 dx = -\varepsilon \int_{-D}^{D} v^2 v_{x^6} dx - 2\varepsilon \int_{-D}^{D} v u_{xx} v_{xxxx} dx$$
$$-2 \int_{-D}^{D} u\,((a+1) v v_{xxxxx} + (5a+2) v_x v_{xxxx} + (10a+1) v_{xx} v_{xxx})\, dx$$
$$+ 6a \int_{-D}^{D} v^2 (u u_x)_{xx} dx - 2c \int_{-D}^{D} u u_{xx} v v_x dx \qquad (A_4)$$

$$\frac{d}{dt}\int_{-D}^{D} u u_x^2 dx = -\varepsilon \int_{-D}^{D} u_{xxxx} u_x^2 dx - 2\varepsilon \int_{-D}^{D} u u_x u_{xxxxx} dx$$

$$+3a\int_{-D}^{D}u_x u_{xx}^2dx+2b\int_{-D}^{D}u_x u_{xx}^2dx+2b\int_{-D}^{D}u_x^2 v v_x dx$$

$$+4b\int_{-D}^{D}uu_x(vv_x)_x dx \qquad (A_5)$$

We make a linear combination of the five formulas to eliminate

$$\int_{-D}^{D}uvv_{xxxxx}dx,\ \int_{-D}^{D}uv_x v_{xxxx}dx\ \int_{-D}^{D}uv_{xx}v_{xxx}dx \text{ and } \int_{-D}^{D}u_x u_{xx}^2dx,$$

namely, multiply the formulas A_1, A_2, A_3, A_4 and A_5 by $a+1$, $-6b/ac$, $3b/a$, b and $-5(a+1)$ respectively and sum up the products, then obtain

$$\frac{d}{dt}\int_{-D}^{D}\left(\frac{a+1}{2}u_{xx}^2-\frac{3b}{ac}v_{xx}^2+bu_{xx}v^2-5(a+1)uu_x^2\right)dx=$$

$$-(a+1)\varepsilon\int_{-D}^{D}u_{xxxx}^2dx+\frac{6b}{ac}\varepsilon\int_{-D}^{D}v_{xxxx}^2dx-\frac{3b}{a}\varepsilon\int_{-D}^{D}v_x^2u_{xxxx}dx$$

$$-\frac{6b}{a}\varepsilon\int_{-D}^{D}uv_xv_{xxxxx}dx-b\varepsilon\int_{-D}^{D}u_{x5}v^2dx-2b\varepsilon\int_{-D}^{D}u_{xx}vv_{xxxx}dx$$

$$+5(a+1)\varepsilon\int_{-D}^{D}u_{xxxx}u_x^2dx+10(a+1)\varepsilon\int_{-D}^{D}uu_xu_{x5}dx$$

$$+18b\int_{-D}^{D}uu_xv_x^2dx+\frac{6b^2}{a}\int_{-D}^{D}vv_x^3dx+3ab\int_{-D}^{D}v^2(v^2)_{xxx}dx$$

$$-2cb\int_{-D}^{D}uvv_xu_{xxx}dx-30(a+1)\int_{-D}^{D}uu_x^3dx-10(a+1)b\int_{-D}^{D}vv_xu_x^2dx$$

$$-10(a+1)b\int_{-D}^{D}uu_x(v^2)_{xx}dx \qquad (3.\ 1)$$

Using the interpolating inequality, we have

$$-\frac{6b}{a}\int_{-D}^{D}uv_xv_{xxxxx}dx\le\frac{1}{14}\Big((a+1)\int_{-D}^{D}u_{xxxx}^2dx-\frac{6b}{ac}\int_{-D}^{D}v_{xxxx}^2dx\Big)+C_1$$

$$-b\int_{-D}^{D}u_{x5}v^2dx\le\frac{1}{14}\left((a+1)\int_{-D}^{D}u_{xxxx}^2dx-\frac{6b}{ac}\int_{-D}^{D}v_{xxxx}^2dx\right)+C_1$$

