J. Partial Differential Equations
Vol. 2, No. 1 (1989), 9—15

VACUUM STATES AND EQUIDISTRIBUTION OF THE RANDOM SEQUENCE FOR GLIMM'S SCHEME (CONTINUATION) ①

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(Received April 20, 1987; revised October 29, 1987)

Abstract

This is a continuation of paper [1]. The difference between this paper and paper [1] is that the initial functions considered here are step functions and those considered in [1] are Lipschitz continuous. Since there are centered rarefaction waves here, more delicate techniques are needed. It may be a necessary step in solving p-System with general initial functions by Glimm's scheme. Notice that this paper can not be deduced from [1].

Consider the initial value problem for isentropic gas dynamics in Lagrangian coordinates, so call p-System,

$$v_t - u_x = 0, \quad u_t + p(v)_x = 0, \quad (0, \infty) \times (-\infty, \infty) \tag{P}$$

$$(v(0, x), u(0, x)) = (v_0(x), u_0(x)), \quad (-\infty, \infty) \tag{I}$$

where the pressure $p = p(v) > 0$ is a C^2 function of the specific volume $v > 0$ and u is the velocity of the gas. We assume that $p'(v) < 0$, $p''(v) > 0$ and $\int_1^\infty \sqrt{-p'(v)}\, dv < \infty$. The Riemann invariants are taken as

$$r(u, v) = u + \Phi(v), \quad s(u, v) = u - \Phi(v), \quad \Phi(v) = \int_1^v \sqrt{-p'(s)}\, ds$$

Theorem *If $u_0(x)$ and $v_0(x)$ are bounded step functions,* $0 < V_* \le v_0(x) \le V^* < \infty$, *satisfying conditions*

$$|\Phi(v_0(x_2)) - \Phi(v_0(x_1))| < u_0(x_2) - u_0(x_1), \quad x_1 < x_2 \tag{M_1}$$

i. e.

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① This work is supported in part by the Foundation of Zhongshan University Advanced Research Centre.

$$r_0(x_1) \leq r_0(x_2), \qquad s_0(x_1) \leq s_0(x_2), \qquad x_1 < x_2 \tag{M_2}$$

and

$$r_0(x+0) - s_0(x-0) < 2\Phi(\infty), \qquad -\infty < x < \infty, \tag{V}$$

where $r_0(x) = r(u_0(x), v_0(x))$, $s_0(x) = s(u_0(x), v_0(x))$. Suppose that the random sequence $a \equiv \{a_n\}$ is uniformly equidistributed on the interval $(-1, 1)$. For given $T > 0$, if the mesh lengths $l > 0$, $h > 0$ are sufficiently small, the ratio $\delta \equiv lh^{-1} > \lambda_*$, where $\lambda_* \equiv \sqrt{-p'(V_*)}$, λ is a constant, then the Glimm's approximations $(u_h(t, x), v_h(t, x))$ of (P), (I) are uniformly bounded with respect to h in the zone $(0, T) \times (-\infty, \infty)$.

Condition (V) assures that there is no vacuum at the initial instant.

We refine the definition of uniformly equidistributed sequence given in [1].

Definition A sequence $a \equiv \{a_n\}$ is uniformly equidistributed on the interval $(-1, 1)$, if there is a constant e, $0 < e < \frac{1}{3}$, and a constant $D = D(e) > 0$, such that

$$|B(j, n, I) - 2^{-1} n\mu(I)| < Dn^e \tag{D}$$

$n = 1, 2, \ldots$ holds for any integer $j \geq 1$ and any subinterval I in the interval $(-1, 1)$, where $B(j, n, I)$ denotes the number of m, $j \leq m \leq j + n - 1$, with $a_m \in I$, and $\mu(I)$ is the length of I. The constant $D(e) > 0$ is independent of j and I.

Uniformly equidistributed sequence can easily be constructed.

