J. Partial Differential Equations
Vol. 2, No. 1, (1989), 1—8

THE DIRICHLET PROBLEM FOR THE DEGENERATE MONGE-AMPERE EQUATION

Zou Henghui
(Tsinghua University)

(Received December 17, 1986; revised May 27, 1987)

1. Introduction

In this paper, we discuss the Dirichlet problem for the degenerate Monge-Ampere equation.

Let $\Omega \subset R^n$ be a bounded smooth strictly convex domain, and let $r(x) \in C^\infty(R^n)$ be a strictly convex function. We call $r(x)$ the defining function if

$$\Omega = \{x \in R^n \mid r(x) < 0\}$$

The problem is to find a convex function $u(x) \in C^{k+2+\alpha}(\Omega)$ which satisfies the equation:

$$\begin{cases} \det(u_{ij} + \sigma_{ij}) = \psi(x, u, \nabla u) & \text{in} \quad \Omega \\ u|_{\partial\Omega} = \phi(x) \end{cases} \tag{1. 1}$$

where $\psi(x, t, p) \in C^{k+\alpha}(\Omega \times R \times R^n)$, $\psi \geq 0$, $(0 < \alpha < 1)$, $\{\sigma_{ij}\}$ is a real symmetry matrix. $\sigma_{ij}(x) \in C^{k+\alpha}(\Omega)$, $\varphi(x) \in C^{k+\alpha}(\partial\Omega)$, ($k \geq 2$ is an integer), and $u_{ij} = \partial_i \partial_j u$, $u_i = \partial_i u$, $\nabla u = (u_1, \ldots, u_n)$. In the following we use the notations: $\nabla^2 u = (u_{ij})$, $\nabla^3 u = (u_{ijk})$, $i, j, k = 1, \ldots, n$.

We say that $v(x)$ is a sub-solution of (1. 1) if $v \in C^2(\Omega)$ and satisfies:

$$\begin{cases} \det(v_{ij} + \sigma_{ij}) \geq \psi(x, v, \nabla v) & \text{in} \quad \Omega \\ v|_{\partial\Omega} = \varphi(x) \end{cases} \tag{1. 2}$$

When $\psi(x, t, p) \geq C > 0$, $\psi_t(x, t, p) \geq 0$, and the equation (1. 1) has a sub-solution, the existence and uniqueness of the solution of (1. 1) has been proved by Caffarelli, Nirenberg and Spruck in [1].

Under the above conditions, the equation is uniformly elliptic. The main contribution in [1] is to prove the global estimation of $C^{2,\alpha}$ norm of the solutions $u(x)$ of (1. 2). The crucial point in [1] is to estimate the logarithmic modulus of continuity of u_{ij} at every point x:

$$\sum_{i,j} |u_{ij}(x) - u_{ij}(y)| \leq \frac{k}{1 + |\ln|x - y||} \qquad \begin{matrix} \forall x \in \partial\Omega \\ \forall y \in \overline{\Omega} \end{matrix} \tag{1. 3}$$

In fact, combining the above inequality with the interior estimations of u_{ijk} we obtain the global estimation of the Hölder norm of u_{ij}.

In the works of Pogorelov, Cheng S. Y. and Yau S. T. the existence of generalized solution of (1. 1) was proved, only the local regularity of the solution, i. e. $u \in C^{k+\alpha}(\Omega) \cap C^0(\Omega)$ was given.

The Monge-Ampere equation (1. 1) originates from geometrical problems e. g. the Minkowski problem. When the Gauss curvature, say $\psi(x, t, p)$ of the right hand of (1. 1), is nonnegative but zero in some points, the equation (1. 1) is a degenerate 2nd order elliptic equation.