$$-2b\int_{-D}^{D}u_{xx}vv_{xxxx}dx\le\frac{1}{14}\left((a+1)\int_{-D}^{D}u_{xxxx}^2dx-\frac{6b}{ac}\int_{-D}^{D}v_{xxxx}dx\right)+C_1$$

$$5(a+1)\int_{-D}^{D}u_x^2u_{xxxx}dx\le\frac{1}{14}\left((a+1)\int_{-D}^{D}u_{xxxx}^2dx-\frac{6b}{ac}\int_{-D}^{D}v_{xxxx}^2dx\right)+C_1$$

$$10(a+1)\int_{-D}^{D}uu_xu_{x5}dx\le\frac{1}{14}\left((a+1)\int_{-D}^{D}u_{xxxx}^2dx-\frac{6b}{ac}\int_{-D}^{D}v_{xxxx}^2dx\right)+C_1$$

$$\frac{6b^2}{a}\int_{-D}^{D}vv_x^3dx\le C_1\left(\int_{-D}^{D}v_{xx}^2dx+1\right)$$

$$3ab\int_{-D}^{D}v^2(u^2)_{xxx}dx\le C_1\left(\int_{-D}^{D}U_{xx}^2dx+1\right)-2cb\int_{-D}^{D}uvv_xu_{xx}dx$$

$$\leq C_1\left(\int_{-D}^{D} U_{xx}^2 dx + 1\right)$$

$$30(a+1)\int_{-D}^{D} u u_x^3 dx \leq C_1\left(\int_{-D}^{D} u_{xx}^2 dx + 1\right)$$

$$10(a+1)\int_{-D}^{D} v v_x u_x^2 dx \leq C_1\left(\int_{-D}^{D} u_{xx}^2 dx + 1\right)$$

$$10(a+1)\int_{-D}^{D} u u_x (v^2)_{xx} dx \leq C_1\left(\int_{-D}^{D} U_{xx}^2 dx + 1\right)$$

where C_1 is a constant depending only on a, b, c and U_0. From all the above inequalities we can simplify formula (3. 1) to

$$\frac{d}{dt}\int_{-D}^{D}\left(\frac{a+1}{2}u_{xx}^2 - \frac{3b}{ac}v_{xx}^2 + \frac{3b}{a}uv_x^2 + bu_{xx}v^2 - 5(a+1)uu_x^3\right)dx$$

$$+\varepsilon\left(\frac{a+1}{2}\int_{-D}^{D} u_{xxxx}^2 dx - \frac{3b}{ac}\int_{-D}^{D} v_{xxxx}^2 dx\right) \leq C_1\left(\int_{-D}^{D} U_{xx}^2 dx + 1\right)$$

Integrating the last inequality with respect to t over domain $[0, T]$ and then using the Gronwall inequality, we get

$$\| U_{xx} \|_{L_\infty(0,T;L_2(-D,D))} + \sqrt{\varepsilon}\, \| U_{xxxx} \|_{L_2(Q_T)} \leq K_2$$

where K_2 is a constant depending on a, b, c and $\| U_0 \|_{H^2(-D,D)}$.

Corollary *Under the conditions of Lemma 4, the solutions U_ε to the problem (1. 4), (1. 2) satisfy*

$$\| U_{\varepsilon x} \|_{L_\infty(Q_T)} \leq K_2^*$$

where K_2^* is a constant independent of ε, D and T.

Lemma 5 *If $-1 < a < 0$, $bc > 0$, $U_0 \in H^k(-D, D)$ $(k \geq 3)$ is periodic with period $2D$, then there are inequalities*

$$\| U_{x^k} \|_{L_\infty(0,T;L_2(-D,D))} + \| U_{x^{k+2}} \|_{L_2(Q_T)} \leq C_k \quad (k = 3,\ 4,\ \dots) \tag{3. 2}$$

where C_k are constants independent of ε and D.

