J. Partial Differential Equations
Vol. 3, *No.* 1(1990), 35—46

SINGULARITIES PRODUCED BY THE REFLECTION AND INTERACTION OF TWO PROGRESSING WAVES[①]

Wang Yaguang
(Institute of Mathematics, Fudan University)
(Received Feb. 1, 1988; revised Sept. 24, 1988)

Abstract We give an example to show that there will be anomalous singularities on the forward half light cone issuing from the reflection point after the reflection at the boundary of two progressing waves carrying singularities. It perfects the results of [1].

Key Words Wave equations; two progressing waves; the reflection and interaction of singularities; mixed problems.
Classifications 35B65; 35L05; 35L20.

1. Introduction to Questions and the Main Results

There have been many works on the propagation of singularities of the solutions to semilinear wave equations so far. In [2] and [3], J. M. Bony considered the case of two progressing waves after intersection, and the elementary fact of his conclusions is that there could be anomalous singularities on the other characteristic hypersurfaces issuing from $H_1 \cap H_2$ after the interaction of two progressing waves propagating on characteristic hypersurfaces H_1 and H_2 as shown in Figure 1. In particular, for the following 2-dimensional wave equation:

$$\Box u = f(u) \tag{1.1}$$

where $u = u(t, x_1, x_2)$, $(t, x_1, x_2) \in R_t \times R_x^2$, we know that there does not exist any anomalous singularities after the interaction of two progressing waves by J. M. Bony's conclusions. But, J. Rauch and M. Reed presented an example to show there are exactly anomalous singularities after the interaction of three progressing waves in [4].

In this paper we consider the case that two progressing waves carrying singularities intersect at the boundary. For this case, Chen Shuxing ([1]) has proved for conormal distributions that there could be anomalous singularities on the forward half light cone issuing from the reflection point after the reflection on the boundary of these two progressing waves. This paper will give an example to show the existence of such singularities.

Denote by (t, x_1, x_2) any point of $R_t \times R_x^2$. We consider the following problem in $(R_t \times R_x^2) \cap \{x_2 > 0\}$

$$\begin{cases} \Box u_1 = 0 & (1.2) \\ \Box u_2 = 0 \ , \qquad x_2 > 0 & (1.3) \\ \Box u_3 = u_1 u_2 & (1.4) \\ u_i|_{x_2=0} = 0 \ , \qquad i = 1,2,3 & (1.5) \end{cases}$$

① Research partially supported by the National Natural Science Foundation of China

where $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$.

Suppose that u_1, u_2 are as follows

$$u_i(t,x_1,x_2) = \begin{cases} h(t - w_i \cdot x), & t \leqslant 0, x_2 > 0 \\ h(w_i \cdot x - t), & t > 0, x_2 > 0 \end{cases} \qquad (i = 1,2)$$

where

$$x = (x_1, x_2), w_1 = - w_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), w_1 = - w_2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

h is the Heaviside function.

We will consider the singularities of the solution u_3 to (1.2)—(1.5) on the forward half light cone $C_0 = \{(t,x_1,x_2) \mid t = \sqrt{x_1^2 + x_2^2}, x_2 > 0\}$ as $t > 0$. For simplicity, we introduce some notations as follows

$\Sigma_1 = \{(t,x_1,x_2) \mid x_1 + x_2 + \sqrt{2}\, t = 0\}$, i. e. the plane $OB'C'$ in Figure 2.

$\Sigma_1^+ = \{x_1 + x_2 + \sqrt{2}\, t \geqslant 0\}$; $\Sigma_1^- = \{x_1 + x_2 + \sqrt{2}\, t < 0\}$

$\Sigma_2 = \{(t,x_1,x_2) \mid -x_1 + x_2 + \sqrt{2}\, t = 0\}$, i. e. the plane $OA'B'$ in Figure 2.

$\Sigma_2^+ = \{-x_1 + x_2 + \sqrt{2}\, t \geqslant 0\}$; $\Sigma_2^- = \{-x_1 + x_2 + \sqrt{2}\, t < 0\}$

$\Sigma_3 = \{(t,x_1,x_2) \mid x_1 - x_3 + \sqrt{2}\, t = 0\}$, i. e. the reflection plane OBC of $OB'C'$ about $\{x_2 = 0\}$ in Figure 2.

$\Sigma_3^+ = \{x_1 - x_3 + \sqrt{2}\, t \geqslant 0\}$; $\Sigma_3^- = \{x_1 - x_3 + \sqrt{2}\, t < 0\}$

$\Sigma_4 = \{(t,x_1,x_2) \mid -x_1 - x_2 + \sqrt{2}\, t = 0\}$, i. e. the reflection plane OAB of $OA'B'$ about $\{x_2 = 0\}$ in Figure 2.

$\Sigma_4^+ = \{-x_1 - x_2 + \sqrt{2}\, t \geqslant 0\}$; $\Sigma_4^- = \{-x_1 - x_2 + \sqrt{2}\, t < 0\}$

$\Sigma_5 = \{(t,x_1,x_2) \mid x_1 = 0\}$, i. e. the plane OBB'.

$\Sigma_5^+ = \{x_1 \geqslant 0\}$; $\Sigma_5^- = \{x_1 < 0\}$

$\Sigma_6 = \{(t,x_1,x_2) \mid t = 0\}$, i. e. the plane OMN.

$\Sigma_6^+ = \{t \geqslant 0\}$; $\Sigma_6^- = \{t < 0\}$

$\mathscr{A} = \Sigma_1^+ \cap \Sigma_2^+ \cap \Sigma_3^- \cap \Sigma_4^-$, i. e. the pyramid $O\text{-}BMB'N$ in Figure 2.

