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BOUNDARY-VALUE PROBLEMS FOR INTEGRO-DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE①

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Abstract In this paper we study the existence of solutions to the Dirichlet problem for a class of integro-differential equations of elliptic type by using the weakly continuous method.

Key Words Integro-differential equations; weakly continuous operator; Choquard equation; weak solutions.

Classifications 45K05; 35J60.

0. Introduction

The integro-diffferential equations of elliptic type occur in many practical models in nuclear physics, theory of quantum field and mechanics.

Ugowski [1] and Tsai Longyi [2] considered the following problem

$$a_{ij}(x)D_{ij}u+b_i(x)D_iu=f(x,u,K(u)),\quad x\in\Omega \tag{0.1}$$

$$u|_{\partial\Omega}=\varphi(x) \tag{0.2}$$

where $K(u)$ denotes an integral operator, and $\Omega\subset R^m$ is a bounded region.

Ugowski discussed the existence of (0.1), (0.2) by using a successive approximation. Tsai Longyi discussed the existence of (0.1), (0.2) by combining methods of supersolution-subsolution and topological degree. Politjukov [3] defined a concept concerning ε-supersolution and ε-subsolution, and discussed parabolic equations by using this method.

What we shall discuss is the following problem

$$\sum_{|\alpha|,|\beta|=n}(-1)^{n}D_\alpha(a_{\alpha,\beta}(x,\Lambda u,R(u))D_\beta u)+\sum_{|\gamma|\leqslant n}(-1)^{|\gamma|}D_\gamma b_\gamma(x,\Lambda u,R(u))=0,\quad x\in\Omega \tag{0.3}$$

$$D_\gamma u|_{\partial\Omega}=0,\quad \forall\ |\gamma|\leqslant n-1 \tag{0.4}$$

where $\Lambda u=(D_\gamma u,|\gamma|\leqslant n-1)$, $R(u)$ is an integral operator acting on Λu, and $\Omega\subset R^m$ is an arbitrary region.

1. The Existence Theorem of the Weakly Continuous Operator Equations

Let X be a linear space, X_1, X_2 be the completions of X with respect to the norm

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$\|\cdot\|_1$ and $\|\cdot\|_2$ respectively, X with respect to $\|\cdot\|_2$ be a separable linear normed space. X_1 be a reflexive Banach space. $x_n \rightharpoonup x_0$ denotes weak convergence and $x_n \to x_0$ denotes strong convergence.

Definition 1.1 *A mapping $G: X_1 \to X_2^*$ is called weakly continuous if for any $x_n, x_0 \in X_1, x_n \rightharpoonup x_0$, there is*

$$\lim_{n\to\infty} \langle Gx_n, y\rangle = \langle Gx_0, y\rangle, \quad \forall\, y \in X_2$$

Theorem 1.2 *Let $G: X_1 \to X_2^*$ be a weakly continuous mapping. If there exists a bounded open set Ω of $X_1, O \in \Omega$, such that*

$$\langle Gu, u\rangle \geqslant 0, \quad \forall\, u \in \partial\Omega \cap X \tag{1.1}$$

then $Gu=0$ has a solution u_0 in X_1, and $u_0 \in \overline{\mathrm{co}\Omega}$.

Proof Take $\{e_i\} \subset X$, such that it is dense in X_2, and denote $\widetilde{X}_n = \mathrm{span}\{e_1, \cdots, e_n\}$, $\widetilde{X}_n$ has the same norm as that of X_1. Define the mapping $A_n: \widetilde{X}_n \to \widetilde{X}_n^*$ as

$$\langle A_n u, v\rangle = \langle Gu, v\rangle, \quad \forall\, u, v \in \widetilde{X}_n$$

It is easy to derive the continuity of A_n from the weak continuity of G. By (1.1) we have

$$\langle A_n u, u\rangle = \langle Gu, u\rangle \geqslant 0, \quad \forall\, u \in \partial\Omega \cap \widetilde{X}_n$$

Using the acute angle principle [4] of the topological degree, there exists $u_n \in \overline{\Omega} \cap \widetilde{X}_n$ such that $\langle A_n u_n, v\rangle = \langle Gu_n, v\rangle = 0, \quad \forall\, v \in \widetilde{X}_n$.