Proof We show it by induction axioms. From Lemma 3 and 4 the inequality (3. 2) holds as $k \leq 2$. Suppose that inequality (3. 2) holds as $k \leq n-1$ then we prove the conclusion as $k = n$. From system (1. 4) we obtain

$$\frac{d}{dt}\int_{-D}^{D}\frac{1}{2}u_{x^n}^2 dx = -\varepsilon\int_{-D}^{D} u_{x^{n+2}}^2 dx + 6a\int_{-D}^{D} u u_{x^n} u_{x^{n+1}} dx$$

$$+ 2b\int_{-D}^{D} u_{x^n} v v_{x^{n+1}} dx + \sum_{i=0}^{n}\int_{-D}^{D} u_{x^n}\left(q_i^1 u_{x^i} u_{x^{n-i}} + q_i^2 v_{x^i} v_{x^{n-i}}\right) dx \tag{B_1}$$

$$\frac{d}{dt}\int_{-D}^{D}\frac{1}{2}v_{x^n}^2 dx = -\varepsilon\int_{-D}^{D} v_{x^{n+2}}^2 dx - c\int_{-D}^{D} u v_{x^n} v_{x^{n+1}} dx$$

$$+ \sum_{i=1}^{n} q_i^3 \int_{-D}^{D} u_{x^i} v_{x^n} v_{x^{n+1-i}} dx \tag{B_2}$$

$$\frac{d}{dt}\int_{-D}^{D}u_{x^{n-1}}(v^2)_{x^{n-1}}dx=\varepsilon\int_{-D}^{D}u_{x^{n+2}}(v^2)_{x^n}dx$$

$$+2\varepsilon\int_{-D}^{D}u_{x^n}(vv_{x^4})_{x^{n-2}}dx+2(a+1)\int_{-D}^{D}u_{x^n}vv_{x^{n+1}}dx$$

$$+\sum_{i=0}^{n}q_i^4\int_{-D}^{D}u_{x^n}v_{x^i}v_{x^{n-i}}dx+3a\int_{-D}^{D}(u^2)_{x^n}(v^2)_{x^{n-1}}dx$$

$$+2c\int_{-D}^{D}u_{x^n}(uvv_x)_{x^{n-2}}dx \qquad (B_2)$$

where q_i^j ($j=1, 2, 3, 4; i=0, 1, 2, \ldots, n$) are constants depending on a, b, c and n only. Combining the three formulas to eliminate the term $\int_{-D}^{D}u_{x^n}vv_{x^{n+1}}dx$ and calculating the combination equality by the interpolation formulas, we obtain

$$\frac{d}{dt}\int_{-D}^{D}((a+1)u_{x^n}^2+v_{x^n}^2-2bu_{x^{n-1}}(v^2)_{x^{n-1}})\,dx$$

$$+(a+1)\varepsilon\int_{-D}^{D}u_{x^{n+2}}^2dx+\varepsilon\int_{-D}^{D}v_{x^{n+2}}^2dx\leq C_1\left(\int_{-D}^{D}U_{x^n}^2dx+1\right)$$

From the inequality we obtain the inequality (3. 2) as $k=n$. By induction the lemma is true.

Corollary *Let $-1<a<0$, $bc>0$, $U_0(x)\in H^{n+4}(-D, D)$ be periodic with period $2D$. Then we have*

$$\| U_{tx^n} \|_{L_\infty(0,T;L_2(-D,D))}\leq K_n$$

where n is a nonnegative integer, K_n are constants, independent of ε and D.

4. Solutions to the Problem (1. 1), (1. 2)

Definition *A two-dimensional vector function $U(x, t)\in L_2(0, t; H^1(-D, D))$ is called a weak solution to the periodic initial-value problem (1. 1), (1. 2), if for any test function $W(x, t)=(w_1(x, t), W_2(x, t))\in W_2^{(3,1)}$, there are*

$$\int_0^T\int_{-D}^{D}(w_{1t}u+aw_{1xx}u_x+6aw_1uu_x+2bw_1vv_x)\,dxdt$$

$$+\int_{-D}^{D}w_1(x, 0)\,u_0(x)\,dx=0 \qquad (4. 1)$$

$$\int_0^T\int_{-D}^{D}(w_{2t}v-w_{2xx}v_x-cw_2uv_x)\,dxdt$$

$$+\int_{-D}^{D}w_2(x, 0)\,v_0(x)\,dx=0 \qquad (4. 2)$$

where $W(x, t)$ is periodic with respect to x with period $2D$ and $W(X, T)=0$.

