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SINGULARITIES OF SOLUTIONS TO CAUCHY PROBLEMS FOR SEMILINEAR WAVE EQUATIONS IN TWO SPACE DIMENSIONS

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Abstract This paper concerns the Cauchy problem for semilinear wave equations with two space variables, of which the initial data have conormal singularities on finite curves intersecting at one point on the initial plane. It is proved that the solution is of conormal distribution type, and its singularities are contained in the union of the characteristic surfaces through these curves and the characteristic cone issuing from the intersection point.

Key Words Semilinear wave equations; propagation of singularities; conormal distributions.

Classifications 35L70; 35B65.

1. Introduction

Consider the semilinear Cauchy problem

$$Pu = f(t,x,u), \qquad (t,x) \in \Omega \subset R^{1+2} \tag{1.1}$$

$$u(0,x) = g(x), \qquad \partial_t u(0,x) = h(x), \quad x \in \Omega \cap \{t=0\} \subset R^2 \tag{1.2}$$

for a second order strictly hyperbolic partial differential operator $P=P(t,x,\partial_t,\partial_x)$. We are concerned with the singularities of the solution u knowing the singularities of the Cauchy data g and h.

There has been a considerable amount of work in this direction, e. g. Bony [1], Ritter [10], Rauch and Reed [9], Metivier [6,7] and other papers cited there. The results applied to (1. 1)—(1. 2) show that the singularities of u will lie in the characteristic surfaces through the curve to which conormal singularities of g or h are confined. In case there are more than one such curves present in $\Omega \cap \{t=0\}$, the singularities of u may spread to the characteristic cones issuing from the intersection points of these curves, as was illustrated in an example given by Rauch and Reed [8]. The relevant analyses to treat this type of phenomenon were later carried out independently and by different methods in Bony [2] and in Melrose and Ritter [4], where the singularities of the solution u to (1. 1) for $t>0$ were studied knowing its singularities for $t<0$.

We study the case where the Cauchy data g and h have conormal singularities along finite C^∞ curves which can intersect each other transversally. The result shows that, locally in t, the singularities of the solution u to (1. 1)—(1. 2) are localized in the characteristic surfaces through these curves and in the characteristic cones from the intersection points of these curves. Bony [3] has announced a similar result which assumes $u \in H^s, s > 3/2$, and deals with weak singularities. In our result, besides the difference of the methods, it is assumed only $u \in L^\infty$ and the Cauchy datum h may have jump discontinuities (i. e. strong singularities) over curves.

The spaces of conormal distributions used to describe the singularities of solutions are equivalent to the spaces in [2] and [4]. We prove the main theorem by an improvement of the approach in [4], where there is loss of derivatives for the result. This defect is overcome in our treatment.

2. Notations and Statement of the Result

For the sake of simplicity, we will state and prove our result only for the special wave operator $\partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2$. The proof in this paper is valid for general second order strictly hyperbolic operators with C^∞ coefficients.

Let Ω be a bounded region of R^3 containing $O, \omega = \Omega \cap \{t=0\}$, and

$$P = \partial_t^2 - \partial_{x_1}^2 - \partial_{x_2}^2 \tag{2.1}$$

Suppose Ω is a domain of determinacy of ω with respect to P. Let $C_i, i=1, \cdots, N, N > 1$, be C^∞ curves in ω intersecting transversally at one point, say $O = (0,0,0)$. We assume that there is no other intersection of any two of these curves. The two characteristic surfaces through C_i are denoted by $S_i, S_{N+i}, i = 1, \cdots, N$, the characteristic cone from O is

$$S_0 = \{(t, x_1, x_2) : t^2 - x_1^2 - x_2^2 = 0\}$$

It is assumed that $S_j, j = 0, \cdots, 2N$, are all regular C^∞ surfaces in $\Omega \backslash O$ and have no triple intersection there, otherwise we can shrink Ω.

Following the notations in [4], [5], for any Lie algebra $\mathscr{V}_*$ of vector fields (i. e. homogeneous first order differential operators) on a C^∞ manifold $M \subset R^n$, we define the associated space of conormal distributions

$$I_k L^p(M, \mathscr{V}_*) = \{v \in L^p(M) : V_1 \cdots V_i v \in L^p(M)$$
$$\text{for any} \quad V_j \in \mathscr{V}_*, 1 \leqslant j \leqslant i \leqslant k\}, \quad (1 < p < \infty)$$

For any finite collection $\mathscr{G}$ of C^∞ submanifolds of M, define the Lie algebra of C^∞ vector fields

$$\mathscr{V}(\mathscr{G}) = \{V \in C^\infty(M, TM) : V \text{ is tangent to each submanifold in } \mathscr{G}\}$$

Now let

$$\mathscr{C}_i = \{C_i \backslash O, O\}, \quad i = 1, \cdots, N$$
$$\mathscr{S}_{ij} = \{S_i \backslash O, S_j \backslash O, O\}, \quad 0 \leqslant i < j \leqslant 2N$$

The main theorem which will be proved in Section 6 is

Theorem 2.1 *Suppose $u \in L^\infty$ is a solution to the Cauchy problem*

$$Pu = f(t, x, u), \qquad (t, x) \in \Omega \tag{2.2}$$
$$u(0, x) = g(x), \qquad \partial_t u(0, x) = h(x), \qquad x \in \omega \tag{2.3}$$

where $f \in C^\infty(\bar{\Omega} \times C^1)$, P is given by (2.1). If for integer $0 \leqslant k \leqslant \infty$, there exists some $p > 2$ such that

$$g,\partial_z g,h \in L^\infty(\omega) \cap \sum_{i=1}^{N} I_k L^p(\omega,\mathscr{V}(\mathscr{C}_i)) \tag{2.4}$$

then

$$u,\partial_{t,z}u \in \sum_{0\leqslant i<j\leqslant 2N} I_k L^2(\Omega,\mathscr{V}(\mathscr{S}_{ij})) \tag{2.5}$$

There is loss from $p>2$ in (2.4) to 2 in (2.5), while there is loss of the Sobolev space index in Bony's result. In view of Proposition 3.3 in the next section, the loss can be made up when $k=\infty$.