Since $\{u_n\}$ is bounded in X_1 and X_1 is reflexive, let, say, $u_n \rightharpoonup u_0 \in X_1$, hence it follows that

$$\lim_{k\to\infty} \langle Gu_k, v\rangle = \langle Gu_0, v\rangle = 0, \quad \forall\, v \in \widetilde{X}_n$$

Because $\bigcup_n \widetilde{X}_n$ is dense in X_2, we have

$$\langle Gu_0, v\rangle = 0, \quad \forall\, v \in X_2$$

i.e., $Gu_0 = 0$. Therefore the theorem is proved.

2. The Elliptic Dirichlet Problem

We consider the following problem

$$\sum_{|\alpha|,|\beta|=n} (-1)^n D_\alpha(a_{\alpha,\beta}(x, \Lambda u, R(u)) D_\beta u) + \sum_{|\gamma|\leqslant n} (-1)^{|\gamma|} D_\gamma b_\gamma(x, \Lambda u, R(u)) = f(x), \quad x \in \Omega \tag{2.1}$$

$$D_\gamma u|_{\partial\Omega} = 0, \quad |\gamma| \leqslant n-1 \tag{2.2}$$

where $\Lambda u = \{D_\alpha u \mid |\alpha| \leqslant n-1\}$, $R(u)$ is an integral operator acting on Λu and $\Omega \subset R^m$ is any region.

First of all, some comments must be made for the related notations of the anisotropic Sobolev space. We denote

$$W^{p_\alpha}_{|\alpha|\leqslant k}(\Omega) = \{u \in L^{p_0}(\Omega), p_0 \geqslant 1 \mid D_\alpha u \in L^{p_\alpha}(\Omega), |\alpha| \leqslant k, p_\alpha \geqslant 1 \text{ or } p_\alpha = 0\}$$

with the norm

$$\|u\| = \sum_{|\alpha|\leqslant k} \mathrm{sign}\ p_\alpha \|D_\alpha u\|_{L^{p_\alpha}}$$

Note that when $\forall\ |\alpha|>0, p_\alpha=0, W^{p_\alpha}_{|\alpha|\leqslant k}(\Omega)=L^{p_0}(\Omega)$.

When $|\alpha|=k$, and all $p_\alpha=p$, it is denoted by $W^{p,p_\alpha}_{k,|\alpha|\leqslant k-1}(\Omega)$. $\dot{W}^{p_\alpha}_{|\alpha|\leqslant k}(\Omega)$ denotes a completion of $C_0^\infty(\Omega)$ under the norm $W^{p_\alpha}_{|\alpha|\leqslant k}(\Omega)$.

$q_\theta(|\theta|\leqslant k)$ is called the critical embedding index from $W^{p_\alpha}_{|\alpha|\leqslant k}(\Omega)$ to $L^p(\Omega)$, if q_θ is the maximum number of index p of $D_\theta u\in L^p(\Omega)$ for any $u\in W^{p_\alpha}_{|\alpha|\leqslant k}(\Omega)$ and it is continuous as an embedding operator.

For example, when Ω is bounded and smooth, the space

$$Y=\{u\in L^k(\Omega), k\geqslant 1\,|\,D_i u\in L^2(\Omega), 1\leqslant i\leqslant m\}$$

with the norm $\|u\|=\|\nabla u\|_{L^2}+\|u\|_{L^k}$ is an anisotropic Sobolev space with the critical embedding indexes $q_i=2(1\leqslant i\leqslant m), q_0=\max\left\{k,\dfrac{2m}{m-2}\right\}$ from Y to $L^p(\Omega)$.

We assume that all functions $f:\Omega\times\boldsymbol{R}^N\to\boldsymbol{R}$ satisfy the Caratheodory condition. We shall introduce several lemmas as follows.

Lemma 2.1 *Assume that* mes $G<\infty, G\subset\boldsymbol{R}^m$ *is measurable. If the sequence* $(v_{1k}(x),\cdots,v_{Nk}(x))$ *converges to* $(v_1(x),\cdots,v_N(x))$ *in measure on* G, *then* $f(x,v_{1k},\cdots,v_{Nk})$ *converges to* $f(x,v_1,\cdots,v_N)$ *in measure on* G *(see [4]).*

Lemma 2.2 *Assume that* $E\subset\boldsymbol{R}^m$ *is a measurable set of finite measure and* $\{f_k\}\subset L^p(E)$ *is bounded,* $p>1$. *If* $f_0\in L^p(E)$ *such that* f_n *converges to* f_0 *in measure on* E, *then for any* $1\leqslant q<p, f_n\to f_0$ *is in* $L^q(E)$.

This lemma is a particular case of Theorem 8.22 in [5].