Lemma 7 *If the conditions in Lemma 2 hold, then the solutions $U_\varepsilon(x, t)$ to the problem (1.*

4), (1. 2) *satisfy*

$$\sup_{0\le t\le T}\|U_{\varepsilon t}(\cdot,\ t)\|_{H^{-3}(-D,\ D)}\le K_1 \tag{4. 3}$$

where the constant K_1 *is independent of* ε, *D and T.*

Proof Let $U(x)$ belong to space $H^3(-D,\ D)$, $V(x)=(v_1(x),\ v_2(x))$, then

$$\left|\int_{-D}^{D}v_1u_t dx\right|\le\varepsilon\left|\int_{-D}^{D}v_{1xxx}u_x dx\right|+|a\int_{-D}^{D}v_{1xxx}u_x dx|$$

$$+|2b\int_{-D}^{D}v_1vv_x dx|+|6a\int_{-D}^{D}v_1uu_x dx|\le C_1\|U\|_{L_\infty(0,\ T;\ H^1(-D,\ D))}\cdot$$

$$\cdot\|v_1\|_{H^3(-D,\ D)}\leqslant C_1\|v_1\|_{H^3(-D,\ D)}$$

where the constants appearing here are independent of ε, T and D. With the same computation, there is

$$\left|\int_{-D}^{D}v_2u_t dx\right|\le C_1\|v_2\|_{H^3(-D,\ D)}$$

Hence estimate (4. 3) holds.

Lemma 8 *Under the conditions in Lemma 7, there are*

$$\|U(\cdot,\ t+\Delta t)-U(\cdot,\ t)\|_{L_\infty(-D,\ D)}\le K_1\Delta t^{1/8}$$

$$\|U(x+\Delta x,\ \cdot)-u(x,\ \cdot)\|_{L_\infty(0,\ T)}\le K_2\Delta x^{1/2}$$

where the constants K_1, K_2 *are independent of* ε *and D.*

Proof We have

$$\|u(\cdot,\ t+\Delta t)-u(\cdot,\ t)\|_{L_\infty(-D,\ D)}$$

$$\le C_1\|u(\cdot,\ t+\Delta t)-u(\cdot,\ t)\|^{7/8}_{H^1(-D,\ D)}\|u(\cdot,\ t+\Delta t)-u(\cdot,\ t)\|^{1/8}_{H^{-3}(-D,\ D)}$$

$$\le C_1\|u\|_{L_\infty(0,\ T;\ H^1(-D,\ D))}\|u_t\|^{1/8}_{L_\infty(0,\ T;\ H^{-3}(-D,\ D))}\Delta t^{1/8}$$

$$\le K_1\Delta t^{1/8},$$

$$\|u(x+\Delta x,\ \cdot)-u(x,\ \cdot)\|_{L_\infty(0,\ T)}\le C_1\|u\|^{1/2}_{L_\infty(0,\ T;\ H^1(-D,\ D))}$$

$$\sup_{0\le t\le T}\left(\int_{-D}^{D}(u(x+\Delta x,\ t)-u(x,\ t))^2dx\right)^{1/4}$$

$$\le C_1\Delta x^{1/2}\|u\|_{L_\infty(0,\ T;\ H^1(-D,\ D))}\le K_2\Delta x^{1/2}$$

where K_1 and K_2 are independent of ε, D and T. For function v there are the same estimates. So we have got the lemma.