Before proving the theorem, we give the following lemmas. Set $f_k^n = f_k(nh, kl)$, here $f = u, v, r, s$, etc., $n + k =$ even.

Lemma 1 *For given integers $n \geq 0$, $q > 0$ and constant b, if*

$$0 \leq r_{k+2}^n - r_k^n, \qquad 0 \leq s_{k+2}^n - s_k^n,$$
$$r_{k+2q}^n - r_k^n \leq b, \qquad s_{k+2q}^n - s_k^n \leq b$$

hold for every k, then

$$0 \leq r_{k+1}^{n+1} - r_{k-1}^{n+1}, \qquad 0 \leq s_{k+1}^{n+1} - s_{k-1}^{n+1},$$
$$r_{k+2q-1}^{n+1} - r_{k-1}^{n+1} \leq b, \qquad s_{k+2q-1}^{n+1} - s_{k-1}^{n+1} \leq b$$

The lemma in paper [1] is a special case of above Lemma 1 as $q = 1$. The proofs of the two lemmas are similar.

The following lemma is trivial.

Lemma 2 *For given $0 < \underline{V} < \overline{V} < \infty$, there are constants $0 < c_* < c^* < \infty$, such that*

$$c_*(\Phi(v_2) - \Phi(v_1)) < \Phi'(v_1) - \Phi'(v_2) \leq c^*(\Phi(v_2) - \Phi(v_1))$$

hold for all v_1, v_2, $0 < \underline{V} \leq v_1 < v_2 \leq \overline{V} < \infty$.

According to condition (M), the Glimm's approximations under consideration consist of rarefaction waves. Hereafter rarefaction waves are simply called waves. If wave γ is issued from point (nh, kl), $n + k =$ odd, the (nh, kl) is the starting point of γ, denoted by $P(\gamma) = (n, k)$. The maximum (minimum) value of wave γ is defined by

$\bar{\gamma} = \max_{v\in\gamma}\Phi(v)$, ($\underline{\gamma} = \min_{v\in\gamma}\Phi(v)$) . The strength of γ is defined by $\|\gamma\| = \bar{\gamma} - \underline{\gamma}$.

Following T. P. Liu [2], we partition the waves into subwaves so that the locality, upper bounded and strength of these subwaves can be traced.

Consider a 2-subwave β^n, $P(\beta^n) = (n, k-1)$, $n+k$ =even, $\beta^n = \{(r, s) \mid r = r(n)$, $s_- \leq s \leq s_+\}$, $\bar{\beta}^n = \frac{1}{2}(r(n) - s_-)$, $\underline{\beta}^n = \frac{1}{2}(r(n) - s_+)$. $\|\beta^n\| = \frac{1}{2}(s_+ - s_-)$. Hereafter, for simplicity, denote $\beta^n = \{r = r(n), s_- \leq s \leq s_+\}$.

Case 1 If the random choice $a_{n+1} > \Phi'(v_+^n)\delta^{-1}$, here v_+^n satisfies $\Phi(v_+^n) = \frac{1}{2}(r(n) - s_+)$, then the corresponding 2-subwaves $\beta^{n+1} = \{r = r(n+1)$, $s_- \leq s \leq s_+\}$, $p(\beta^{n+1}) = (n+1, k-2)$ (move backward!), $r(n+1) = r(n)$, i. e. $\beta^{n+1} = \beta^n$.

Case 2 If the random choice $a_{n+1} \leq \Phi'(v_-^n)\delta^{-1}$, here v_-^n satisfies $\Phi(v_-^n) = \frac{1}{2}(r(n) - s_-)$, then the corresponding 2-subwaves $\beta^{n+1} = \{r = r(n+1)$, $s_- \leq s \leq s_+\}$, $P(\beta^{n+1}) = (n+1, k)$; (move forward!) $r(n+1) = r(n) + 2\|\alpha_n^n\|$, where α_n^n is the 1-wave, which interacts with β^n, $\bar{\beta}^{n+1} - \bar{\beta}^n = \frac{1}{2}(r(n+1) - r(n)) = \|\alpha_n^n\|$, $\|\beta^{n+1}\| = \|\beta^n\| = \frac{1}{2}(s_+ - s_-)$.