Lemma 2.3 *Assume that* $f:\Omega\times\boldsymbol{R}^N\to\boldsymbol{R}$ *satisfies the following condition*

$$|f(x,z_1,\cdots,z_N)|\leqslant C\sum_{i=1}^N|z_i|^{p_i/q}+b(x) \tag{2.3}$$

where $p_i>1, q>1, C>0$ *are constants,* $b\in L^q(\Omega)$. *If* $\{u_{ik}\}\subset L^{p_i}(\Omega)(1\leqslant i\leqslant N)$ *are bounded and* $u_i\in L^{p_i}(\Omega)$, *for any bounded subregion* Ω_0 *of* Ω, $u_{ik}(x)$ *converges to* u_i *in measure on* Ω_0, *then for any* $v\in L^{q'}(\Omega), 1/q+1/q'=1$, *there is*

$$\lim_{k\to\infty}\int_\Omega f(x,u_{1k},\cdots,u_{Nk})v\,dx=\int_\Omega f(x,u_1,\cdots,u_N)v\,dx \tag{2.4}$$

Proof Define a mapping $f:L^{p_1}(\Omega)\oplus\cdots\oplus L^{p_N}(\Omega)\to L^q(\Omega)$ by

$$\langle fu,v\rangle=\int_\Omega f(x,u_1,\cdots,u_N)v\,dx,\quad v\in L^{q'}(\Omega)$$

By (2.3), f is a bounded mapping. Since $C_0^\infty(\Omega)$ is dense in $L^{q'}(\Omega)$, it suffices to prove that (2.4) is true for any $v\in C_0^\infty(\Omega)$.

For any $v\in C_0^\infty(\Omega)$, there exists a bounded subregion Ω_0 of Ω such that supp$v\subset\Omega_0$. Therefore what we must do is to verify the following equality

$$\lim_{k\to\infty}\int_{\Omega_0}f(x,u_{1k},\cdots,u_{Nk})v\,dx=\int_{\Omega_0}f(x,u_1,\cdots,u_N)v\,dx \tag{2.5}$$

According to the assumption and (2.3), $\{f(x,u_{1k},\cdots,u_{Nk})\}$ is bounded in $L^q(\Omega_0)$. By Lemma 2.1, $f(x,u_{1k},\cdots,u_{Nk})$ converges to $f(x,u_1,\cdots,u_N)$ in measure on Ω_0. Hence, by Lemma 2.2, $\forall\ \tilde{q}\in(1,q), f(x,u_{1k},\cdots,u_{Nk})$ converges to $f(x,u_1,\cdots,u_N)$ in $L^{\tilde{q}}(\Omega_0)$. This shows that (2.5) holds. Therefore the proof of the lemma is completed.

For (2.1),(2.2), we need some assumptions as follows.

(A_1) There exists a set of differentiable functions $B_i(x,\eta,R)$, $0\leqslant i\leqslant m$ such that $B_i(x,0,R)=0$ for $1\leqslant i\leqslant m$, and

$$\sum_{|\gamma|=n} b_\gamma(x,\Lambda u,R(u))D_\gamma u=\sum_{i=1}^{m}D_iB_i(x,\Lambda u,R(u))+B_0(x,\Lambda u,R(u))$$

(A_2) There exists a constant $C_1>0$ such that

$$C_1|\xi|^2\leqslant\sum_{|\alpha|,|\beta|=n}a_{\alpha,\beta}(x,\eta,R)\xi_\alpha\xi_\beta \tag{2.6}$$

$$C_1\sum_{|\lambda|\leqslant n-1}|D_\lambda u|^{p_\lambda}-f_1(x)\leqslant\sum_{|\lambda|\leqslant n-1}b_\lambda(x,\Lambda u,R(u))D_\lambda u+B_0(x,\Lambda u,R(u)) \tag{2.7}$$

where $p_0>1$, either $p_\lambda>1$, or $p_\lambda=0$ for $0<|\lambda|\leqslant n-1$, and $f_1\in L^1(\Omega)$.