Theorem 1 *Let it be supposed that the conditions in Lemma 2 are true. Then the periodic initial-value problem* (1. 1), (1. 2) *has at least one global weak solution* $U(x,\ t)$ *and*

$$U(x,\ t)\in L_\infty(0,\ T;\ H^1(-D,\ D))\cap C^{(\frac{1}{2},\frac{1}{8})}(Q_T)$$

Proof The set of two-dimensional vector valued solutions $U_\varepsilon(x,\ t)$ $(\varepsilon>0)$ to the problem (1. 4), (1. 2) is uniformly bounded in the functional space $L_\infty(0,\ T;\ H^1(-D,\ D))$ or $C^{(\frac{1}{2},\frac{1}{8})}(Q_T)$. Hence we can select a subsequence $\{U_{\varepsilon_i}(x,\ t)\}$ from $\{U_\varepsilon(x,\ t)\}$ and exists $U(x,\ t)$, such that as i tends to infinite, $\varepsilon_i\to0$, the subsequence $\{U_{\varepsilon_i}(x,\ t)\}$

converges to $U(x,t)$ uniformly in $C^{(\frac{1}{2}-\delta,\frac{1}{8}-\delta)}$ for $\delta>0$ and converges weakly to $U(x,t)$ in $L_q(0,T;H^1(-D,D))$ for $1<q<\infty$. Moreover for any $t\in[0,T]$, $\{U_{\varepsilon_i}(x,t)\}$ converges uniformly to $U(x,t)$ with respect to x. Hence

$$U(x,t)\in L_\infty(0,T;H^1(-D,D))\cap C^{(\frac{1}{2},\frac{1}{8})}(Q_T)$$

From Lemma 2 there are

$$\int_0^T\int_{-D}^{D}(w_{1t}u_{\varepsilon_i}+\varepsilon_i w_{1xxx}u_{\varepsilon_i x}+aw_{1xx}u_{\varepsilon_i x}+6aw_1u_{\varepsilon_i}u_{\varepsilon_i x}$$

$$+2bw_1v_{\varepsilon_i}v_{\varepsilon_i x})\,dxdt+\int_{-D}^{D}w_1(x,0)\,u_0(x)\,dx=0$$

$$\int_0^T\int_{-D}^{D}(w_{2t}v_{\varepsilon_i}+\varepsilon_i w_{2xxx}v_{\varepsilon_i x}-w_{2xx}v_{\varepsilon_i x}-cu_{\varepsilon_i}v_{\varepsilon_i x}w_2)\,dxdt$$

$$+\int_{-D}^{D}w_2(x,0)\,v_0(x)\,dx=0$$

where $W(x,t)=(w_1(x,t),w_2(x,t))$, $W\in W_2^{(3,1)}(Q_T)$ is periodic with respect to x with period $2D$. Let i tend to infinite, the limit of the two integral relations is the integral equalities (4.1), (4.2). This shows that $U(x,t)$ is a weak solution of the periodic initial-value problem (1.1), (1.2).

Such proof as is used in Lemma 7 gives

Lemma 9 *Let $-1<a<0$, $bc>0$, $U_0(x)\in H^3(-D,D)$ be periodic with period $2D$. Then for the solutions U_ε to the periodic initial-value problem (1.4), (1.2) there stands*

$$\|U_{\varepsilon t}\|_{L_\infty(0,T;H^{-1}(-D,D))}\le K_0$$

where K_0 is a constant independent of ε and D.

Theorem 2 *If the conditions in Lemma 9 hold, then the periodic initial-value problem (1.1), (1.2) has a unique generalized global solution $U(x,t)$ and*

$$U\in L_\infty(0,T;H^3(-D,D))\cap W_\infty^{(1)}(0,T;L^2(-D,D))$$

Hence U, U_x and U_{xx} are Hölder continuous in Q_T.

Proof The set of the generalized solutions $U_\varepsilon(\varepsilon>0)$ is uniformly bounded in $L_\infty(0,T;H^3(-D,D))$ or $W^1_\infty(0,T;H^{-1}(-D,D))$. We have

$$\sup_{-D\le x\le D}|U_{x^k}(x,t_2)-U_{x^k}(x,t_1)|\le C_1\|U(\cdot,t_2)-U((\cdot,t_1)\|^{\frac{5}{8}-\frac{k}{4}}_{H^1(-D,D)}$$

$$\|U(\cdot,t_2)-U(\cdot,t_1)\|^{\frac{3}{8}+\frac{k}{4}}_{H^3(-D,D)}\le C_1|t_2-t_1|^{\frac{5}{8}-\frac{k}{4}},\quad(k=0,1,2)$$