Theorem 2.2 *Theorem 2.1 is still true for $p=2$ if $k=\infty$.*

When $k=\infty$, the conclusion (2.5) implies $u\in C^\infty(\Omega\setminus\bigcup_{0\leqslant i\leqslant 2N} S_i)$. The spaces in (2.4) admit functions with jumps over $C_i, i=1,\cdots,N$, although g is then continuous. Particularly, (2.4) with $k=\infty$ permits $\partial_z g,h$ to be piecewise C^∞ functions with jump discontinuities over $C_i, i=1,\cdots,N$, so we have

Corollary 2.3 *Suppose $u\in L^\infty$ is a solution to (2.2)—(2.3) and*

$$g,\partial_z g,h \in \overline{C}^\infty(\omega\setminus\bigcup_{1\leqslant i\leqslant N} C_i) \quad \text{(i. e. piecewise } C^\infty)$$

Then $u\in C^\infty(\Omega\setminus\bigcup_{0\leqslant i\leqslant 2N} S_i)$.

It is interesting to ask if u in Corollary 2.3 is also piecewise C^∞, i. e. $u\in\overline{C}^\infty(\Omega\setminus\bigcup_{0\leqslant i\leqslant 2N} S_i)$, as was indicated in [9] when $N=1$. An example will be given in Section 7, however, showing that piecewise C^∞ data may not produce piecewise C^∞ solutions when $N>1$.

3. Vector Fields and the Associated Algebras

Definition *The set $M\subset R^m$ is called cone-like set (with respect to O) if*

$$\{\lambda z; 0<\lambda\leqslant 1, z\in M\cap U\}\subset M$$

for some small neighborhood U of O. If M is a cone-like open set of R^m, then define for $\sigma\in R^1$

$$C^{\infty,\sigma}(M,O)=\{\varphi\in C^\infty(M\setminus O); \forall\ \alpha, \partial_z^\alpha\varphi = O(r^{\sigma-|\alpha|}), \quad r=\sqrt{\Sigma_{i=1}^m z_i^2}\}$$

It is easy to see

$$C^\infty(M)\subset C^{\infty,0}(M,O); C^{\infty,\sigma_1}\cdot C^{\infty,\sigma_2}\subset C^{\infty,\sigma_1+\sigma_2}; \partial_z: C^{\infty,\sigma}\to C^{\infty,\sigma-1} \tag{3.1}$$

If cone-like open sets $K_\alpha, \alpha\in\mathscr{A}$, form a finite covering of $M\setminus O$, then there exists a partition of unity $1=\sum_{\sigma\in\mathscr{A}}\chi_\alpha, \chi_\alpha\in C^{\infty,0}(M,O)$, subordinate to that covering.

For any collection $\mathscr{G}$ of finite C^∞ submanifolds in $M\setminus O$, define the Lie algebra of singular vector fields

$$\mathscr{V}_s(\mathscr{G})=\{V=\sum_{i=1}^m a_i\partial_{z_i}; a_i\in C^{\infty,1}(M,O), i=1,\cdots,m, V \text{ is}$$

$$\text{tangent in } M\setminus O \text{ to each submanifold in } \mathscr{G}\}$$

Clearly $\mathscr{V}_s(\mathscr{G})$ and the associated conormal distribution space $I_k L^p(M,\mathscr{V}_s(\mathscr{G}))$ are $C^{\infty,0}(M,O)$-modules. Note that for $\mathscr{C}_i$ (respectively $\mathscr{S}_{ij}$) in the last section, $\mathscr{V}_s(\mathscr{C}_i)$

(resp. $\mathscr{V}_s(\mathscr{S}_{ij})$) is a $C^{\infty,0}$-module generated by $\mathscr{V}(\mathscr{C}_i)$ (resp. $\mathscr{V}(\mathscr{S}_{ij})$), therefore, it is easy to prove by induction on k that

$$\begin{aligned} I_kL^p(\omega,\mathscr{V}_s(\mathscr{C}_i)) &= I_kL^p(\omega,\mathscr{V}(\mathscr{C}_i)), \quad 1\leqslant i\leqslant N \\ I_kL^p(\Omega,\mathscr{V}_s(\mathscr{S}_{ij})) &= I_kL^p(\Omega,\mathscr{V}(\mathscr{S}_{ij})), \quad 0\leqslant i<j\leqslant 2N \end{aligned} \tag{3.2}$$

Setting

$$\mathscr{C} = \{C_i\backslash O: i=1,\cdots,N\}, \mathscr{S} = \{S_i\backslash O: 0\leqslant i\leqslant 2N\}$$

The following lemma is readily verified.

Lemma 3.1 *For any χ_i (resp. χ_{ij}) $\in C^{\infty,0}$, $1\leqslant i\leqslant N$ (resp. $0\leqslant i<j\leqslant 2N$), supported in a cone-like open set K_i (resp. K_{ij}) having the property:*

$$K_i(\textit{resp. } K_{ij}) \textit{ does not intersect with some cone-like neighborhood of } \bigcup_{\substack{1\leqslant i\leqslant N\\ i\neq i}}(C_i\backslash O)\ (\textit{resp. } \bigcup_{\substack{0\leqslant i\leqslant 2N\\ i\neq i,j}}(S_i\backslash O)) \tag{3.3}$$

then one has

$$\chi_i\mathscr{V}_s(\mathscr{C}_i)\subset\mathscr{V}_s(\mathscr{C})\ (\textit{resp. } \chi_{ij}\mathscr{V}_s(\mathscr{S}_{ij})\subset\mathscr{V}_s(\mathscr{S}))\ .$$