The main results of this paper are as follows:

Theorem 1. 1 *If $f(x)$, $\varphi(x)$ and $u_\varepsilon(x)$ are such that*

1) $f(x) \in C^2(\Omega)$, $\quad f(x) \geq 0$, $\quad 0 \overline{\in} f(\partial\Omega)$

2) $\varphi(x) \in C^4(\partial\Omega) \qquad (0 < \alpha < 1)$

3) $u_\varepsilon(x) \in C^4(\Omega) \cap C^2(\bar{\Omega})$ *is convex and satisfies:*

$$\begin{cases} \det((u_\varepsilon)_{ij} + \sigma_{ij}) = (f(x))^n + \varepsilon & in \quad \Omega \\ u_\varepsilon|_{\partial\Omega} = \varphi(x) \end{cases} \tag{1. 4}$$

then we have

1) $\| u_\varepsilon \|_{2, \bar{\Omega}} \leq C_0$ *for some constant* C_0 *depending only on* $\| \phi \|_{4, \partial\Omega}$, $\| f \|_{2, \Omega}$ *and* $(\| f \|_{0, \Omega_{d_0}})^{-1}$.

2) $\forall K', K \subset\subset K' \subset \Omega$, *there is* β, $\beta = \beta(K') > 0$, *such that*

$$\| u_\varepsilon \|_{2+\beta, \overline{\Omega \setminus K'}} \leq C(d(K, K'), \| u_\varepsilon \|_{2, \bar{\Omega}})$$

where $K = \{x \in \Omega \mid f(x) = 0\}$, $d_0 = d(K, \partial\Omega)$ *and* $\Omega_d = \left\{x \in \Omega \mid d(x, \partial\Omega) \leq \frac{1}{2} d_0\right\}$.

Theoerm 1. 2 *Suppose that the conditions of Theorem1. 1 hold, then there is a convex function $u(x) \in C^2(\Omega \setminus K) \cap C^{1,1}(\bar{\Omega})$ satisfying:*

$$\begin{cases} \det(u_{ij} + \sigma_{ij}) = (f(x))^n & in \quad \Omega \setminus K \\ u|_{\partial\Omega} = \varphi(x) \end{cases} \tag{1. 5}$$

All symbols used here are the same as that of[4].

2. The Estimation of C^2-Norm of the Solutions

It is difficult to estimate the solutions of (1. 1) directly when the equation is degenerate. We perturb the right hand of (1. 1) by $\varepsilon (> 0)$. The equation becomes a 2nd order non-degenerate uniformly elliptic equation. We do have the a priori estimation of the perturbed equation, but it might be dependent with $\varepsilon > 0$. To prove the existence of solution of the original equation, we hope that the a priori estimation is independent of ε.

We shall prove that:

(1) The estimation of C^2-norm of solutions of the perturbed equation is independent of ε.

(2) The estimation of $C^{2,\alpha}$-norm of solutions depends only on the condition near the degenerate domain.

Proposition 2.1 *Suppose that the conditions of Theorem 1.1 hold, any $u_\varepsilon(x) \in C^4(\Omega) \cap C^2(\bar{\Omega})$ is convex and satisfies (1.4), then*

$\| u_\varepsilon \|_{1,\bar{\Omega}} \leq C$ *for some const. C depending only on $\| \phi \|_{1,\partial\Omega}$ and $\| f \|_{0,\Omega}$.*

Proof Take $w(x) = \phi(x) + Ar(x)$, here $\phi(x)$ is an extension of (x) in Ω, $A > 0$ (constant to be determined), $r(x)$ is the defining functions of Ω.

It is easy to prove that:

1) $w(x) |_{\partial\Omega} = u_\varepsilon(x) |_{\partial\Omega} = \phi(x)$

2) $\det(w_{ij} + \sigma_{ij}) > \det((u_\varepsilon)_{ij} + \sigma_{ij})$ (A is large enough)

In fact, 1) is obvious.

To prove 2), we notice that

$$\lim_{A \to +\infty} \det(w_{ij} + \sigma_{ij}) = +\infty \quad ((r_{ij}) \text{ is positive})$$

We take A large enough and 2) follows.