so U_ε, $U_{\varepsilon x}$ and $U_{\varepsilon xx}$ are Hölder continuous in domain Q_T. Hence we can select a subsequence $\{U_{\varepsilon_i}(x,t)\}$ from $\{U_\varepsilon(x,t)\}$ and there exists a two-dimensional vector valued function $U(x,t)$ such that as $i\to\infty$, $\varepsilon_i\to0$, U_{ε_i}, $U_{\varepsilon_i x}$ and $U_{\varepsilon_i xx}$ are uniformly convergent to U, U_x and U_{xx} respectively in Q_T. $U_{\varepsilon_i t}$ and $U_{\varepsilon_i xxx}$ are weakly convergent to U_t and U_{xxx} respectively in space $L_q(0, T; H^2(-D, D))$ for $1 < q < \infty$. Hence $U(x, t) \in L_\infty(0, T; H^3(-D, D))$. Let $W(x, t) = (w_1(x, t), w_2(x, t))$ be periodic with period $2D$ with respect to x. For the solutions U_{ε_i}, we have

$$\int_0^t \int_{-D}^D (u_{\varepsilon_i t} + \varepsilon_i u_{\varepsilon_i xxxx} - a(u_{\varepsilon_i xxx} + 6u_{\varepsilon_i} u_{\varepsilon_i x}) - 2bv_{\varepsilon_i} v_{\varepsilon_i x}) w_1 dxdt$$

$$+ \int_0^t \int_{-D}^D (v_{\varepsilon_i t} + \varepsilon_i v_{\varepsilon_i xxxx} + v_{\varepsilon_i xxx} + cu_{\varepsilon_i} v_{\varepsilon_i x}) w_2 dxdt = 0 \tag{4. 4}$$

From Lemma 5, $\sqrt{\varepsilon_i} \| U_{\varepsilon_i xxxx} \|_{L_2(Q_T)}$ is uniformly bounded with respect to ε. Let $i \to \infty$. From integral inequality (4. 4) we obtain

$$\int_0^t \int_{-D}^D ((u_t - a(u_{xxx} + 6uu_x) - 2bvv_x) w_1 +$$

$$+ (v_t + v_{xxx} + cuv_x) w_2) dxdt = 0 \tag{4. 5}$$

This means that $U(x, t)$ is a generalized global solution to the problem (1. 2), (1. 1).

Let it be supposed that the problem (1. 1), (1. 2) has two solutions $U(x, t)$ and $\bar{U}(x, t)$. Denote $Q = U - \bar{U}$, $q_1 = u - \bar{u}$, $q_2 = v - \bar{v}$. Then Q satisfies

$$\int_0^t \int_{-D}^D (q_{1t} - aq_{1xx} - 6a(\bar{u}_x q_1 + uq_{1x}) - 2b(v_x q_2 + \bar{v} q_{2x})) w_1 dxdt + \int_0^t \int_{-D}^D (q_{2t} + q_{2xxx}$$

$$+ c(v_x q_1 + \bar{u} q_{2x})) w_2 dxdt = 0 \tag{4. 6}$$

Taking W to be (q_1, q_2), $(0, q_{2xx})$, $(q_{1xx}, 0)$ and $(\bar{v} q_2, \bar{v} q_1)$, we obtain

$$\| Q(\cdot, t) \|^2_{L_2(-D, D)} \le C_1 \| Q \|^2_{L_2(Q_t)} + C_2 \| q_{2x} \|^2_{L_2(Q_t)} \tag{4. 7}$$

$$\| q_{2x}(\cdot, t) \|^2_{L_2(-D, D)} \le C_3 \| Q \|^2_{L_2(0, t; H^1(-D, D))} \tag{4. 8}$$

$$\| q_{1x}(\cdot, t) \|^2_{L_2(-D, D)} \le C_4 \| Q \|_{L_2(0, t; H^1(-D, D))}$$