(A_3) The constructivity condition

$$|a_{\alpha,\beta}(x,\Lambda u,R(u))|\leqslant C_3\sum_{|\gamma|\leqslant n-1}|D_\gamma u|^{S_\gamma}+g_1(u) \tag{2.8}$$

$$|b_\lambda(x,\Lambda u,R)|\leqslant C_2\sum_{|\gamma|\leqslant n-1}|D_\gamma u|^{T_\gamma}+g_2(u) \tag{2.9}$$

where $T_\gamma<q_\gamma$, $S_\gamma<q_\gamma/2$. q_γ is a critical embedding index from $\dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega)$ to $L^\gamma(\Omega)$. $g_1:\dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\tilde\Omega)\to L^{\tilde p_1}(\tilde\Omega)$ and $g_2:\dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\tilde\Omega)\to L^{\tilde p_2}(\tilde\Omega)$ are bounded integral operators, $2<\tilde p_1\leqslant\infty$, $1<\tilde p_2\leqslant\infty$, and $\tilde\Omega\subset\Omega$ is any bounded subregion.

(A_4) The integral operator $R(u)=\int_\Omega G(x,y,\Lambda u(y))dy$ satisfies the following condition

$$|G(x,y,\eta)|\leqslant\sum_{|\gamma|\leqslant n-1}|f_\gamma(x,y)||\eta_\gamma|^{\bar S_\gamma}+f_2(x,y) \tag{2.10}$$

where $2\leqslant\bar S_\gamma<q_\gamma$, q_γ is the same as in (A_3), $f_\gamma(x_0,y)\in L^{t_\gamma}(\Omega\backslash\Omega_0)$, $f_\gamma(x_0,y)\in L^{r_\gamma}_{loc}(\Omega)$, for any $x_0\in\Omega$, and Ω_0 the neighbourhood of x_0. $t_\gamma,r_\gamma>q_\gamma(q_\gamma-\bar S_\gamma)^{-1}$. $F_2(x)=\int_\Omega f_2(x,y)dy\in L^1_{loc}(\Omega)$.

Let $X_1=\dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega)$. $u\in X_1$ is called the weak solution of the problem (2.1), (2.2), if for any $v\in C_0^\infty(\Omega)$,

$$\int_\Omega\Big[\sum_{|\alpha|,|\beta|=n}a_{\alpha,\beta}(x,\Lambda u,R(u))D_\beta uD_\alpha v+\sum_{|\gamma|\leqslant n}b_\gamma(x,\Lambda u,R(u))D_\gamma v-fv\Big]dx=0 \tag{2.11}$$

Remark 2.4 If $p_\lambda=0$ in (2.7) and $\forall\ 0<|\lambda|\leqslant n-1$, X_1 is the completion of $C_0^\infty(\Omega)$ under the following norm: $\|u\|=\|D^n u\|_{L^2}+\|u\|_{L^{p_0}}$.

Theorem 2.5 *Under the assumptions of* (A_1)–(A_4), *if* $f\in L^{p'_0}(\Omega)$, *then the problem* (2.1), (2.2) *has a weak solution in* X_1.

Proof Take a bounded subregion $\Omega_k\subset\Omega$, $\Omega_k\subset\Omega_{k+1}$, $\bigcup_k\Omega_k=\Omega$, for any k. We shall prove first that there exists $u_k\in\dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega_k)$ which satisfies

$$\int_{\Omega_k}\Big[\sum_{|\alpha|,|\beta|=n} a_{\alpha,\beta}(x,\Lambda u_k,R(u_k))D_\beta u_k D_\alpha v + \sum_{|\gamma|\leqslant n} b_\gamma(x,\Lambda u_k,R(u_k))D_\gamma v - fv\Big]dx = 0 \tag{2.12}$$

$\forall\, v\in C_0^\infty(\Omega)$, and there is

$$\Big[\int_{\Omega_k}\sum_{|\alpha|=n}|D_\alpha u_k|^2 dx\Big]^{1/2} + \sum_{|\lambda|\leqslant n-2}\|D_\lambda u_k\|_{L^{p_\lambda}(\Omega_k)} \leqslant C \tag{2.13}$$

where the constant $C>0$ is independent of k.

Denote X_2 as the completion of $C_0^\infty(\Omega_k)$ under the norm C^n and the left side of (2.12) as $\langle Fu,v\rangle_k$. By the conditions (A_3) and (A_4), it is easy to derive that $\langle Fu,v\rangle_k$ defines a bounded mapping $F:\dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega_k)\to X_2^*$.