Proposition 3.2 *The spaces of conormal distributions in (2.4), (2.5) can be described as*

$$\sum_{i=1}^{N} I_kL^p(\omega,\mathscr{V}(\mathscr{C}_i)) = I_kL^p(\omega,\mathscr{V}_s(\mathscr{C})) \tag{3.4}$$

$$\sum_{0\leqslant i<j\leqslant 2N} I_kL^p(\Omega,\mathscr{V}(\mathscr{S}_{ij})) = I_kL^p(\Omega,\mathscr{V}_s(\mathscr{S})) \tag{3.5}$$

Proof It is easy to see $\mathscr{V}_s(\mathscr{C})\subset\mathscr{V}_s(\mathscr{C}_i)$, hence $I_kL^p(\omega,\mathscr{V}_s(\mathscr{C}_i))\subset I_kL^p(\omega,\mathscr{V}_s(\mathscr{C}))$, so by (3.2),

$$\sum_{i=1}^{N} I_kL^p(\omega,\mathscr{V}(\mathscr{C}_i))\subset I_kL^p(\omega,\mathscr{V}_s(\mathscr{C}))$$

For the reverse inclusion, take $u\in I_kL^p(\omega,\mathscr{V}_s(\mathscr{C}))$, then $u=\sum_{i=1}^{N}\chi_i u$ where $\chi_i\in C^{\infty,0}(\omega,O)$, $i=1,\cdots,N$, form a partition of unity subordinate to some cone-like open covering K_i of $\omega\backslash O$ satisfying the property (3.3) for $i=1,\cdots,N$. According to (3.1) and Lemma 3.1, $\chi_i u\in I_kL^p(\omega,\mathscr{V}_s(\mathscr{C}_i))$, therefore

$$I_kL^p(\omega,\mathscr{V}_s(\mathscr{C}))\subset\sum_{i=1}^{N} I_kL^p(\omega,\mathscr{C}_i))$$

thus follows (3.4). (3.5) can be proved in the same way.

The next proposition is a consequence of an inequality of Gagliardo-Nirenberg type and essentially contained in [4].

Proposition 3.3 *Suppose $\mathscr{V}_*$ is a Lie algebra of C^∞ (or singular) vector fields on M, a bounded (or bounded cone-like) open set of R^m. If $v\in L^\infty(M)\cap I_kL^p(M,\mathscr{V}_*)$ and $V_1,\cdots,V_i\in\mathscr{V}_*$, then*

$$V_1\cdots V_i v\in L^{kp/i}(M),\quad 1\leqslant i\leqslant k$$

consequently, for $f\in C^\infty(M,C^1)$, $f(\cdot,v)\in I_kL^p(M,\mathscr{V}_*)$.

This proposition means that the space $I_kL^p(M,\mathscr{V}_*)$ is an algebra closed unde nonlinear composition.

Lemma 3. 4 *For any $V \in \mathscr{V}_s(\mathscr{S})$, there is $V' \in \mathscr{V}_s(\mathscr{C})$ such that*

$$V = V' + (a_0 + b)\partial_t + a_1\partial_{z_1} + a_2\partial_{z_2} \tag{3. 6}$$

where $a_i, b \in C^{\infty,1}(\Omega, O)$, $i=0,1,2$, and $a_i|_{t=0}=0$, $b|_{C_j}=0$ for $j=1,\cdots,N$.

Proof It is easy to see $\mathscr{V}_s(\mathscr{S}) \subset \mathscr{V}_s(\mathscr{S}_{ij})$, $0 \leqslant i < j \leqslant 2N$, so from Lemma 3.1,

$$\chi_{ij}\mathscr{V}_s(\mathscr{S}) = \chi_{ij}\mathscr{V}_s(\mathscr{S}_{ij})$$

for a partition of unity $1 = \sum_{0 \leqslant i < j \leqslant 2N} \chi_{ij}, \chi_{ij} \in C^{\infty,0}(\Omega, O)$, subordinate to some cone-like open covering K_{ij} of $\Omega \backslash O$ satisfying the property (3. 3) for $0 \leqslant i < j \leqslant 2N$. Therefore

$$\mathscr{V}_s(\mathscr{S}) = \sum_{0 \leqslant i < j \leqslant 2N} \chi_{ij}\mathscr{V}_s(\mathscr{S}_{ij}) \tag{3. 7}$$

and $V = \sum_{0 \leqslant i < j \leqslant 2N} \chi_{ij}V_{ij}$ for some $V_{ij} \in \mathscr{V}_s(\mathscr{S}_{ij})$. By an appropriate choice, we can assume that $K_{ij} \cap \{t=0\} = \varnothing$ if $i=0$ or $i+N \neq j$(i. e. $S_i \cap S_j \cap (\omega \backslash O) = \varnothing$), then for such i, j, $\chi_{ij}V_{ij}$ has the form (3. 6) with $b=0$. If $i+N=j, i \geqslant 1$, then $S_i \cap S_j = C_i$ and there are only S_i, S_{i+N}, C_i in suppχ_{ij}, so $\chi_{ij}V_{ij}$ is generated there by $r\partial_{y_2}$, $(t-y_1)(\partial_t - \partial_{y_1})$, $(t+y_1)(\partial_t + \partial_{y_1})$, where $r = \sqrt{t^2+y_1^2+y_2^2}$, and $S_i = \{t-y_1=0\}$, $S_{i+N} = \{t+y_1=0\}$, $C_i = \{t=0, y_1=0\}$ after a C^∞ local coordinate change to (t, y_1, y_2). As $\mathscr{V}_s(\mathscr{C})$ is generated by $r\partial_{y_2}, y_1\partial_{y_1}$ in suppχ_{ij}, $j=i+N$, $\chi_{ij}V_{ij}$ has the form (3. 6) with $b = \chi_{ij}y_1 c$ for some $c \in C^{\infty,0}$, thereby $b \in C^{\infty,1}$ and $b=0$ on C_i. Since $K_{i,i+N} \cap C_m = \varnothing$ for $m \neq i$ from (3. 3), b satisfies the requirement of the lemma.