$$+ 2b \int_0^t \int_{-D}^D \bar{v} q_{1x} q_{2xx} dxdt \tag{4. 9}$$

$$\int_{-D}^D \bar{v} q_1 q_2 dx = \int_0^t \int_{-D}^D \bar{v}_t q_1 q_2 dxdt + (a+1) \int_0^t \int_{-D}^D \bar{v} q_{1x} q_{2xx} dxdt$$

$$+ \int_0^t \int_{-D}^D (((2a-1)\bar{v}_x + 6a\bar{u}\bar{v}) q_{1x} q_{2x} + (6a\bar{v}u_x - \bar{v}_{xx} - c\overline{uv}) q_1 q_{2x}$$

$$+ a\bar{v}_{xx} q_2 q_{1x} - c\bar{v} v_x q_1^2 + v^2 q_{2x}^2 + 2b\bar{v} v_x q_2 q_{2x}) dxdt \tag{4. 10}$$

respectively, where the constants appearing are independent of ε and D. We make the linear combination $-\frac{2b}{a+1}(4.10) + (4.8) + (4.9)$ and obtain

$$\| Q_x(\cdot, t) \|^2_{L_2(-D, D)} - \frac{2b}{a+1} \int_{-D}^D \bar{v} q_1 q_2 dx \le C_5 \| Q \|^2_{L_2(0, t; H^1(-D, D))}$$

$$+ C_6 \left| \int_0^t \int_{-D}^D \bar{v}_t q_1 q_2 dxdt \right| \tag{4. 11}$$

From inequality (4. 6), we also have

$$\int_0^t\int_{-D}^{D}\bar{v}_t q_1 q_2 dxdt = -\int_0^t\int_{-D}^{D}(\bar{v}_{xxx}+c\bar{u}\bar{v}_x)q_1q_2dxdt \tag{4.12}$$

From formulas (4.11), (4.12), we get the following relation

$$\begin{aligned}\|Q_x(\cdot,t)\|^2_{L_2(-D,D)} &\leq C_7\|Q(\cdot,t)\|^2_{L_2(-D,D)}\\ &+C_8\|Q\|^2_{L_2(0,t;H^1(-D,D))}\end{aligned} \tag{4.13}$$

Taking the sum of (4.13) and (C_7+1) (4.7) we get

$$\int_{-D}^{D}(Q^2(x,t)+Q_x^2(x,t))dx \leq C_9\int_0^t\int_{-D}^{D}(Q^2+Q_x^2)dxdt$$

from which we obtain

$$\int_{-D}^{D}(Q^2+Q_x^2)dx=0$$

Hence the periodic initial-value problem (1.1), (1.2) has only one solution.

Therefore the theorem has been proved.

Theorem 3 *If the conditions in Theorem 2 hold, then as ε tends to zero, there are*

$$\|U-U_\varepsilon\|_{L_\infty(Q_T)}=O(\varepsilon^{\frac{1}{2}})$$

$$\|(U-U_\varepsilon)_{x^k}\|_{L_\infty(Q_T)}=O(\varepsilon^{\frac{1}{2}-\frac{k-\frac{1}{2}}{4}}),\quad (k=1,2)$$

Proof Let $Z_1=u_\varepsilon-u$, $Z_2=v_\varepsilon-v$, $Z=(Z_1,Z_2)$. From integral relation (4.5) and system (1.1) we have

$$\begin{aligned}\int_0^t\int_{-D}^{D}&((Z_{1t}+\varepsilon u_{\varepsilon xxxx}-aZ_{1xxx}-6a(u_{\varepsilon x}Z_1+uZ_{1x})\\ &-2b(v_{\varepsilon x}Z_2+vZ_{2x}))w_1+(Z_{2t}+\varepsilon v_{\varepsilon xxxx}+Z_{2xxx}\\ &+c(Z_1v_{\varepsilon x}+uZ_{2x}))w_2)dxdt=0\end{aligned}$$