According to the maximum principle, we have

$$u_\varepsilon(x) \geq w(x) \qquad \text{in}\Omega \tag{2.1}$$

Because $u_\varepsilon(x)$ is convex, so we have

$$\max_{\bar{\Omega}} | u_\varepsilon(x) | \leq C \tag{2.2}$$

for some C depending only on $\| \phi \|_{0,\partial\Omega}$ and $\| f \|_{0,\Omega}$.

$u_\varepsilon(x)$ is convex, it admits the maximum of $| (u_\varepsilon(x))_i |$ on the boundary. According to 1), 2), (2.1), it is easy to prove that $| (u_\varepsilon(x))_i |$ is dominated by $| w_i(x) |$ on the boundary. Then we complete the proof of Proposition 2.1.

Proposition 2.2 *If $u_\varepsilon(x) \in C^4(\Omega) \cap C^2(\bar{\Omega})$ is convex and satisfies (1.4) and the conditions of Theorem 1.1 hold, then we have*

$$\| u_\varepsilon \|_{2,\Omega} \leq C \quad \textit{for } C \textit{ is the same as } C_0 \textit{ of Theorem 1.1} \tag{2.3}$$

Proof We first estimate $| (u_\varepsilon)_{ij} |$ near the boundary.

Consider any boundary point; without loss of generality we may take it to be the origin and the x_n-axis to be interior normal. According to [1], we can obtain the following inequalities: (the subscript ε is omitted)

I. $\quad | u_{\alpha\beta}(0) | \leq c, \quad$ for $\alpha, \beta < n$

II. $\quad | u_{\alpha n}(0) | \leq c, \quad$ for $\alpha < n$

Because $0 \overline{\in} f(\partial\Omega)$, we know that $f(x)$ has a uniformly positive upper and lower bound near the boundary which is independent of ε. From [1] we have (for a_{ij}, C_i depend only on $\|\phi\|_{4,\partial\Omega}$ and Ω)

$$\tilde{u} = u - \lambda x_n$$

$$\tilde{u}|_{\partial\Omega} \leq \sum_{1<j\leq n} a_{1j}x_1x_j + C\left(\sum_{1<\beta<n} x_\beta^2 + |x|^4\right)$$

$$\leq \sum_{1<j\leq n} a_{1j}x_1x_j + C_1\sum_{1<\beta\leq n} x_j^2 \qquad (*)$$

Denote $\Omega_\delta = \{x \in \Omega \mid d(x, \partial\Omega) < \delta\}$. We take $\delta > 0$ small enough such that $f \geq c_\delta > 0$ in Ω_δ.

Let $\Omega'_\delta = \{x \in \Omega \mid x_n < \delta\} \subset \Omega_\delta$.

Now choose $h(x)$ as a barrier function

$$h(x) = -ax_n + rx_1^2 + \sum_{1<j\leq n} a_{1j}x_1x_j + B\sum_{1<\beta\leq n} x_\beta^2$$

First, with the aid of Proposition 2. 1, we take B so large that

$$\frac{1}{2}B\delta^2 + \sum_{1<j\leq n} a_{1j}x_1x_j \geq \|\tilde{u}\|_{0,\Omega'_\delta} \quad (\text{in} \quad \Omega'_\delta) \text{ and } \quad B \gg C_1 \qquad (*')$$

Then, we take r so that $\left(2r - \frac{1}{2B}\sum_{1<j\leq n} a_{1j}^2\right) > 0$ is so small that

$$\det(h_{ij}) \leq c_\delta^n < (f)^n + \varepsilon \qquad \text{in} \quad \Omega'_\delta$$