4. Commutator Relations

Lemma 4. 1 *Suppose $p(\zeta)$ is a real nondegenerate quadratic form on $\boldsymbol{R}^m$ and it is neither positively definite nor negatively definite. If $q(\zeta)$ is another quadratic form on $\boldsymbol{R}^m$ satisfying*

$$\{\zeta \in \boldsymbol{R}^m : p(\zeta) = 0\} \subset \{\zeta \in \boldsymbol{R}^m : q(\zeta) = 0\} \tag{4. 1}$$

then there is a constant c such that $q(\zeta) = cp(\zeta)$.

Proof Without loss of generality, we can assume

$$p(\zeta) = \sum_{i=1}^{m_+} \zeta_i^2 - \sum_{j=m_++1}^{m} \zeta_j^2, \quad 1 \leqslant m_+ < m \tag{4. 2}$$

$$q(\zeta) = \sum_{i,j=1}^{m} a_{ij}\zeta_i\zeta_j, \quad a_{ij} = a_{ji} \tag{4. 3}$$

From (4. 2), p vanishes at $\zeta_i = 1, \zeta_j = \pm 1, \zeta_l = 0$ for $l \neq i, j, 1 \leqslant i \leqslant m_+ < j \leqslant m$, so (4. 1) results in $a_{ii} \pm 2a_{ij} + a_{jj} = 0$, hence

$$a_{ij} = 0, a_{ii} = -a_{jj} \quad \text{for} \quad 1 \leqslant i \leqslant m_+ < j \leqslant m \tag{4. 4}$$

As p also vanishes at $\zeta_1 = 1, \zeta_{j_1} = \zeta_{j_2} = 1/\sqrt{2}, \zeta_l = 0$ for $l \neq 1, j_1, j_2, m_+ < j_1 \neq j_2 \leqslant m$, (4. 1) together with (4. 4) result in

$$a_{j_1 j_2} = 0 \quad \text{for} \quad m_+ < j_1 \neq j_2 \leqslant m \tag{4. 5}$$

Similarly we have

$$a_{i_1 i_2} = 0 \quad \text{for} \quad 1 \leqslant i_1 \neq i_2 \leqslant m_+ \tag{4.6}$$

(4.3)—(4.6) mean just $q(\zeta) = a_{11} p(\zeta)$.

Now consider $\Omega, P, S_i, 0 \leqslant i \leqslant 2N, \mathscr{S}_{ij}, 0 \leqslant i < j \leqslant 2N$, as in Section 2, let

$$\mathscr{S}'_{ij} = \{S_i, S_j\}, \quad 0 \leqslant i < j \leqslant 2N$$

We know from [4] that all the vector fields $\mathscr{V}(\mathscr{S}_{ij}), \mathscr{V}(\mathscr{S}'_{ij})$ are finitely generated C^∞-modules. Denote by Ψ^m the space of m-th order pseudo-differential operators, by Diff^m the space of m-th order differential operators with C^∞ coefficients.

Lemma 4.2 *The following commutator relations hold in* Ω:

$$[\mathscr{V}(\mathscr{S}_{0i}), P] = \Psi^0 \cdot P + \Psi^1 \cdot \mathscr{V}(\mathscr{S}'_{0i}) + \Psi^1, \quad 1 \leqslant i \leqslant 2N \tag{4.7}$$

$$[\mathscr{V}(\mathscr{S}'_{ij}), P] = C^\infty \cdot P + \text{Diff}^1 \cdot \mathscr{V}(\mathscr{S}'_{ij}) + \text{Diff}^1$$

$$0 \leqslant i < j \leqslant 2N \tag{4.8}$$

In addition, the coefficient operator of P in (4.7) can be written in the form

$$b + \sum_{i=1}^{3} A_i \cdot a_i, \text{ where } A_i \in \Psi^0, a_i, b \in C^\infty(\Omega), a_i(O) = 0 \tag{4.9}$$

Proof The assertions except (4.9) have been prove in [4], so what are left for us to do is to show that A_0 has the form (4.9) in the following relation coming from (4.7)

$$[V, P] = A_0 P + \sum_{\beta \in \mathscr{B}} B_\beta V_\beta + B$$

where $A_0 \in \Psi^0, B_\beta, B \in \Psi^1, V \in \mathscr{V}(\mathscr{S}_{0i})$ and $\{V_\beta : \beta \in \mathscr{B}\}$ is a finite basis of $\mathscr{V}(\mathscr{S}_{0i})$. Let $q(t,x;\tau,\xi), p(t,x;\tau,\xi), \psi(t,x;\tau,\xi)$ be the principal symbols of $[V,P], P, A_0$ respectively. By the fact that the principal symbol of any operator of $\mathscr{V}(\mathscr{S}_{0i})$ vanishes at $(t,x) = O$, we get

$$q(O;\tau,\xi) = \psi(O;\tau,\xi) p(O;\tau,\xi)$$

Since q is a quadratic form of (τ,ξ) and p is a real nondegenerate quadratic form satisfying the condition of Lemma 4.1, it implies $\psi(O;\tau,\xi) = \text{constant}$. It follows immediately that

$$\psi(t,x;\tau,\xi) = \psi(O;\tau,\xi) + \psi_1(t,x;\tau,\xi)x_1 + \psi_2(t,x;\tau,\xi)x_2 + \psi_3(t,x;\tau,\xi)t$$

which results in

$$A_0 = b + \sum_{i=1}^{3} A_i \cdot a_i + A_{-1}$$

b is a constant and $A_{-1} \in \Psi^{-1}$. Thus ends the proof, for $A_{-1}P$ can be absorbed in the last term on the right side of (4.7).