A lengthy computation analogous to the proof of the uniqueness in Theorem 2 provides the following inequality

$$\begin{aligned}\|Z(\cdot,t)\|^2_{H^1(-D,D)} &\leq C_1\|Z\|^2_{L_2(0,t;H^1(-D,D))}\\ &+C_1\varepsilon^2(\|U_{x^6}\|^2_{L_2(Q_t)}+\|U^2_{x^4}\|_{L_2(Q_t)})\end{aligned}$$

where C_1 is a constant, independent of ε, D. From Lemma 5,

$$\varepsilon\|U_{x^6}\|^2_{L_2(Q_t)}+\varepsilon\|U_{\varepsilon xxxx}\|^2_{L_2(Q_t)}$$

is uniformly bounded with respect to ε. Hence there are

$$\|Z\|_{L_\infty(0,T;H^1(-D,D))}\leq C_1 \quad\text{and}\quad \|Z\|_{L_\infty(Q_T)}=O(\varepsilon^{\frac{1}{2}})$$

Then using the interpolation formula, we have

$$\|Z_{x^k}(\cdot,t)\|_{L_\infty(-D,D)}\leq C_1\|Z(\cdot,t)\|^{1-(k-\frac{1}{2})/2}_{H^1(-D,D)}\|Z(\cdot,t)\|^{(k-\frac{1}{2})/2}_{H^3(-D,D)}$$

Therefore

$$\|Z_{x^k}\|_{L_\infty(Q_T)}=O(\varepsilon^{\frac{1}{2}-\frac{k-\frac{1}{2}}{4}}),\quad (k=1,2)$$

Also the classical solution to the periodic initial-value problem (1. 2), (1. 1) is obtained by the limiting process of the solutions to the perturbed problem (1. 4), (1. 2) as $\varepsilon \to 0$.

Theorem 4 *If the conditions in Theorem 2 hold, $U_0(x) \in H^k(-D, D)$ ($k = 4, 5, 6, \ldots$) is $2D$-periodic, then the periodic initial-value problem (1. 1), (1. 2) has one and only one classical global solution $U(x, t)$ and*

$$U(x, t) \in L_\infty(0, T; H^k(-D, D)) \cap W_\infty^{(1)}(0, T; H^{k-3}(-D, D))$$

5. Solutions to the Initial-value Problem

The all priori estimates stated above of the generalized solutions to the perturbed system (1. 4) under the periodic initial-value conditions (1. 2) is independent of not only ε but also D. Considering the usual approach of the solutions U_ε to the perturbed periodic initial value problem (1. 4), (1. 2) as D converges to infinite as given in [4, 5] etc. we can get the following results.

Theorem 5 *Let $-1 < a < 0$, $bc > 0$, $U_0(x)$ belong to space $H^3(R)$. Then the initial value problem (1. 1), (1. 3) has a unique generalized global solution $U(x, t)$ and*

$$U(x, t) \in L_\infty(0, T; H^3(R)) \cap W_\infty^{(1)}(0, T; L_2(R))$$

Theorem 6 *Under the conditions in Theorem 5, for the solutions U_ε to the perturbed initial value problem (1. 4), (1. 2) and the solution U to the initial value problem (1. 1), (1. 3), we have*

$$\| U - U_\varepsilon \|_{L_\infty(Q_T^*)} = O(\varepsilon^{\frac{1}{2}})$$

$$\| (U - U_\varepsilon)_{x^k} \|_{L_\infty(Q_T^*)} = O(\varepsilon^{\frac{1}{2} - \frac{k - \frac{1}{2}}{4}}), \quad (k = 1, 2)$$

Theorem 7 *Let $a + 1 > 0$, $bc > 0$, $U_0(x) \in H^2(R)$. For the initial value problem (1. 1), (1. 3), there exists at least one weak solution.*

Theorem 8 *Suppose that system (1. 1) satisfies $-1 < a < 0$, $bc > 0$, $U_0(x) \in H^n(R)$ ($n \geq 4$). Then the problem (1. 1), (1. 3) has one and only one classical solution $U(x, t)$ and*

$$U(x, t) \in L_\infty(0, T; H^n(R)) \cap W_\infty^{(1)}(0, T; H^{(n-3)}(R))$$

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