Case 3 If $\Phi'(v_-^n)\delta^{-1} < a_{n+1} < \Phi'(v_+^n)\delta^{-1}$, then the corresponding 2-subwave β^{n+1} consists of two parts β_1^{n+1}, β_2^{n+1} (partition!) $P(\beta_1^{n+1}) = (n+1, k-2)$, (backward!), $P(\beta_2^{n+1}) = (n+1, k)$, (forward!) $\beta_1^{n+1} = \{r = r_1(n+1)$, $s_- \leq s \leq s_1\}$, $\beta_2^{n+1} = \{r = r_2(n+1)$, $s_1 \leq s \leq s_+\}$, where s_1 satisfies $\frac{1}{2}(r_1(n) - s_1) = \Phi(v_1^n)$, v_1^n satisfies $\Phi'(v_1^n) = a_{n+1}\delta^{-1}$, and $r_1(n+1) = r(n)$, $r_2(n+1) = r(n) + 2\|\alpha_n^n\|$, α_n^n is the 1-wave interacted with β_2^n (see below!). Correspondingly, we partition β^n into two subwaves β_1^n, β_2^n, $\beta_1^n = \{r = r(n), s_- \leq s \leq s_1\}$, $\beta_2^n = \{r = r(n), s_1 \leq s \leq s_+\}$. Hence $\beta_1^{n+1} = \beta_1^n$, $\bar{\beta}_2^{n+1} - \bar{\beta}_2^n = \|\alpha_n^n\|$, $\|\beta_2^{n+1}\| = \|\beta_2^n\| = \frac{1}{2}(s_+ - s_1)$.

Case 4 Suppose that β_l^n, β_r^n are 2-subwaves issued from two adjacent points. If the left one β_l^n moves forward and the right one β_r^n moves backward, then their corresponding waves β_l^{n+1}, β_r^{n+1} issued from the same point, (combine!), $\bar{\beta}_l^{n+1} - \bar{\beta}_l^n = \|\alpha_n^n\|$, where α_n^n is the 1-wave interacted with β_l^n, $\beta_r^{n+1} = \beta_r^n$, $\|\beta_l^{n+1}\| = \|\beta_l^n\|$.

Obviously, similar arguments are available for 1-waves.

For a given random sequence, initial waves and $M > 0$, we partition the initial waves, such that there is no more partition until $t = Mh$, i. e. , Case 3 no longer arises for partitioned subwaves.

Consider an initial 2-partitioned subwave β^0 and its corresponding partitioned subwave $\beta^1, \ldots, \beta^s$.

$\beta^n = \{r = r(n), s_- < s < s_+\}$, $n = 0, \ldots, N$, $P(\beta^n) = (n, K_n - 1)$, $n + K_n =$ even. Correspondingly sequence of 1-waves is denoted by α_k^n, $k = 1, 2, \ldots$, if $p(\alpha_k^n) = (n, K_n + 2k - 1)$. Above arguments imply

Lemma 3 *(A) The strength of β^n is invariant $\| \beta^n \| = \| \beta^0 \|$, $n = 1, \ldots, M$.*

(B) (1) $a_+ \leq a_{n+1} < 1 \Leftrightarrow \beta^n$ moves backward and α_1^n moves forward $\Rightarrow \beta^{n+1} = \beta^n$,

$$\alpha_1^{n+1} = \alpha_1^n, \text{ where } a_+ = \Phi'(v_+^n)\delta^{-1}, \ \Phi(v_+^n) = \frac{1}{2}(r(n) - s_+).$$

(2) β^n and $\dot{\alpha}_1^n$ move forward $\Leftrightarrow \overline{\beta}^{n+1} - \beta^n = \| \dot{\alpha}_1^n \| \Rightarrow -1 < a_{n+1} < a_+$, where $\dot{\alpha}_1^n$, which interacts with β^n, is a subwave of α_1^n and the complement $\widetilde{\alpha}_1^n = \alpha_1^n \backslash \dot{\alpha}_1^n$ does not interact with β^n. It is possible, $\| \dot{\alpha}_1^n \| = 0$ or $\| \alpha_1^n \| = 0$.

