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THE UNIQUENESS OF STEADY-STATE SOLUTION FOR TWO-PHASE CONTINUOUS CASTING PROBLEM

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Abstract Concerning steady-state continuous casting problem, we know that if the number of phases is one, both existence and uniqueness had been solved ([1],[2],[3]), if the number of phases is two, the existence had been proved ([4]), but the uniqueness of weak solution is an open problem all the time. This paper is devoted to solving this problem.

Key Words Partial differential equation; free boundary problem; uniqueness.
Classification 35R35.

1. The Steady-state Continuous Casting Problem

The portion of ingot considered is supposed to include the solid-liquid interface (Figure) and occupies a cylindrical open domain $\Omega=\Gamma\times(0,H)$ of R^n ($\Gamma=(0,a)$ for $n=2$ and Γ is an open bounded domain of R^2 with Lipschitz boundary for $n=3$.)

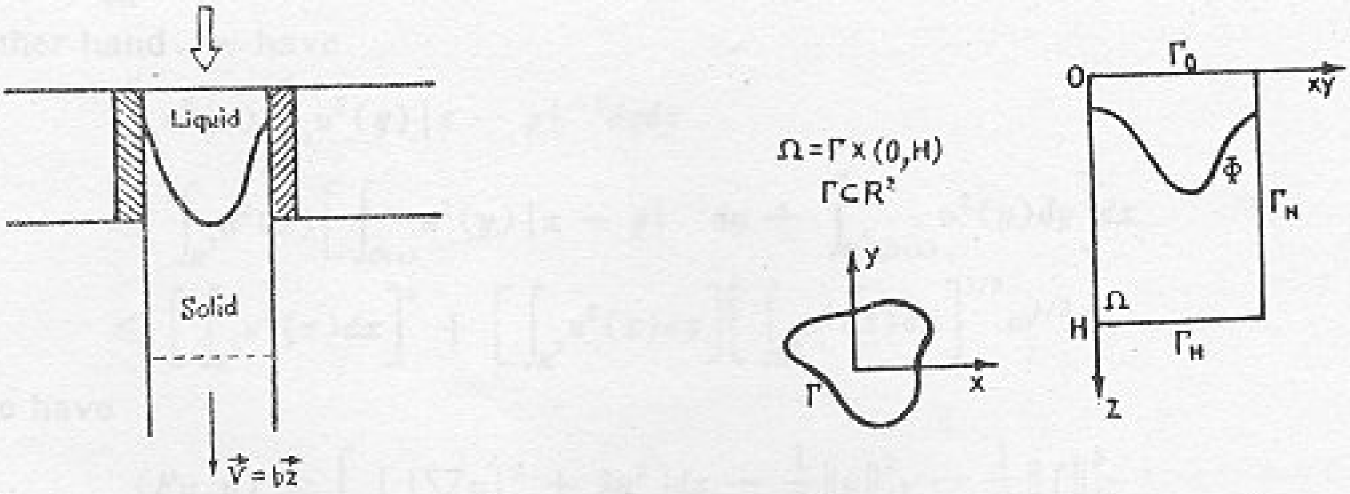

Figure (a) Ingot solidification in continuous casting (b) Ingot geometry in R^n

We set $\Gamma_i=\Gamma\times\{i\}$, $i=0,H$, $\Gamma_D=\Gamma_0\cup\Gamma_H$ and $\Gamma_N=\partial\Gamma\times(0,H)$, we denote the gradient by $\nabla=(\partial_x,\partial_y,\partial_z)$, so $\Delta=\nabla\circ\nabla$. We shall assume free boundary $\Phi=\{(x,y,z)\in\Omega: z=\Phi(x,y)\}$ fixed with respect to the mould and the casting velocity given by $\vec{v}=b\vec{z}$ with constant $b>0$. The metal temperature $T=T(x,y,z)$ verifies stationary heat equation

$$bC(T)\partial_z T=\nabla(k(T)\nabla T) \qquad \text{in} \qquad \Omega\backslash\Phi \tag{1}$$