Now, by taking $a \ll \min(r, \delta)$ small enough, and from $(*)$, $(*')$, we can ensure that

$$\tilde{u}|_{\partial\Omega'_\delta\cap\partial\Omega} \leq \sum_{1<j\leq n} a_{1j}x_1x_j + C_1\sum_{1<\beta\leq n} x_\beta^2$$

$$\leq -ax_n + rx_1^2 + \sum_{1<j\leq n} a_{1j}x_1x_j + B\sum_{1<\beta\leq n} x_\beta^2 = h|_{\partial\Omega\cap\Omega'_\delta}$$

and

$$h|_{\partial\Omega'_\delta\cap\{x_n=\delta\}} \geq -\delta a + B\delta^2 + \sum_{1<j\leq n} a_{1j}x_1x_j$$

$$\geq \frac{B\delta^2}{2} + \sum_{1<j\leq n} a_{1j}x_1x_j \geq \tilde{u}|_{\partial\Omega'_\delta\cap\{x_n=\delta\}}$$

Thus, by the maximum principle,

$$\tilde{u} \leq h \qquad \text{in} \qquad \Omega'_\delta$$

Consequently,

$$\tilde{u}_n(0) \leq h_n(0) = -a$$

(a depends only on $\|\phi\|_{4,\partial\Omega}$, $(\|f\|_{0,\Omega_{\delta_0}})^{-1}$ and Ω). By the above construction, we have

$$\frac{\partial^2}{\partial x_1^2}u(x', \rho(x')) = 0, \qquad \text{at the origin}$$

i. e.

$$\tilde{u}_{11} + \tilde{u}_n\rho_{11} = 0, \qquad \text{at } 0$$

Thus

$$u_{11}(0) = \tilde{u}_{11}(0) = -\tilde{u}_n(0)\,\rho_{11}(0) \geq a\rho_{11}(0)$$

So we get (without loss of generality)

$$\sum_{\alpha,\beta<n} u_{\alpha\beta}(0)\,\xi_\alpha\xi_\beta \geq C_0 > 0 \quad \text{for} \quad \xi \in R^{n-1}$$

By combining this with the equation and I, II, we have

$|u_{nn}(0)| \leq C$, (C depends only on $\|\phi\|_{4,\partial\Omega}$, $(\|f\|_{a\Omega_{t_0}})^{-1}$ and Ω)

So we have already estimated $|(u_\varepsilon(x))_{ij}|$ on the boundary.

The next step is to get a local estimation of $|(u_\varepsilon)_{ij}|$. We take for convenience that $\sigma_{ij} = 0$.

Lemma 2. 3 *If* (a_{ij}) *is a* $n\times n$ *positive real matrix;* (b_{ij}) *is a* $n\times n$ *symmetry real matrix, then*

$$\sum_{i,j,k,l} a_{ij}a_{kl}b_{ik}b_{jl} \geq \frac{1}{n}\Big(\sum_{i,j} a_{ij}b_{ij}\Big)^2 \quad (i,\ j,\ k,\ l = 1,\ 2,\ \dots,\ n) \qquad (2.\ 4)$$

Proof Take $A = (a_{ij})$, $B = (b_{ij})$,

$$C = AB = \Big(\sum_k a_{ik}b_{kj}\Big) = (c_{ij})$$

then

$$\begin{aligned}\sum_{i,j,k,l} a_{ij}a_{kl}b_{ik}b_{jl} &= \sum_{i,j,k,l} a_{ji}b_{ik}a_{kl}b_{lj}\\ &= \sum_{j,k}\Big(\sum_i a_{ji}b_{ik}\Big)\Big(\sum_l a_{kl}b_{lj}\Big)\\ &= \sum_{j,k}\Big(\sum_i a_{ji}b_{ik}\Big)^2\\ &= \sum_{i,j}(c_{ij})^2\\ &\geq \sum_i (c_{ii})^2\end{aligned}$$

and

$$\begin{aligned}\Big(\sum_{i,j} a_{ij}b_{ij}\Big)^2 &= (\operatorname{tr}(c_{ij}))^2\\ &= \Big(\sum_i c_{ii}\Big)^2\\ &\leq n\Big(\sum_i (c_{ii})^2\Big)\end{aligned}$$

We are going to prove Proposition 2. 2 with the aid of lemma 2. 3.