(C) Let $\dot{a}_n^i$ be the corresponding wave of $\dot{\alpha}_1^n$, $n = 1, \ldots, M$. $\dot{a}_n^i$ lies on $t = ih$, $0 \leq i \leq n$, $\dot{a}_n^n \equiv \dot{\alpha}_1^n$. Then $\dot{a}_i^i, \ldots \dot{a}_n^i$ arrange from left to right, and $\dot{a}_i^i \cup \ldots \cup \dot{a}_n^i$ includes all 1-partitioned subwaves between $\dot{a}_i^i$ and $\dot{a}_n^i$.

Consider a union of ordered 2-partitioned subwaves on $t = nh$, $\beta^n = \bigcup_{i=1}^{k} \beta_i^n$, $\beta_i^n = \{r = r_n, s_i \leq s \leq s_{i+1}\}$ are partitioned subwaves, $p(\beta_i^n) = (n, m_i), m_1 \leq \ldots \leqslant m_k$. The number of starting points of β^n is $d(\beta^n) = m_k - m_1 + 1$. Set $a_\pm = \Phi'(v_\pm)\delta^{-1}$, where v_+, v_- satisfy $\Phi(v_+) = \underline{\beta}_k$, $\Phi(v_-) = \overline{\beta}_1$. Then we get

Lemma 4 *(1) $a_- < a_{n+1} < a_+ \Leftrightarrow d(\beta^{n+1}) = d(\beta^n) + 1$,*

(2) $a_{n+1} \leq a_-$, or $a_{n+1} \geq a_+ \Leftrightarrow d(\beta^{n+1}) = d(\beta^n)$, where a_{n+1} is the random choice.

Proof of Theorem Let

$$(v_n(x), u_n(x)) = \begin{cases} (v_0(-n), u_0(-n)), & \text{as } x < -n \\ (v_0(x), u_0(x)), & \text{as } |x| < n \\ (v_0(n), u_0(n)), & \text{as } x > n \end{cases} \qquad (I_n)$$

Since the initial functions in (I) are bounded, $0 < v_* \leq v_0(x) \leq v^* < \infty$, it is sufficient to prove the theorem for initial value problem (P), (I_n). We omit the subscript "n" for simplicity. Now the number of discontinuities in initial functions is finite, $-\infty = z_0 < z_1 < \ldots < z_i < z_{i+1} = \infty$. Denote $b = \min_{1 \leq j \leq i-1} (z_{j+1} - z_j) > 0$.

Choose sufficiently large natural number N. Let mesh length $l = \frac{b}{2}(N^2 + 2)^{-1}$, $h = l\delta^{-1}$; $T_1 \equiv N^2 h < 2^{-1} b \delta^{-1}$. There are not more than i pairs of initial waves, they cannot interact with each other before $t = N^2 h \equiv T_1$. Consider an initial 2-wave β. Let $\beta^0 \equiv \bigcup_{m=1}^{k} \beta_m^0$ be a subwave of β, and $\beta^n = \bigcup_{m=1}^{k} \beta_m^n$ be the corresponding waves of β^0, β_m^n are partitioned waves $\beta_m^* = \beta_m^0 = \{r = r_0,\ s_{m-1} \leq s \leq s_m\}$, $s_0 < \dots < s_k$. Let v_-, v_+ satisfy $\Phi(v_-) = \bar{\beta}_1^0 = \frac{1}{2}(r_0 - s_0)$, $\Phi(v_+) = \beta_k^0 = \frac{1}{2}(r_0 - s_k)$ respectively. Condition (V) implies $\Phi(v_-) < \Phi(\infty)$. We now prove that if