By (A_1) and (A_2), for any $u\in C_0^\infty(\Omega_k)$, we have

$$\begin{aligned}\langle Fu,u\rangle_k &= \int_{\Omega_k}\Big[\sum_{|\alpha|,|\beta|=n} a_{\alpha,\beta}(x,\Lambda u,R(u))D_\alpha u D_\beta u \\ &\quad + \sum_{|\gamma|\leqslant n-1} b_\gamma(x,\Lambda u,R(u))D_\gamma u + B_0(x,\Lambda u,R(u)) - f(x)u\Big]dx \\ &\geqslant \int_{\Omega_k}\Big[C_1\sum_{|\alpha|=n}|D_\alpha u|^2 + C_1\sum_{|\lambda|\leqslant n-1}|D_\lambda u|^{p_\lambda} \\ &\quad - f_1(x) - \frac{1}{p_0'\varepsilon}|f|^{p_0'} - \frac{\varepsilon}{p_0}|u|^{p_0}\Big]dx\end{aligned} \tag{2.14}$$

where $\varepsilon>0$ is an arbitrary positive number. Since $f_1\in L^1(\Omega)$, there exists a constant $M>0$ independent of k, such that when $u\in C_0^\infty(\Omega_k)$, $\|u\|_{\dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}}=C$, there is $\langle Fu,u\rangle\geqslant 0$.

By Theorem 1.2, if it can be proved that $F:\dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega_k)\to X_2^*$ is weakly continuous, then there exists $u_k\in \dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega_k)$ which satisfies (2.12). From the inequality (2.14) and the result of Theorem 1.2, it follows again that u_k satisfies (2.13).

In what follows we shall prove the weak continuity of F.

Assume that $u^N(x)\rightharpoonup u_k(x)$ is in $\dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega_k)$ (k is fixed), hence it is obvious that Λu^N converges to Λu_k in measure on Ω_k. First, we shall prove that the integral operator $R(u^N)$ converges to $R(u_k)$ in measure on Ω_k. What we have to do is to verify that for each term of $R(u)$

$$\lim_{N\to\infty}\int_{\Omega_k} G(x_0,y,\Lambda u^N(y))dy = \int_{\Omega_k} G(x_0,y,\Lambda u_k(y))dy \tag{2.15}$$

holds for any $x_0\in\Omega_k$.

In fact if we make an extension for $u\in \dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega_k)$ that

$$\tilde u(x) = \begin{cases} u(x), & x\in\Omega_k \\ 0, & x\in\Omega\backslash\Omega_k\end{cases} \tag{2.16}$$

then $\tilde u\in \dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega)$. Note that

$$\int_\Omega [G(x,y,\Lambda\tilde u^N(y)) - G(x,y,\Lambda\tilde u_k(y))]dy = \int_{\Omega_k}[G(x,y,\Lambda u^N(y)) - G(x,y,\Lambda u_k(y))]dy$$

Thus (2. 15) can be derived from Lemma 2. 1—2. 3 and the condition (A_4).

Therefore it follows that $R(u^N)$ converges to $R(u_k)$ in measure on Ω_k.

We verify next that

$$\lim_{N\to\infty}\int_{\Omega_k}\sum_{|\alpha|,|\beta|=n} a_{\alpha,\beta}(x,\Lambda u^N,R(u^N))D_\alpha u^N D_\beta v dx$$
$$=\int_{\Omega_k}\sum_{|\alpha|,|\beta|=n} a_{\alpha,\beta}(x,\Lambda u_k,R(u_k))D_\alpha u_k D_\beta v dx \tag{2.17}$$

$$\lim_{N\to\infty}\int_{\Omega_k} b_\gamma(x,\Lambda u^N,R(u^N))D_\gamma v dx=\int_{\Omega_k} b_\gamma(x,\Lambda u_k,R(u_k))D_\gamma v dx \tag{2.18}$$

We obtain (2. 18) from Lemmas 2. 1—2. 3 and (2. 9). For (2. 17), we note that

$$\int_{\Omega_k}\sum_{|\alpha|,|\beta|=n}[a_{\alpha,\beta}(x,\Lambda u^N,R(u^N))D_\alpha u^N D_\beta v-a_{\alpha,\beta}(x,\Lambda u_k,R(u_k))D_\alpha u_k D_\beta v]dx$$
$$=\int_{\Omega_k}\sum_{|\alpha|,|\beta|=n}[a_{\alpha,\beta}(x,\Lambda u^N,R(u^N))-a_{\alpha,\beta}(x,\Lambda u_k,R(u_k))]D_\alpha u^N D_\beta v dx$$
$$+\int_{\Omega_k}\sum_{|\alpha|,|\beta|=n} a_{\alpha,\beta}(x,\Lambda u_k,R(u_k))D_\beta v(D_\alpha u^N-D_\alpha u_k)dx$$