Differentiate the equation (1. 4) twice with respect to x_k, we get

$$F^{ij}(D_k^2u)_{ij} = (f^n+\varepsilon)\,u^{il}u^{jm}(D_ku)_{ij}(D_ku)_{lm} + D_k^2(f^n) - (f^n+\varepsilon)\,(D_k(f^n))^2 \qquad (2.\ 5)$$

where $F^{ij} = \dfrac{\partial(\det(u_{ij}))}{\partial u_{ij}}$ and the subscript ε is omitted.

Because (u_{ij}) is positive, so (u^{ij}), which is the inverse matrix of (u_{ij}), is also positive.

We know that

$F^{ij} = (f^n + \varepsilon)u^{ij} \qquad (i, j = 1, \ldots, n)$, therefore (F^{ij}) is positive.

On the other hand, we have

$$
\begin{aligned}
&D_k^2(f^n) - (f^n+\varepsilon)^{-1}(D_k(f^n))^2 \\
&= nf^{n-1}f_{kk} + n(n-1)f^{n-2}f_k^2 - n^2(f^n+\varepsilon)^{-1}f^{2n-2}f_k^2 \\
&\geq nf^{n-1}f_{kk} + n(n-1)f_k^2 f^{2n-2}(f^n+\varepsilon)^{-1} - n^2(f^n+\varepsilon)^{-1}f^{2n-2}(f_k)^2 \\
&= nf^{n-1}f_{kk} - n(f^n+\varepsilon)^{-1}f_k^2 f^{2n-2} \\
&= nf^{n-1}f_{kk} - (n(f^n+\varepsilon))^{-1}(D_k(f^n))^2
\end{aligned} \tag{2. 6}
$$

According to (2. 4), it follows that

$$
\begin{aligned}
(f^n+\varepsilon)u^{il}u^{jm}(D_k u)_{ij}(D_k u)_{lm} \\
&= (f^n+\varepsilon)^{-1}F^{il}F^{jm}(D_k u)_{ij}(D_k u)_{lm} \\
&\geq (n(f^n+\varepsilon))^{-1}(F^{ij}(D_k u)_{ij})^2 \\
&= (n(f^n+\varepsilon))^{-1}(D_k(f^n))^2
\end{aligned} \tag{2. 7}
$$

Combining (2. 5), (2. 6), (2. 7), we have

$$
F^{ij}(D_k^2 u)_{ij} \geq nf^{n-1}f_{kk} \tag{2. 8}
$$

Take $w(x)$ as the following:

$$
w(x) = D_k^2 u(x) + C\sum_{i=1}^{n} x_i^2
$$

(C is constant to be fixed)

Let λ_1, λ_2, ... and λ_n be the eigenvalues of (u_{ij}), then (according to (2. 8))

$$
\begin{aligned}
F^{ij}(w)_{ij} &= F^{ij}(D_k^2 u + C\sum_i x_i^2)_{ij} \\
&= F^{ij}(D_k^2 u)_{ij} + 2C\sum_i F^{ii} \\
&= F^{ij}(D_k^2 u)_{ij} + 2C(f^n+\varepsilon)\sum_i \lambda_i^{-1} \\
&= nf^{n-1}f_{kk} + 2nC(f^n+\varepsilon)^{1-\frac{1}{n}} \\
&\geq nf^{n-1}(C - |f_{kk}|) \\
&\geq 0
\end{aligned}
$$

(if C is large enough).

By the maximum principle, observing that (F^{ij}) is positive and u is convex (i. e. $D_k^2 u \geq 0$), we have

$$
\max_{\Omega} |D_k^2 u| \leq \max_{\partial\Omega} |D_k^2 u| + C \qquad k = 1, 2, \ldots, n \tag{2. 9}
$$

It is easy to derive (2. 3) from (2. 9).

Proposition 2. 2 is proved.

Corollary 2. 3 Suppose that the conditions of Theorem 1. 1 hold, but we take $\phi(x) = c$ to replace of $o \overline{\in} f(\partial\Omega)$, the result of proposition 2. 2 follows.

Proof The proof is same as that of proposition 2. 2.