We denote by $\text{Diff}^{1,\sigma}$ the space of first order differential operators with $C^{\infty,\sigma}$ coefficients. Note that the singular vector fields $\mathscr{V}_s(\mathscr{S})$ in the last section is in $\text{Diff}^{1,1}$ and is a finitely generated $C^{\infty,0}$-module.

Lemma 4.3 *Suppose* $K \subset \Omega$ *is a set which does not intersect with* $\Gamma_\varepsilon = \{(t,x) : (1+\varepsilon) \cdot t^2 \geqslant x_1^2 + x_2^2\}$ *for some* $\varepsilon > 0$. *Then*

$$[\mathscr{V}_s(\mathscr{S}), P] = C^{\infty,0} \cdot P + \text{Diff}^{1,-1} \cdot \mathscr{V}_s(\mathscr{S}) + \text{Diff}^{1,-1} \tag{4.10}$$

holds in K.

Proof From [4], there is a finite generating set $\{V_{a'}, V_{a''}: a' \in \mathscr{A}', a'' \in \mathscr{A}''\}$ of $\mathscr{V}(\mathscr{S}_{ij})$ such that $\{V_{a'}, rV_{a''}: a' \in \mathscr{A}', a'' \in \mathscr{A}''\}$ generates $\mathscr{V}_s(\mathscr{S}_{ij})$, where $r = \sqrt{t^2+x_1^2+x_2^2} \in C^{\infty,1}$, therefore we obtain from (4.8) and (3.1)

$$[\mathscr{V}_s(\mathscr{S}_{ij}), P] = C^{\infty,0} \cdot P + \mathrm{Diff}^{1,-1} \cdot \mathscr{V}_s(\mathscr{S}_{ij}) + \mathrm{Diff}^{1,-1}, \quad 1 \leqslant i < j \leqslant 2N \tag{4.11}$$

If we take $\sum_{i=1}^{2N} \chi_{0i} = 1$ on Γ_ε and $\chi_{0i} = 0$ on K in (3.7), we get

$$\mathscr{V}_s(\mathscr{S}) = \sum_{1 \leqslant i < j \leqslant 2N} \chi_{ij} \mathscr{V}_s(\mathscr{S}_{ij}) \quad \text{in } K$$

The relation (4.10) then follows from (4.11).

5. L^2-estimates

Let P be as before, $G \subset \{t \geqslant 0\}$ be any bounded region which is a domain of determinacy of $G \cap \{t=0\}$. Introducing the notation $G(\delta) = G \cap \{t=\delta\}$, we know the following classical estimate with parameter η for $v \in H^2(G)$:

$$\int_{G(T)} |\nabla_{t,x} v(T,x)|^2 e^{-\eta T} dx + \eta \int_0^T \int_{G(t)} |\nabla_{t,x} v|^2 e^{-\eta t} dx dt$$
$$\leqslant C \int_{G(0)} |\nabla_{t,x} v(0,x)|^2 dx + C \int_0^T \int_{G(t)} |\partial_t v \cdot Pv| e^{-\eta t} dx dt, \qquad \eta > \eta_0$$

where η_0 depends only on P. Here and below in this section, C will always mean a sufficiently large constant independent of v and η. Combining the above inequality with the following obvious estimate

$$\int_{G(T)} |v(T,x)|^2 e^{-\eta T} dx + \eta \int_0^T \int_{G(t)} |v|^2 e^{-\eta t} dx dt$$
$$\leqslant C \int_{G(0)} |v(0,x)|^2 dx + C\eta^{-1} \int_0^T \int_{G(t)} |\partial_t v|^2 e^{-\eta t} dx dt, \qquad \eta > 0$$

we obtain

Lemma 5.1 *Suppose $v \in H^1(G)$ is a solution to*

$$Pv + (a_0\partial_t + a_1\partial_{x_1} + a_2\partial_{x_2})v + bv = f, \quad a_i, b \in C^\infty(G), \quad i = 0,1,2$$

Then v satisfies the estimate

$$(\eta - \eta_1) \iint_G |\nabla_{t,x} v|^2 e^{-\eta t} dx dt + (\eta^3 - \eta_1^3) \iint_G |v|^2 e^{-\eta t} dx dt$$
$$\leqslant C \int_{G(0)} |\nabla_{t,x} v|^2 dx + C\eta^2 \int_{G(0)} |v|^2 dx + C\eta^{-1} \iint_G |f|^2 e^{-\eta t} dx dt, \qquad \eta > \eta_1$$

where $\eta_1 = C(\sum_{i=0}^{2} \| a_i \|_{C^0(G)} + \sqrt{\| b \|_{C^0(G)}} + \eta_0)$ and C is also independent of G varying in a fixed bounded set.

Now let K be a cone-like open set in $\Omega \cap \{t \geqslant 0\}$. We assume that K is a domain of determinacy of $K(0) = K \cap \{t=0\}$ and has no intersection with $\Gamma_\varepsilon = \{(t,x): (1+\varepsilon)t^2 \geqslant x_1^2 + x_2^2\}$ for some $\varepsilon > 0$.