Since $u^N\rightharpoonup u_k$ is in $\dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega_k)$, then for any $|\alpha|=n$, $D_\alpha u^N\rightharpoonup D_\alpha u_0$ is in $L^2(\Omega_k)$. For any $v\in C_0^\infty(\Omega)$ and $u\in \dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega_k)$, by the condition (2. 8), $D_\beta v a_{\alpha,\beta}(x,\Lambda u,R(u))\in L^2(\Omega_k)$. Therefore we have

$$\lim_{N\to\infty}\int_{\Omega_k}\sum_{|\alpha|,|\beta|=n} a_{\alpha,\beta}(x,\Lambda u_k,R(u_k))D_\beta v(D_\alpha u^N-D_\alpha u_k)dx=0$$

On the other hand,

$$\left|\int_{\Omega_k}\sum_{|\alpha|,|\beta|=n}(a_{\alpha,\beta}(x,\Lambda u^N,R(u^N))-a_{\alpha,\beta}(x,\Lambda u_k,R(u_k)))D_\alpha u^N D_\beta v dx\right|$$
$$\leqslant C\sum_{|\alpha|,|\beta|=n}\left[\int_{\Omega_k}|a_{\alpha,\beta}(x,\Lambda u^N,R(u^N))-a_{\alpha,\beta}(x,\Lambda u_k,R(u_k))|^2dx\right]^{1/2}$$
$$\cdot\left[\int_{\Omega_k}|D_\alpha u^N|^2dx\right]^{1/2}$$

By (2. 8), for any $u\in \dot W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega_k)$, $a_{\alpha,\beta}(x,\Lambda u,R(u))\in L^p(\Omega_k)$, $p>2$ is a constant. Again by Lemma 2. 1 and 2. 2, we have

$$\lim_{N\to\infty}\int_{\Omega_k}|a_{\alpha,\beta}(x,\Lambda u^N,R(u^N))-a_{\alpha,\beta}(x,\Lambda u_k,R(u_k))|^2dx=0$$

This implies that

$$\lim_{N\to\infty}\sum_{|\alpha|,|\beta|=n}\int_{\Omega_k}[a_{\alpha,\beta}(x,\Lambda u^N,R(u^N))-a_{\alpha,\beta}(x,\Lambda u_k,R(u_k))]D_\alpha u^N D_\beta v dx=0$$

Therefore (2. 17) holds.

Let $\{u_k(x)\}$ be a sequence satisfying (2. 12)and (2. 13). For $\{u_k\}$ we make an extension of (2. 16) which we still denote by $\{u_k(x)\}$, then $\{u_k(x)\}\subset W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega)$ is bounded. We may assume that $u_k(x)\rightharpoonup u_0(x)$ is in $W^{2,p_\lambda}_{n,|\lambda|\leqslant n-1}(\Omega)$, then Λu_k converges to Λu_0 in measure on any bounded subregion of Ω. From the condition (A_3) it follows that $a_{\alpha,\beta}(x,\Lambda u_k,R(u_k))\in L^T_{loc}(\Omega)$ is bounded, $T>2$ is a real number. $b_\gamma(x,\Lambda u_k,R(u_k))\in$

$L^S_{loc}(\Omega)$ is bounded, $S>1$ is a real number. Again since for any $v\in C_0^\infty(\Omega)$, there exists a natural number k such that $\text{supp}\,v\subset\Omega_k$, what we shall do is to prove that for any bounded subregion $\widetilde{\Omega}$ of Ω, $R(u_k)$ converges to $R(u_0)$ in measure on $\widetilde{\Omega}$.

For any $\varepsilon>0$ and any $x_0\in\Omega$, there exists $\Omega_\varepsilon\subset\Omega$ which is bounded, $x_0\in\Omega_\varepsilon$, such that

$$\left[\int_{\Omega\setminus\Omega_\varepsilon}|f_\gamma(x_0,y)|^{t_\gamma}dy\right]^{1/t_\gamma}<\varepsilon \tag{2.19}$$

$$\int_{\Omega\setminus\Omega_\varepsilon}|f_2(x_0,y)|dy<\varepsilon \tag{2.20}$$

Because $t_\gamma>q_\gamma(q_\gamma-\overline{S}_\gamma)^{-1}$, $2\leqslant t'_\gamma\overline{S}_\gamma<q_\gamma$, $\left(\frac{1}{t_\gamma}+\frac{1}{t'_\gamma}=1\right)$, by the interpolation inequality and the embedding theorem, it is obtained that

$$\int_{\Omega\setminus\Omega_\varepsilon}|f_\gamma(x_0,y)D_\gamma u(y)|^{\overline{S}_\gamma}dy\leqslant\varepsilon C\|u\|_{W^{2,r_\lambda}_{\alpha,|\lambda|\leqslant\alpha-1}} \tag{2.21}$$

where $C>0$ is a Sobolev embedding constant.