Corollary 2.4 Suppose that the conditions of Theorem 1.1 hold, but we change the right hand of (1.4) into:

$$g(x)\cdot h(x,\ t,\ p) \tag{2.10}$$

where $g(x)=(f(x))^m$, ($f(x)$ is as above), and

$$h(x,\ t,\ p)\geq c>0,\ m\geq n$$

Then the result of Proposition 2.2. follows.

Proof Omitted.

3. The Existence Of The Solution

Now, for obtaining the existence of the solution u_ε of the equation (1.4), we need a priori estimation of the $C^{2\alpha}$-norm of u_ε i.e. the following proposition.

Proposition 3.1 If $f(x)\in C^2(\Omega)$, $f(x)\geq 0$, $\phi(x)\in C^4(\partial\Omega)$, and $u_\varepsilon(x)\in C^4(\Omega)\cap C^2(\bar{\Omega})$ is convex and satisfies (1.4), then for $\Omega'\subset\subset\Omega$, there is α, $0<\alpha<1$, depending only on ε, such that

$$\| u_\varepsilon\|_{2,\alpha,\bar{\Omega}'}\leq C \tag{3.1}$$

for some constant C depending only on $\| u_\varepsilon\|_{2,\bar{\Omega}}$, Ω' and ε.

This proposition is referred to [4].

With the aid of (1.3) and (3.1), we can obtain the following inequalities:

$$\| u_\varepsilon\|_{2,\bar{\Omega}}\leq C \tag{3.2}$$

and

$$\| u_\varepsilon\|_{2,\alpha(a),\overline{\Omega\backslash K'}}\leq C(a) \tag{3.3}$$

where $K\subset\subset K'\subset\Omega$, $a=d(\Omega\backslash K',K)$. α and $\alpha(a)$ depend only on $\| u_\varepsilon\|_{2,\bar{\Omega}}$, ε and $\| u_\varepsilon\|_{2,\bar{\Omega}}$, a respectively, and so do C and $C(a)$.

According to [1], by the use of the continuity method, with the aid of (3.2), the solution u_ε of $(1.4)_\varepsilon$ is obtained for $\varepsilon>0$.

We take $\varepsilon_n=\frac{1}{n}$, $n=1, 2, \ldots$, then we have a sequence $\{u_{\varepsilon_n}\}_1^\infty$. With the aid of (3.3), we get $\{u_{\varepsilon_{n_j}}\}_1^\infty$, a subsequence of $\{u_{\varepsilon_n}\}_1^\infty$, which is convergent to $u(x)$ in $C^2(\Omega\backslash K)$.

According to (2.3) and (3.3), $u(x)\in C^2(\Omega\backslash K)\cap C^{1,1}(\bar{\Omega})$ and is convex and it follows easily that $u(x)$ satisfies (1.5). Hence the proof of Theorem 1.2 is complete.

The author wishes to thank his teacher K. C. Chang for his kind instruction.

References

[1] Caffarelli L., Nirenberg L., Spruck J., The Dirichlet problem for nonlinear second

order elliptic equations. I. Monge-Ampere equation. *Comm. Pure. Appl. Math.*, *Vol.* **37** (1984).

〔2〕 Cheng S. Y. and Yau S. T., On the regularity of the solution of the n-dimensional Minkowski problem. *Comm. Pure. Appl. Math.*, **29** (1976).

〔3〕 Cheng S. Y. and Yau S. Y., On the regularity of the Monge-Ampere equations det $(u_{ij}) = F(x, u)$. *Comm. Pure. Appl. Math.*, **30** (1977).

〔4〕 Gilbarg D., Trudinger N. S., Elliptic partial differential equation of second order, 2nd edition, Springer—Verlag, Berlin, Heidelberg, New York (1983).

〔5〕 Hildebrandt, Stefan, Nonlinear elliptic systems and harmonic mappings. Universitat BONN, (1982).

〔6〕 Yau S. Y., Survey on Partial Differential Equations in Differential Geometry.