Proposition 5.2 *Suppose $w \in \mathscr{D}'(K)$ is a solution to*

$$Pw + (a_0\partial_t + a_1\partial_{x_1} + a_2\partial_{x_2})w + bw = f$$

where $a_i \in C^{\infty,-1}(K), i=0,1,2, \quad b \in C^{\infty,-2}(K)$. *If for* $\sigma \in R^1, r = \sqrt{t^2+x_1^2+x_2^2}$,

$$r^{-1-\sigma}w(0,x), r^{-\sigma}\partial_{t,x}w(0,x) \in L^2(K(0)), \quad r^{1/2-\sigma}f \in L^2(K)$$

then

$$r^{-3/2-\sigma}w, r^{-1/2-\sigma}\partial_{t,x}w \in L^2(K)$$

Proof Decompose

$$K = \bigcup_{i \geqslant i_0} K_i, \quad K_i = \{(t,x): 2^{-(i+1)} \leqslant r < 2^{-i}\} \cap K$$

There are $K_i', i \geqslant i_0$, domains of determinacy, so that

$$K_i \subset K_i' \subset \{(t,x): C^{-1}2^{-(i+1)} \leqslant r < C2^{-i}\}$$

thanks to the support properties of the solution operator to P (i. e. property of finite propagation speed). It results from usual regularity theorem that $w \in H^1(K_i')$. Applying Lemma 5. 1 to K_i' with $\eta = 2^i\eta_2$, where η_2 is to be decided, we obtain

$$(\eta_2 - C)2^i \iint_{K_i'} |\nabla_{t,x}w|^2 e^{-2^i\eta_2 t}dxdt + (\eta_2^3 - C^3)2^{3i}\iint_{K_i'} |w|^2 e^{-2^i\eta_2 t}dxdt$$

$$\leqslant C\int_{K_i'(0)} |\nabla_{t,x}w|^2 dx + C\eta_2^2 2^{2i}\int_{K_i'(0)} |w|^2 dx + C\eta_2^{-1}2^{-i}\iint_{K_i'} |f|^2 e^{-2^i\eta_2 t}dxdt \quad (5.1)$$

by the fact that $\sum_{j=0}^{2} \| a_j \|_{C^0(K_i')} + \sqrt{\| b \|_{C^0(K_i')}} \leqslant C2^i$. Now fix $\eta_2 > C$, there is another C such that $C^{-1} \leqslant e^{-2^i\eta_2 t} \leqslant 1$ on K_i', so with still another C, (5. 1) gives

$$2^i \iint_{K_i'} |\nabla_{t,x}w|^2 dxdt + 2^{3i}\iint_{K_i'} |w|^2 dxdt$$

$$\leqslant C\int_{K_i'(0)} |\nabla_{t,x}w|^2 dx + C2^{2i}\int_{K_i'(0)} |w|^2 dx + C2^{-i}\iint_{K_i'} |f|^2 dxdt$$

$$\leqslant C\sum_{j=i-m_0}^{i+m_0}\left(\int_{K_j(0)} |\nabla_{t,x}w|^2 dx + 2^{2j}\int_{K_j(0)} |w|^2 dx + 2^{-j}\iint_{K_j} |f|^2 dxdt\right) \quad (5.2)_i$$

where in the last inequality, we have used the fact that $K_i' \subset \bigcup_{j=i-m_0}^{i+m_0} K_j$ for a fixed m_0. Multiplying $(5.2)_i$ by $2^{2\sigma i}$ and summing them up, we find

$$\sum_{i \geqslant i_0}\left[2^{2(1/2+\sigma)i}\iint_{K_i} |\nabla_{t,x}w|^2 dxdt + 2^{2(3/2+\sigma)i}\iint_{K_i} |w|^2 dxdt\right]$$

$$\leqslant C\sum_{i \geqslant i_0}\left[2^{2\sigma i}\int_{K_i(0)} |\nabla_{t,x}w|^2 dx + \right.$$

$$\left. + 2^{2(1+\sigma)i}\int_{K_i(0)} |w|^2 dx + 2^{2(-1/2+\sigma)i}\iint_{K_i} |f|^2 dxdt\right] \quad (5.3)$$

Since $r \sim 2^{-i}$ on K_i, (5. 3) implies

$$\iint_K (|r^{-1/2-\sigma}\nabla_{t,x}w|^2 + |r^{-3/2-\sigma}w|^2)dxdt$$

$$\leqslant C\int_{K(0)} (|r^{-\sigma}\nabla_{t,x}w|^2 + |r^{-1-\sigma}w|^2)dx + C\iint_K |r^{1/2-\sigma}f|^2 dxdt$$

The proposition follows.

Remark 1 If P is replaced by a second order sytem $\mathscr{P}$ with diagonal principal part P, Proposition 5.2 still holds.

Remark 2 The result is valid in $t\leqslant 0$.

6. Proof of Theorem 2.1

Lemma 6.1 *The condition* (2.4) *implies that there is a constant* c *such that*

$$r^{-1}(g-c)\in L^{\infty}(\omega)\cap I_kL^p(\omega,\mathscr{V}_s(\mathscr{C})) \tag{6.1}$$

where $r=\sqrt{x_1^2+x_2^2}$.

Proof After a coordinate change with $C^{\infty,1}$ coefficients we can assume all C_i, $1\leqslant i\leqslant N$, are straight lines, hence there exists a finite basis $\{V_\alpha:\alpha\in\mathscr{A}\}$ of $\mathscr{V}_s(\mathscr{C})$, having the property that the coefficients are all homogeneous of degree one with respect to (x_1,x_2). Such singular coordinate change does not affect (2.4), so we still have $g\in W^{1,\infty}(\omega)\subset C^0(\omega)$. We will prove that $c=g(O)$ suffices to give (6.1).

Without loss of generality, we assume ω to be star shaped with respect to O, since we are only concerned with some neighborhood of O. Define the operator $T: C_c^\infty(\omega)\to C_c^\infty(\omega)$ by $T\varphi(x)=\int_0^1\varphi(\lambda x)d\lambda$. It is readily seen that T extends to $L^q(\omega)\to L^q(\omega)$ for $2<q\leqslant\infty$. Obviously we have

$$g(x)-g(O)=\sum_{i=1}^{2}x_i\cdot T(\partial_{x_i}g) \tag{6.2}$$

Because the coefficients of V_α are homogeneous of degree one, we have

$$V_\alpha T=TV_\alpha \quad \text{on} \quad \{v\in L^q(\omega):V_\alpha v\in L^q(\omega)\},\quad 2<q<\infty$$

so (2.4) implies $T(\partial_{x_i}g)\in I_kL^p(\omega,\mathscr{V}_s(\mathscr{C}))$, $i=1,2$. (6.1) then follows from (6.2).