On the other hand, $r_\gamma>q_\gamma(q_\gamma-\overline{S}_\gamma)^{-1}$, $r'_\gamma\overline{S}_\gamma<q_\gamma$. By Young's inequality, from (2.10) we have

$$|G(x_0,y,\eta)|\leqslant C_1\sum_{|\gamma|\leqslant\alpha-1}|\eta_\gamma|^{S^*_\gamma}+\widetilde{f}(x_0,y) \tag{2.22}$$

where $S^*_\gamma=r'\overline{S}_\gamma$, $\widetilde{f}(x_0,y)\in L^1(\Omega_\varepsilon)$. By the property of Caratheodory operator, (2.22) implies that there exists $N_0>0$ such that when $k>N_0$,

$$\left|\int_{\Omega_\varepsilon}[G(x_0,y,\Lambda u_k(y))-G(x_0,y,\Lambda u_0(y))]dy\right|<\varepsilon$$

Combining (2.19)—(2.21), it gives

$$\begin{aligned}&\left|\int_\Omega[G(x_0,y,\Lambda u_k(y))-G(x_0,y,\Lambda u_0(y))]dy\right|\\ \leqslant&\left|\int_{\Omega_\varepsilon}[G(x_0,y,\Lambda u_k(y))-G(x_0,y,\Lambda u_0(y))]dy\right|\\ &+\left|\int_{\Omega\setminus\Omega_\varepsilon}G(x_0,y,\Lambda u_k(y))dy\right|+\left|\int_{\Omega\setminus\Omega_\varepsilon}G(x_0,y,\Lambda u_0(y))dy\right|\\ \leqslant&\ \varepsilon+\varepsilon C\|u_k\|_{W^{2,r_\lambda}_{\alpha,|\lambda|\leqslant\alpha-1}}+\varepsilon C\|u_0\|_{W^{2,r_\lambda}_{\alpha,|\lambda|\leqslant\alpha-1}}\end{aligned}$$

It follows that $R(u_k)$ converges to $R(u_0)$ in measure on any bounded subregion. Therefore the theorem is true.

The following theorem is obvious.

Theorem 2.6 *Under the assumptions of* (A_3) *and* (A_4), $f\in L^{p'_0}(\Omega)$. *If there exists a constant* $R>0$ *such that* $\langle Fu,u\rangle\geqslant0$, $\forall\ u\in C_0^\infty(\Omega)$, $\|u\|_{X_1}=R$, *then the problem* (2.1), (2.2) *has a weak solution* u_0 *in* X_1 *and* $\|u_0\|_{X_1}\leqslant R$, *where* $\langle Fu,v\rangle$ denotes the left side of (2.11).

As an application, we consider the following Choquard equation

$$-\Delta u+\lambda u-u\int_{R^3}\frac{u^2(y)}{|x-y|}dy=f(x),\quad x\in R^3 \tag{2.23}$$

where $\lambda>0$ is a constant, $f\in L^2(R^3)$.

Remark 2.7 The results of Theorem 2.8 still holds, if instead of $\lambda\geqslant\frac{3}{2}+\|f\|_{L^2}^2+\frac{\omega}{2C^4}\|f\|_{L^2}^4$, we take $\lambda\geqslant 1+\frac{\varepsilon}{2}+\|f\|_{L^2}^2+\frac{\omega}{2C^4}\cdot\frac{2\varepsilon-1}{\varepsilon}\|f\|_{L^2}^4$, $\frac{1}{2}<\varepsilon<1$.