where $C\geq 0$ is the specific heat and $k>0$ the thermal conductivity. The left member in (1) takes into account the heat transfer due to the convection. If T_0 denotes the melting temperature at the interface, after the usual renormalization procedure

$$\theta = \int_T^{T_0} k(\tau)d\tau = K(T) \tag{2}$$

at the solid region $\{\theta > 0\}$ and at the liquid region $\{\theta < 0\}$, equation (1) becomes

$$\partial_z f(\theta) = \Delta\theta \quad \text{in} \quad \Omega\backslash\Phi = \{\theta > 0\} \cup \{\theta < 0\} \tag{3}$$

where $f = C_b \circ K^{-1}$ and $C_b(T) = b\int_T^{T_0} C(\tau)d\tau$. At the interface we have $\theta = 0$ and the stefan condition is given, in terms of the renormalized temperature θ, by

$$-[\nabla\theta]_-^+ \circ \vec{\gamma} = -\lambda\vec{v} \circ \vec{\gamma} = \lambda b \quad \text{on} \quad \Phi = \{\theta = 0\}$$

where $\lambda > 0$ is the latent heat, $\vec{\gamma} = (\partial_x\Phi, \partial_y\Phi, -1)$ is a normal vector to Φ and $[\ ]_-^+$ denotes the jump across Φ.

2. Definition of Weak Solution Existence of Solution to the Two-phase Problem

Problem (P) Find a couple $(\theta, \eta) \in H^1(\Omega) \times L^\infty(\Omega)$, such that

$$\theta = h \quad \text{on} \quad \Gamma_D \tag{4}$$

$$0 \le \chi\{\theta > 0\} \le \eta \le 1 - \chi\{\theta < 0\} \le 1 \quad \text{a. e. in} \quad \Omega \tag{5}$$

$$\int_\Omega \{\nabla\theta\nabla\zeta - [f(\theta) + \lambda b\eta]\partial_z\zeta\} + \int_{\Gamma_N} g(x,y,z,\theta)\zeta = 0$$

$$\forall \zeta \in H^1(\Omega); \zeta = 0 \quad \text{on} \quad \Gamma_D \tag{6}$$

For our existence result, we shall assume that $f = f(\theta): R \to R$ is a continuous function; $g = g(x,y,z,\theta): \Gamma_N \times R \to R$ is a Caratheodory function, i. e., it is measurable in $(x,y,z) \in \Gamma_N$ for all $\theta \in R$ and continuous in θ for a. e. (x,y,z). Furthermore, letting μ and M be given constants, for a. e. $(x,y,z) \in \Gamma_N$, we assume

$$g(x,y,z,\theta)\theta \ge 0 \qquad \text{if} \quad \theta \le \mu < 0 \quad \text{or} \quad \theta \ge M > 0$$

$$\forall L > 0, \exists\ \bar{g}_L \in L^q(\Gamma_N), \quad q > n-1, \text{such that}$$

$$|g(x,y,z,\theta)| \le \bar{g}_L(x,y,z), \quad \text{for} \quad |\theta| < L;$$

$$h \in C^{0,1}(\bar{\Gamma}_D), \mu \le h|_{\Gamma_0} < 0 \quad \text{and} \quad 0 < h|_{\Gamma_H} \le M$$

Under preceding conditions for f, g and h, Rodrigues had proved that there exists a solution $(\theta, \zeta) \in H^1(\Omega) \cap C^{0,\alpha}(\bar{\Omega}) \times L^\infty(\Omega)$ for some fixed $0 < \alpha < 1$ (see Theorem 1 in [4]).