Since the Cauchy problem (2.2)—(2.3) with $g,h\in C^\infty$ will develop C^∞ solution u, we can assume without loss of generality that

$$r^{-1}g\in L^{\infty}(\omega)\cap I_kL^p(\omega,\mathscr{V}_s(\mathscr{C})),\quad p>2 \tag{6.3}$$

in addition to (2.4).

The proof of Theorem 2.1 divides into two steps, the first of which is to do the analysis outside

$$\Gamma_\varepsilon=\{(t,x):(1+\varepsilon)t^2\geqslant x_1^2+x_2^2\},\quad \varepsilon>0 \ \text{ small}$$

while the second is to do near Γ_ε. The first step consists of

Proposition 6.2 *Let K be a cone-like domain of determinacy of $\omega\backslash O$. Suppose $K\cap\Gamma=\varnothing$ for some $\varepsilon>0$, the conditions of Theorem 2.1 and (6.3) are satisfied. Then there exists some $\sigma\in(0,\frac{1}{2}]$ so that*

$$r^{-3/2-\sigma}u,\ r^{-1/2-\sigma}\partial_{t,x}u\in I_kL^2(K,\mathscr{V}_s(\mathscr{S})) \tag{6.4}$$

Proof Let $\{V_i:i=1,\cdots,l\}$ be a finite basis of $\mathscr{V}_s(\mathscr{S})$, and put $U_j=\{V_{i_1}\cdots V_{i_m}u: m\leqslant j\}$. We first prove inductively on $0\leqslant j\leqslant k$ the assertions

$$r^{-1}\mathscr{V}_s(\mathscr{C})^i(U_j|_{t=0}),\quad \mathscr{V}_s(\mathscr{C})^i(\partial_{t,x}U_j|_{t=0})\in L^{pk/(i+j)}(\omega\backslash O),$$

$$\text{for}\quad 0 \leqslant i \leqslant k-j \qquad (6.5)_j$$

$$U_j,\quad \partial_{t,x} U_j \in L^2_{\text{loc}}(K) \qquad (6.6)_j$$

For $j=0$, $(6.5)_0$, $(6.6)_0$ follow from (2.4), (6.3), Proposition 3.3 and a usual regularity theorem.

Proof of $(6.5)_1, \cdots, (6.5)_j, (6.6)_j \Rightarrow (6.5)_{j+1}$: From Lemma 4.3, U_j satisfies

$$\mathscr{P}_j U_j + \text{Diff}^{1,-1} U_j = F_j \quad \text{in} \quad K \qquad (6.7)_j$$

where $\mathscr{P}_j$ is a second order hyperbolic system with diagonal principal part P, F_j has the form

$$F_j = C^{\infty,0} \cdot \sum_{j_1+\cdots+j_m \leqslant j} F_{j_1 \cdots j_m}(t,x,u) U_{j_1} \cdots U_{j_m}, \qquad F_{j_1 \cdots j_m} \in C^\infty$$

Using the equation $(6.7)_j$ and Lemma 3.4, we see that the traces $U_{j+1}|_{t=0}$, $\partial_{t,x} U_{j+1}|_{t=0}$ exist and

$$U_{j+1}|_{t=0} = \mathscr{V}_s(\mathscr{C})(U_j|_{t=0}) + C^{\infty,1} \cdot (\partial_t U_j|_{t=0})$$

$$\partial_{t,x} U_{j+1}|_{t=0} = \mathscr{V}_s(\mathscr{C})(\partial_{t,x} U_j|_{t=0}) + C^{\infty,0} \cdot (\partial_{t,x} U_j|_{t=0}) + (bF_j|_{t=0})$$

where $b=b(t,x)$ is as in Lemma 3.4, hence $b(0,x)\partial_x \in \mathscr{V}_s(\mathscr{C})$. Therefore, $(6.5)_{j+1}$ results from $(6.5)_1, \cdots, (6.5)_j$ and $u|_{t=0} \in L^\infty$.

Proposition 3.3 and $(6.6)_j$ imply $F_{j+1} \in L^2_{\text{loc}}(K)$, so by a usual regularity theorem for $(6.7)_{j+1}$, $(6.5)_{j+1}$ implies $(6.6)_{j+1}$, thus ends the inductive proof of $(6.5)_j$, $(6.6)_j$ for $0 \leqslant j \leqslant k$.

Now that $(6.5)_j$ has been verified for $0 \leqslant j \leqslant k$ and $p>2$, then there exists some $\sigma \in (0, \frac{1}{2}]$ so that

$$r^{-1-\sigma} U_j|_{t=0},\quad r^{-\sigma} \partial_{t,x} U_j|_{t=0} \in L^2(\omega \backslash O), \quad 0 \leqslant j \leqslant k$$

We are in the position to use Proposition 5.2 together with its remarks to get

$$r^{-3/2-\sigma} U_j,\quad r^{-1/2-\sigma} \partial_{t,x} U_j \in L^2(K), \quad 0 \leqslant j \leqslant k$$

which is equivalent to (6.4).