Theorem 2.8 *If $\lambda\geqslant\frac{3}{2}+\|f\|_{L^2}^2+\frac{\omega}{2C^4}\|f\|_{L^2}^4$, then the problem (2.23) has a weak solution u_0 in $\dot{W}_2^1(R^3)$ and $\|u_0\|_{W_2^1}\leqslant\|f\|_{L^2}$, where $\omega=\frac{4}{3}\pi$ is the volume of the unit ball in R^3 and C is the optimal Sobolev embedding constant from $\dot{W}_2^1(R^3)$ to $L^4(R^3)$.*

Proof Denote $f_0(x,y)=|x-y|^{-1}$, then $G(x,y,u)=f_0(x,y)u^2$. A calculation yields $\overline{S}_0=2, q_0=6$ and $q_0(q_0-\overline{S}_0)^{-1}=1.5$. Obviously, when $t_0>3>r_0>1.5$. $\forall\, x_0\in R^3, f_0(x_0,\cdot)\in L_{loc}^{r_0}(R^3), f_0(x_0,\cdot)\in L^{t_0}(\Omega\backslash B(x_0)), B(x_0)=\{y\in R^3\,|\,\|y-x_0\|<1\}$. Thus the condition (A_4) is satisfied. Let

$$b_0(x,u,k(u)) = \lambda u - u\int_{R^3}\frac{u^2(y)}{|x-y|}dy$$

We shall verify the condition (A_3)

$$|b_0(x,u,k(u))|\leqslant u^2+\frac{1}{2}\lambda^2+\frac{1}{2}\left[\int_{R^3}u^2(y)|x-y|^{-1}dy\right]^2$$

Denote $g_0(u)=\frac{1}{2}\lambda^2+\frac{1}{2}\left[\int_{R^3}u^2(y)|x-y|^{-1}dy\right]^2$. It is easy to verify that $g_0:\dot{W}_2^1(R^3)\to L_{loc}^\infty(R^3)$.

Finally we verify the acute angle condition. Let $u\in C_0^\infty(R^3)$, then

$$\langle Fu,u\rangle = \int_{R^3}[|\nabla u|^2+\lambda|u|^2-f(x)u]dx-\int_{R^3}u^2(x)\int_{R^3}u^2(y)|x-y|^{-1}dydx$$

On the other hand, we have

$$\begin{aligned}&\int_{R^3}u^2(x)\int_{R^3}u^2(y)|x-y|^{-1}dydx\\ &\leqslant\int_{R^3}u^2(x)\left[\int_{B(x)}u^2(y)|x-y|^{-1}dy+\int_{R^3\backslash B(x)}u^2(y)dy\right]dx\\ &\leqslant\left[\int_{R^3}u^2(x)dx\right]^2+\left[\int_{R^3}u^2(x)dx\right]\left[\int_{R^3}u^4(x)dx\right]^{1/2}\omega^{1/2}\end{aligned}$$

Hence we have

$$\begin{aligned}\langle Fu,u\rangle\geqslant&\int_{R^3}[|\nabla u|^2+\lambda u^2]dx-\frac{1}{2}\|u\|_{L^2}^2-\frac{1}{2}\|f\|_{L^2}^2\\ &-\frac{\varepsilon}{2}\omega\|u\|_{L^4}^4-(1+\frac{1}{2\varepsilon})\|u\|_{L^2}^4\end{aligned}\tag{2.24}$$

Denote $\|f\|_{L^2}^2=M$. Take $\|u\|_{W_2^1}^2=M$, $\varepsilon=\frac{C^4}{M\omega}$, where $C>0$ is the optimal Sobolev embedding constant of $\|u\|_{W_2^1}\geqslant C\|u\|_{L^4}$. Hence it follows from (2.24) that

$$\begin{aligned}\langle Fu,u\rangle\geqslant&\int_{R^3}(|\nabla u|^2+u^2)dx\\ &+\left(\lambda-\frac{3}{2}-\left(1+\frac{1}{2\varepsilon}\right)\|u\|_{L^2}^2\right)\|u\|_{L^2}^2-\frac{1}{2}M-\frac{\varepsilon}{2}\omega\|u\|_{L^4}^4\end{aligned}$$

$$\geqslant\left(\lambda-\frac{3}{2}-M-\frac{\omega}{2C_4}M^2\right)\|u\|_{L^2}^2+\frac{1}{2}M-\frac{C^4}{2M}\|u\|_{L^4}^4$$

By the condition in the theorem, $\lambda\geqslant\frac{3}{2}+M+\frac{\omega}{2C^4}M^2$, and the fact; from $\|u\|_{W_2^1}=M^{1/2}$ it follows that $M^2\geqslant C^4\|u\|_{L^4}^4$, there is

$$\langle Fu,u\rangle\geqslant 0,\quad \forall\ u\in C_0^\infty(R^3), \|u\|_{W_2^1}=\|f\|_{L^2}$$

Using Theorem 2. 5, this theorem follows.

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