$$\| \beta^0 \| = \frac{1}{2}(s_k - s_0) > N^{2\varepsilon-1}, \qquad 0 < \varepsilon < \frac{1}{3} \tag{3}$$

then

$$d(\beta^{N^2}) > G_N N^{2\varepsilon+1} \tag{4}$$

where $G_N \equiv C_* \delta^{-1}(1 - O(N^{-\varepsilon}))$, $O(N^{-\varepsilon}) > 0$. C_* is defined in Lemma 2.

Consider interval $I = (\Phi'(v_-)\delta^{-1},\ \Phi'(v_+)\delta^{-1})$. From Lemma 4, Lemma 2 and (D), (3)

$$\begin{aligned} d(\beta^{(i+1)N}) - d(\beta^{iN}) &\geq B(iN, N, I) \geq N\delta^{-1}(\Phi'(v_+) - \Phi'(v_-)) - DN^{\varepsilon} \\ &\geq N\delta^{-1}C_*(\Phi(v_-) - \Phi(v_+)) - DN^{\varepsilon} > C_*\delta^{-1}N^{2\varepsilon} - DN^{\varepsilon} \end{aligned} \tag{5}$$

$i = 0, \dots, N-1$, then we obtain

$$d(\beta^{N^2}) > C_*\delta^{-1}N^{2\varepsilon+1} - DN^{\varepsilon+1}$$

(4) has been proven. Obviously, above argument is available for 1-wave.

From (3), (4), we get

$$0 \leq r_{k+2q}^{N^2} - r_k^{N^2} \leq 2N^{2\varepsilon-1}, \qquad 0 \leq s_{k+2q}^{N^2} - s_k^{N^2} \leq 2N^{2\varepsilon-1}$$

hold for all integers k, $N^2 + k =$ even, where natural number $q \leq G_N N^{1+2\varepsilon}$. From Lemma 1①

$$0 \leq r_{k+2q}^n - r_k^n \leq 2N^{2\varepsilon-1},\ 0 \leq s_{k+2q}^n - s_k^n \leq 2N^{2\varepsilon-1} \tag{$*$}$$

hold for all integers $n \geq N^2$, k, $n + k =$ even. Consequently,

$$0 \leq r_{k+2p}^n - r_k^n \leq 2iN^{2\varepsilon-1}, \qquad 0 \leq s_{k+2p}^n - s_k^n \leq 2iN^{2\varepsilon-1} \tag{6}$$

as $0 < p \leq iG_N N^{2\varepsilon+1}$, where $i = 1, 2, \dots$.

In order to prove the theorem, it is sufficient to prove

$$0 \leq \Phi(v_1) - \Phi(v_0) \leq L(T - T_1)M^{-1}\Phi'(v_0)(1 + O(N^{-\varepsilon})) \tag{7}$$

where $L = C_*^{-1}T_1^{-1}$,

$$v_m \equiv \sup_{D_m} v_m(x,\ t+0)$$

region $D_m \equiv \{(t, x) \mid 0 \leq t \leq (N^2 + m\bar{N})h\}$, $\bar{N} \equiv [N_1] + 1$, $N_1 \equiv N^{1+2\varepsilon}$, $m = 1, 2, \dots, M$, $M \equiv [M_1] + 1$, $M_1 \equiv (T - T_1)T_1^{-1}N^{1-2\varepsilon}$. Let $\tau = \bar{N}h$, then $T - T_1 \leq M\tau \leq (T - T_1) + 2\tau$. Condition (V) implies $\Phi(v_0) < \Phi(\infty)$. The remainder of the proof is the same as that in paper [1].

We regard the level $t = T_1 = N^2h$ as a new initial level $t' = 0$, i. e. let $t' = t - T_1$.