The next step, as in [4], is to show that there is $\chi \in C^{\infty,0}$, $\chi = 1$ on Γ_ε for some $\varepsilon > 0$, such that

$$\chi u, \partial_{t,x}(\chi u) \in \sum_{i=1}^{2N} I_k L^2(\Omega, \mathscr{V}(\mathscr{S}_{0i})) \subset I_k L^2(\Omega, \mathscr{V}_s(\mathscr{S})) \qquad (6.8)$$

Consider it in $t \geqslant 0$ for instance, take χ with $\text{supp}\chi \subset \{t \geqslant 0\}$, then $\bar{u} = \chi u$ satisfies

$$P\bar{u} = f(t,x,\bar{u}) + \bar{f} \quad \text{in} \quad \Omega \qquad (6.9)$$

$$\bar{u} \in L^\infty(\Omega), \quad \text{supp}\bar{u} \subset \{t \geqslant 0\} \qquad (6.10)$$

where

$$\bar{f} = [P, \chi]u + \chi f(t,x,u) - f(t,x,\chi u)$$

If $\text{supp}\chi$ is contained in $\Gamma_{2\varepsilon}$ for sufficiently small $\varepsilon > 0$ so that $\Gamma_{4\varepsilon} \cap (\bigcup_{1 \leqslant i < j \leqslant 2N} (S_i \cap S_j)) = \{O\}$, we obtain from Proposition 6.2.

$$r^{1/2-\sigma} \bar{f} \in \sum_{i=1}^{2N} I_k L^2(\Omega, \mathscr{V}(\mathscr{S}_{0i})) \qquad (6.11)$$

We fix this ε and χ from now on.

The assertion (6.8) is proved from (6.9)—(6.11) by an argument in Section 7 of [4] with slight modifications, which we sketch below.

First note the following wellknown

Lemma 6.3 *Let $\mathscr{P}$ be a second order pseudo-differential system with diagonal principal part P. The solution $U \in \mathscr{D}'(\mathbf{R}^3)$ to*

$$\mathscr{P}U = F \in L^1_{loc}(\mathbf{R}^1, L^2_{loc}(\mathbf{R}^2)), \quad \text{supp } U \subset \{t \geqslant 0\}$$

is in $H^1_{loc}(\mathbf{R}^3)$.

Without loss of generality, we assume $f(t,x,0)=0$. Consider

$$P\bar{u}_i = \chi_i f(t,x,\bar{u}) + \bar{f}_i, \quad \text{supp}\bar{u}_i \subset \{t \geqslant 0\}, \quad i = 1,\cdots,2N \tag{6.12}$$

where

$$r^{1/2-\sigma}\bar{f}_i \in I_k L^2(\Omega, \mathscr{V}(\mathscr{S}_{0i})), \quad \sum_{i=1}^{2N}\bar{f}_i = \bar{f}$$
$$\chi_i \in C^{\infty,0}(\Omega, O), \quad \sum_{i=1}^{2N}\chi_i = 1 \tag{6.13}$$

and $\text{supp}\chi_i$ does not intersect with some cone-like neighborhood of $\bigcup_{\substack{1\leqslant j \leqslant 2N \\ j \neq i}} ((S_0 \cap S_j)\backslash O)$. Then $\bar{u} = \sum_{i=1}^{2N}\bar{u}_i$. Because $r^{-1/2+\sigma}L^2_{loc}(R^3) \subset L^1_{loc}(R^1, L^2_{loc}(R^2))$ for $\sigma > 0$, it is easy to see $\bar{u}_i \in H^1(\Omega)$ by means of Lemma 6.3. To prove

$$\bar{u}_i, \partial_{t,x}\bar{u}_i \in I_k L^2(\Omega, \mathscr{V}(\mathscr{S}_{0i})) \tag{6.14}$$

from (6.12) and (6.13), an iterative commutator argument as in the proof of Proposition 6.2 is needed. The corresponding commutator relation is (4.7). Since the act on $\bar{f}_i$ of any operator in the form (4.9) preserves (6.13), the iteration works by making use of Lemma 4.2 and 6.3. Thus we prove (6.14) for $i=1,\cdots,2N$, and (6.8) follows.

Now take $K_\varepsilon = \Omega \backslash \Gamma_\varepsilon$, then $\text{supp}(1-\chi) \subset K_\varepsilon$. Proposition 6.2 for this K_ε means that

$$(1-\chi)u, \quad \partial_{t,x}((1-\chi)u) \in I_k L^2(\Omega, \mathscr{V}_\varepsilon(\mathscr{S}))$$

which, together with (6.8), proves Theorem 2.1.

7. Example with Piecewise C^∞ Data

Consider the Cauchy problem:

$$(\partial_t^2 - \partial_x^2 - \partial_y^2)u = 0 \tag{7.1}$$
$$u(0,x,y) = 0, \quad \partial_t u(0,x,y) = \mu(x,y) \tag{7.2}$$

where

$$\mu(x,y) = \begin{cases} 1, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases} \tag{7.3}$$

is a piecewise C^∞ function. The solution is given by

$$u(t,x,y) = \frac{1}{2\pi}\int_0^t \frac{\rho d\rho}{\sqrt{t^2-\rho^2}}\int_0^{2\pi}\mu(x+\rho\cos\theta, y+\rho\sin\theta)d\theta$$

Direct calculation gives

$$u = \frac{t}{4} + \frac{(x+y)}{4} + \frac{1}{2\pi}\left[x\arcsin\frac{y}{\sqrt{t^2-x^2}} + y\arcsin\frac{x}{\sqrt{t^2-y^2}}\right]$$

$$-\frac{t}{4\pi}\left[\arcsin\left(\frac{2t^2y^2}{(t^2-x^2)(x^2+y^2)}-1\right)+\arcsin\left(\frac{2t^2x^2}{(t^2-y^2)(x^2+y^2)}-1\right)\right)$$

in the region: $x<0, y<0, t^2>x^2+y^2$; and

$$\partial_t^2 u(t,x,y)=\frac{1}{2\pi}\frac{xy}{(t^2-x^2)\sqrt{t^2-x^2-y^2}}$$

Clearly $\partial_t^2 u$ is not bounded when $t^2-x^2-y^2\to 0$ in the above mentioned region, so the solution u to (7.1)—(7.3) is not piecewise C^∞.

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