3. Proof of Uniqueness of Weak Solution with Linear Cooling

We shall assume

$$f \in C^0(R) \cap C^1(R\backslash\{0\}) \quad , \quad \beta_1 \ge f' \ge \beta_2 > 0 \quad \text{in} \quad (R\backslash\{0\}) \tag{7}$$

$$g(x,y,z,\theta) = \gamma(\theta - \rho) \qquad \text{on} \quad \Gamma_H \tag{8}$$

$$h \in C^{0,1}(\bar{\Gamma}_D) \qquad \mu \le h < 0 \text{ on } \Gamma_0 \text{ and } 0 < h \le M \text{ on } \Gamma_H \tag{9}$$

$$\rho \in L^\infty(\Gamma_N), \qquad \mu \le \rho \le M \tag{10}$$

Here γ is a positive constant denoting the cooling coefficient, $\rho \in L^\infty$ is a given function representing known temperature, μ and M are constants.

If $(\hat{\theta}, \hat{\eta})$ is a weak solution gotten in existence theorem; (θ, η) is an arbitrary weak solution, we shall prove that

$$\theta = \hat{\theta}, \quad \eta = \hat{\eta}, \quad \text{a. e. in } \Omega$$

Lemma 1 *Arbitrary weak solution is bounded*: $\mu \leq \theta \leq M$.

Proof In (6), we take $\zeta = (\theta - M)^+$, then

$$\int_\Omega \{\nabla \theta \circ \nabla (\theta - M)^+ - (f(\theta) + \lambda b \eta) \partial_z (\theta - M)^+\} + \gamma \int_{\Gamma_N} (\theta - \rho)(\theta - M)^+ = 0$$

From (10), we get

$$\int_\Omega |\nabla (\theta - M)^+|^2 = \int_{\{\theta > M\}} [f(\theta) + \lambda b \eta] \partial_z \theta - \int_{\Gamma_N} \gamma (\theta - \rho)(\theta - M)^+$$

$$\leq \int_{\{\theta > M\}} [f(\theta) + \lambda b \eta] \partial_z \theta$$

From (5), we know that $\eta = 1$ in $\{\theta > M\}$, therefore

$$\int_\Omega |\nabla (\theta - M)^+|^2 \leq \int_{\{\theta > M\}} [f(\theta) + \lambda b] \partial_z \theta$$

$$= \int_\Omega \partial_z F_M(\theta)$$

$$= \int_{\Gamma_D} F_M(h) n_z = 0$$

where

$$F_M(\theta) = \begin{cases} \int_M^\theta [f(\tau) + \lambda b] d\tau & \text{if} \quad \theta \geq M \\ 0 & \text{if} \quad \theta < M \end{cases}$$

and $\theta = h$ on Γ_D.

From this, we conclude $\theta \leq M$ a. e. in Ω. Similarly we get $\theta \geq \mu$ in Ω by taking $\xi = (\theta - \mu)^-$. (Lemma 1 has been proved.)

From the definition of weak solution, we have

$$\int_\Omega \{\nabla (\theta - \hat{\theta}) \nabla \zeta - [f(\theta) - f(\hat{\theta}) + \lambda b (\eta - \hat{\eta})] \partial_z \zeta\} + \int_{\Gamma_N} \gamma (\theta - \hat{\theta}) \zeta = 0, \quad \forall \zeta \in H^1(\Omega), \zeta = 0 \text{ on } \Gamma_D \tag{11}$$

If $\Delta \zeta \in L^2(\Omega)$, n is outer normal direction of $\partial \Omega$, after integrating by parts, then

$$-\int_\Omega \{(\theta - \hat{\theta}) \Delta \zeta + [f(\theta) - f(\hat{\theta}) + \lambda b (\eta - \hat{\eta})] \partial_z \zeta\} + \int_{\partial \Omega} (\theta - \hat{\theta}) \frac{\partial \zeta}{\partial n} + \int_{\Gamma_N} \gamma (\theta - \hat{\theta}) \zeta = 0$$

i. e.

$$-\int_\Omega [f(\theta) - f(\hat{\theta}) + \lambda b (\eta - \hat{\eta})](e \Delta \zeta + \partial_z \zeta) + \int_{\Gamma_N} (\theta - \hat{\theta}) \left(\frac{\partial \zeta}{\partial n} + \gamma \zeta \right) = 0 \tag{12}$$