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① In [1], we have $0 \leq r_{k+2}^{(n)} - r_k^{(n)} \leq 2Ll$, $0 \leq s_{k+2}^{(n)} - s_k^{(n)} \leq 2Ll$, but they are no longer true here. We only get $(*)$. This is the critical point of this paper.

For simplicity, we omit the superscript "l". So $D_m \equiv \{(t,x) \mid 0 \le t \le m\bar{N}h\}$.

Consider an initial 2-partitioned wave β^0 and its corresponding waves β^n, $n=1, \dots, \bar{N}$. In order to estimate the upper bound of $\Phi(v_1)$, without loss of generality, we may assume that $\bar{\beta}^{\bar{N}} \ge \Phi(v_0)$. Consequently, there exists n_0, $1 \le n_0 \le \bar{N}$, such that

$$\bar{\beta}^0 \le \bar{\beta}^{n_0-1} \le \Phi(v_0) \le \bar{\beta}^{n_0} \le \bar{\beta}^{\bar{N}} \leqslant \Phi(v_1)$$

Then

$$\bar{\beta}^{\bar{N}} - \Phi(v_0) \le \bar{\beta}^{\bar{N}} - \bar{\beta}^{n_0-1} = (\bar{\beta}^{\bar{N}} - \bar{\beta}^{n_0}) + (\bar{\beta}^{n_0} - \bar{\beta}^{n_0-1}) \tag{8}$$

According to Lemma 3

$$\bar{\beta}^{\bar{N}} - \bar{\beta}^{n_0} \le \sum_{i=n_0}^{\bar{N}-1} \| \dot{\alpha}_1^i \| \le \sum_{j=n_0}^{\bar{N}-1} \| \dot{\alpha}_j^{n_0} \| \tag{9}$$

Let p be the number of starting points of $\bigcup_{j=n_0}^{\bar{N}-1} \dot{\alpha}_j^{n_0}$. According to Lemma 3 and (D),

$$0 < p \le 2B(n_0, \bar{N} - n_0, I) - (\bar{N} - n_0)$$

where interval $I = (-1, \Phi'(v_0)\delta^{-1})$. In view of (6) and (D),

$$0 < p \le (\bar{N} - n_0)\Phi'(v_0)\delta^{-1} + D(\bar{N}n_0)^{\varepsilon} \le \bar{N}\Phi'(v_0)\delta^{-1} + D\bar{N}^{\varepsilon}$$

$$\frac{p}{G_N N^{2\varepsilon+1}} \le \frac{N^{\varepsilon}}{C_*}\Phi'(v_0)(1 + O(N^{-\varepsilon}))$$

$$\begin{aligned} 0 \le r^{*}_{k+2p} - r^{*}_{k} &\le 2C_*^{-1}N^{2\varepsilon-1}\Phi'(v_0)(1 + O(N^{-\varepsilon})) \\ &\le 2L(T - T_1)M^{-1}\Phi'(v_0)(1 + O(N^{-\varepsilon})) \end{aligned} \tag{10}$$

Substituting (10) into (9), from (8), we can get

$$\bar{\beta}^N - \Phi(v_0) \le L(T - T_1)M^{-1}\Phi'(v_0)(1 + O(N^{-\varepsilon})) \tag{11}$$

(11) is available for any 2-partitioned wave. Then (7) has been proven.

Let $(v(t,x), u(t,x))$ be the solution of the initial value problem under consideration. Following paper [1], we can prove

$$v_* \le v(t,x) \le \begin{cases} v_0 & \text{as } 0 < t < T_1 \\ v_0 + (t - T_1)/C_* T_1 & \text{as } T_1 \le t \end{cases}$$

More delicate calculation shows that $C_* = \min\limits_{v_* \le v \le v_0} (-\Phi''(v))/\Phi'(v)$ and $C_* = (\gamma + 1)v_0^{-1}$ for polytropic gas $p(v) = k^2 v^{-\gamma}$, $T_1 \equiv 2^{-1}b\delta^{-1}$.

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