Here

$$e(x,y,z) = \begin{cases} \dfrac{\theta - \hat{\theta}}{f(\theta) - f(\hat{\theta}) + \lambda b(\eta - \hat{\eta})} & \text{if} \quad \theta \neq \hat{\theta} \\ \dfrac{1}{f'(\theta)} & \text{if} \quad \theta = \hat{\theta} \neq 0 \\ 0 & \text{if} \quad \theta = \hat{\theta} = 0 \end{cases}$$

From (7) and Lemma 1, e is non-negative bounded measurable. Let $\{\tilde{e}_m\} \in C^\infty(\bar{\Omega})$, $0 \leq \tilde{e}_m \leq C$, C is independent of m, $\|\tilde{e}_m - e\|_{L^2(\Omega)} \leq \frac{1}{m}$ and let $e_m = \tilde{e}_m + \frac{1}{m}$, then

$$\|e_m - e\|_{L^2(\Omega)} \leq \frac{1}{m}\{1 + (\text{meas}\Omega)^{1/2}\}$$

$$\left\| \frac{e_m - e}{\sqrt{e_m}} \right\|_{L^2(\Omega)} \leq \frac{1}{m}\{1 + (\text{meas}\Omega)^{1/2}\} \tag{13}$$

For any $u(x,y,z) \in C_0^\infty(\Omega)$, we denote by ζ_m the solution of

$$\begin{cases} \Delta\zeta_m + \dfrac{1}{e_m}\partial_z\zeta_m = u & \text{in} \quad \Omega & (14) \\ \zeta_m = 0 & \text{on} \quad \Gamma_D & (15) \\ \dfrac{\partial\zeta_m}{\partial n} + \gamma\zeta_m = 0 & \text{on} \quad \Gamma_N & (16) \end{cases}$$

Lemma 2 *$\{\zeta_m\}$ are uniformly bounded: $|\zeta_m| \leq C$, C is independent of m, depends on u.*

Proof Let Ω lie in the slab $0 < z < d$ and set $\mathscr{L} = \Delta + \frac{1}{e_m}\partial_z$, then $\mathscr{L}e^z = e^z \geq 1$ in Ω. Let $V = \sup\limits_\Omega |u| \left(\frac{\gamma+1}{\gamma}e^d - e^z\right)$, since

$$\mathscr{L}V = \sup_\Omega |u|(-e^z) \leq -\sup_\Omega |u| \qquad \text{in} \quad \Omega$$

and

$$\mathscr{L}(V - \zeta_m) \leq -\sup_\Omega |u| - u \leq 0 \qquad \text{in} \quad \Omega$$

$$V - \zeta_m \geq 0 \qquad \text{on} \quad \Gamma_D$$

we have

$$\frac{\partial(V - \zeta_m)}{\partial n} + \gamma(V - \zeta_m) = \frac{\partial V}{\partial n} + \gamma V$$

$$= \partial_z V\cos(z,n) + \gamma V$$

$$= -\sup_\Omega |u| e^z \cos(z,n) + \gamma \sup_\Omega |u| \left(\frac{\gamma+1}{\gamma}e^d - e^z\right)$$

$$\geq -\sup_\Omega |u| e^z + \gamma \sup_\Omega |u| \left(\frac{\gamma+1}{\gamma}e^d - e^z\right)$$

$$= \sup_\Omega |u| (\gamma + 1)(e^d - e^z) \qquad \text{on} \quad \Gamma_N$$

From minimum principle, we know $V - \zeta_m \geq 0$ in Ω, i. e. $\zeta_m \leq V$. Replacing ζ_m by $-\zeta_m$, we obtain $-V \leq \zeta_m$. It follows that: $|\zeta_m| \leq |V| \leq C$, C is independent of m, depends on u and d. (Lemma 2 has been proved.)

Multiplying (14) by $e_m\Delta\zeta_m$ and integrating over Ω, we get, after using

$$|\int_\Omega e_m u \Delta\zeta_m| \le \frac{1}{2}\int_\Omega e_m u^2 + \frac{1}{2}\int_\Omega e_m (\Delta\zeta_m)^2$$

the inequality

$$\frac{1}{2}\int_\Omega e_m(\Delta\zeta_m)^2 + \int_\Omega \partial_z\zeta_m \Delta\zeta_m \le \frac{1}{2}\int_\Omega e_m u^2 \tag{17}$$

Considering

$$\int_\Omega \partial_z\zeta_m\Delta\zeta_m = \int_{\partial\Omega}\partial_z\zeta_m\frac{\partial\zeta_m}{\partial n} - \frac{1}{2}\int_\Omega \partial_z|\nabla\zeta_m|^2$$

$$= \int_{\Gamma_N}\partial_z\zeta_m\frac{\partial\zeta_m}{\partial n} + \int_{\Gamma_D}|\partial_z\zeta_m|^2 n_z - \frac{1}{2}\int_{\Gamma_D}|\nabla\zeta_m|^2 n_z$$

$$= \int_{\Gamma_N}\partial_z\zeta_m\frac{\partial\zeta_m}{\partial n} + \frac{1}{2}\int_{\Gamma_D}|\partial_z\zeta_m|^2 n_z$$

and

$$\int_{\Gamma_N}\partial_z\zeta_m\frac{\partial\zeta_m}{\partial n} = \int_{\Gamma_N}\partial_z\zeta_m(-\gamma\zeta_m) = -\frac{1}{2}\gamma\int_{\Gamma_N}\partial_z(\zeta_m)^2$$

using (15), we know that

$$\int_{\Gamma_N}\partial_z(\zeta_m)^2 = \int_{\partial\Gamma}\left(\int_0^H \partial_z(\zeta_m)^2 dz\right)ds = \int_{\partial\Gamma}(\zeta_m)^2|_0^H ds = 0$$

Therefore

$$\int_\Omega \partial_z\zeta_m\Delta\zeta_m = \frac{1}{2}\int_{\Gamma_H}|\partial_z\zeta_m|^2 - \frac{1}{2}\int_{\Gamma_0}|\partial_z\zeta_m|^2$$

Substituting it into (17), we obtain

$$\int_\Omega e_m(\Delta\zeta_m)^2 + \int_{\Gamma_H}|\partial_z\zeta_m|^2 \le \int_{\Gamma_0}|\partial_z\zeta_m|^2 + \int_\Omega e_m u^2 \tag{18}$$

In (12), we choose $\zeta=\zeta_m$, then

$$\int_\Omega[f(\theta)-f(\hat\theta)+\lambda b(\eta-\hat\eta)]e_m u$$

$$= \int_\Omega[f(\theta)-f(\hat\theta)+\lambda b(\eta-\hat\eta)](e_m-e)\Delta\zeta_m$$

From Lemma 1 and (13), (18), we have

$$\left|\int_\Omega[f(\theta)-f(\hat\theta)+\lambda b(\eta-\hat\eta)]e_m u\right|$$

$$\le C(\theta,\hat\theta)\int_\Omega|(e_m-e)\Delta\zeta_m|$$

$$\le C\left\|\frac{e_m-e}{\sqrt{e_m}}\right\|_{L^2(\Omega)}\|\sqrt{e_m}\Delta\zeta_m\|_{L^2(\Omega)}$$

$$\le \frac{C}{\sqrt{m}}\left[\int_{\Gamma_0}\left|\frac{\partial\zeta_m}{\partial z}\right|^2 + \int_\Omega e_m u^2\right] \tag{19}$$

Now we estimate $|\partial_z\zeta_m(x,y,0)|$.

Lemma 3 *$|\partial_z\zeta_m(x,y,0)|$ are uniformly bounded; $|\partial_z\zeta_m(x,y,0)|\le C$, C is indepen-*

dent of m.

Proof Since $\hat{\theta}\in C^{0,\alpha}(\overline{\Omega})$, we know that there exists $\delta>0$, such that: $\hat{\theta}\leq r<0$ in $\Omega\cap\{0<z<\delta\}$, here r is a negative constant. Considering (7) in this subset, we obtain

$$e(x,y,z)=\begin{cases}\dfrac{\theta-\hat{\theta}}{f(\theta)-f(\hat{\theta})+\lambda b\eta} & \text{if} \quad \theta\neq\hat{\theta}\\ \dfrac{1}{f'(\theta)}\geq\dfrac{1}{\beta_1} & \text{if} \quad \theta=\hat{\theta}<0\end{cases} \tag{20}$$

when $\theta\neq\hat{\theta}$:

(1) $\mu\leq\theta<0$, then $e(x,y,z)=\dfrac{\theta-\hat{\theta}}{f(\theta)-f(\hat{\theta})}=\dfrac{1}{f'(\bar{\theta})}\geq\dfrac{1}{\beta_1}$ (21)

(2) $0\leq\theta\leq M$, then $e(x,y,z)=\dfrac{\theta-\hat{\theta}}{f(\theta)-f(\hat{\theta})+\lambda b}$

Since $\mu\leq\hat{\theta}\leq r$, we have

$$\theta-\hat{\theta}\geq 0-r=-r$$
$$f(\theta)-f(\hat{\theta})+\lambda b\leq f(M)-f(\mu)+\lambda b$$

Therefore

$$e(x,y,z)\geq\frac{-r}{f(M)-f(\mu)+\lambda b} \tag{22}$$

From (20)—(22), we know that

$$e(x,y,z)\geq C>0 \qquad \text{in} \quad \Omega\cap\{0<z<\delta\}$$

here C is independent of m.

Thus, there exists $\delta'>0$, such that when $0<z<\delta'$, $e_m\geq\dfrac{C}{2}>0$, $m=1,2,\cdots$. Using Lemma 2 and standard barrier function technique, we can obtain $|\partial_z\zeta_m(x,y,0)|\leq C'$, where C' is independent of m and depends on u. (Lemma 3 has been proved.)

From Lemma 3, (19) becomes

$$\left|\int_\Omega[f(\theta)-f(\hat{\theta})+\lambda b(\eta-\hat{\eta})]e_m u\right|\leq\frac{C}{\sqrt{m}}(1+\int_\Omega e_m u^2)$$

Let $m\to\infty$, then

$$\left|\int_\Omega[f(\theta)-f(\hat{\theta})+\lambda b(\eta-\hat{\eta})]eu\right|=0$$

From the arbitrarity of u, we know

$$[f(\theta)-f(\hat{\theta})+\lambda b(\eta-\hat{\eta})]e=0 \qquad \text{a.e.} \quad \text{in } \Omega$$

From the definition of e, we can obtain

$$\theta=\hat{\theta} \qquad \text{a.e. in} \quad \Omega$$

From (11), it follows that

$$\int_\Omega(\eta-\hat{\eta})\partial_z\zeta=0 \qquad \forall\,\zeta\in H^1(\Omega),\zeta=0 \quad \text{on} \quad \Gamma_D$$

then $\partial_z(\eta-\hat{\eta})=0$ in $D'(\Omega)$, i.e. $\eta-\hat{\eta}$ is a function $F(x,y)$.

Since $\eta-\hat{\eta}=0$ in $0<z<\delta$, it follows that $\eta=\hat{\eta}$ a.e. in Ω.

The uniqueness of weak solution has been proved.

Remark 1 In some cases $F(\theta)=\begin{cases}\alpha_1\theta,\theta\geq0\\ \alpha_2\theta,\theta<0\end{cases}$, α_1,α_2 are positive constants. The condition (7) has included this case.

Remark 2 In (9), γ is positive constant denoting the cooling coefficient, therefore when $g(x,y,z,\theta)$ is linear for θ, the uniqueness has been solved, but when $g(x,y,z,\theta)$ is nonlinear for θ, the uniqueness is open.

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