$\mathscr{B}$ = the symmetric region of $\mathscr{A}$ about $\{x_2 = 0\}$, i. e. the pyramid $O\text{-}B_1M_1B_1'N_1$, where B_1, M_1, B_1', N_1 are on the stretched line of $\overline{OB}, \overline{OM}, \overline{OB'}, \overline{ON}$ respectively.

Obviously, u_3 can be considered as the solution to the following linear problem

$$\begin{cases} \Box u_3 = \chi_{\mathscr{A}} - \chi_{\mathscr{B}} & (1.6) \\ u_3 = 0, \quad t < 0 & (1.7) \end{cases}$$

where $\chi_{\mathscr{A}}$ and $\chi_{\mathscr{B}}$ are the characteristic functions of $\mathscr{A}$ and $\mathscr{B}$ respectively.

By the general expression of the solutions to wave equations we know the solution u_3 to (1.6) and (1.7) is

$$u_3(p) = (E * \chi_{C_p^- \cap \mathscr{A}} - E * \chi_{C_p^- \cap \mathscr{B}})(p) \qquad (1.8)$$

where $p = (t,x_1,x_2)$, $t > 0$, C_p^- is the backward light cone issuing from p, E is the fundamental solution to $\Box$.

For the singularities of u_3 we have the main theorem of this paper as follows

Theorem 1 *For (1.2)—(1.5), we suppose u_1, u_2 are the two functions given as above and $u_3=0$ as $t<0$. Then there are new singularities of u_3 to be produced on $C_0=\{(t,x_1,x_2) \mid = \sqrt{x_1^2+x_2^2}, x_2>0\}$ as $t>0$; furthermore, the third total differential of u_3 does not exist on C_0.*

The proof will be given in Section 2 and Section 3.

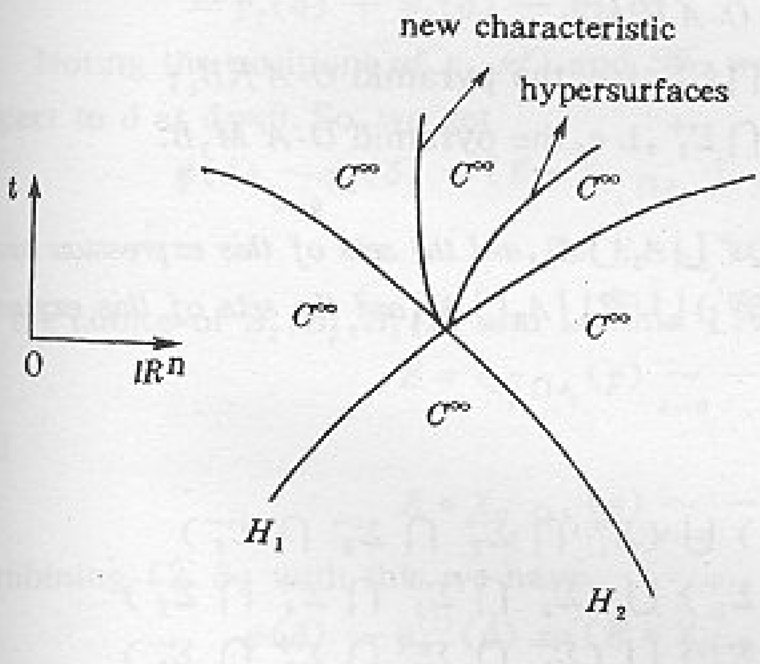

Figure 1

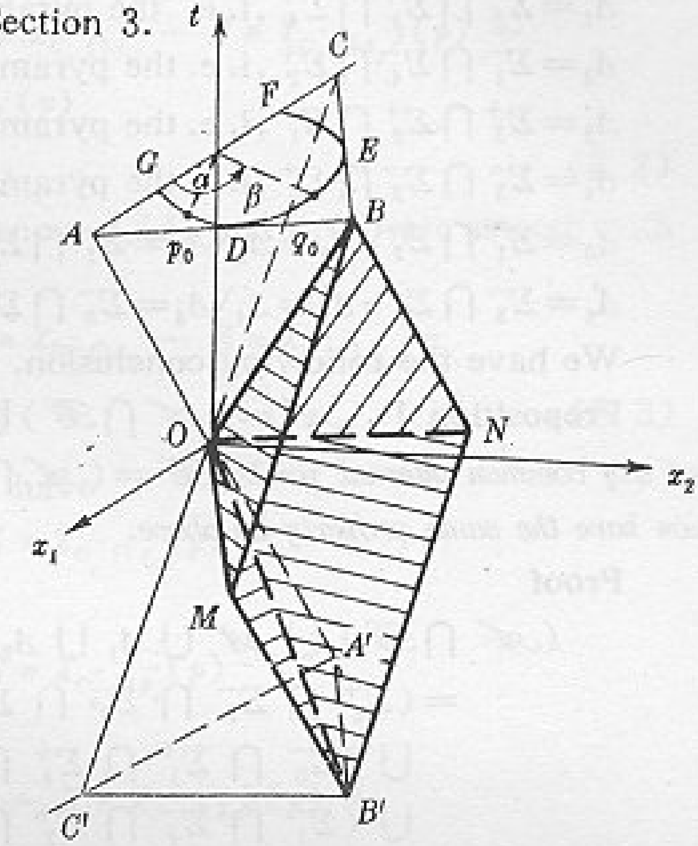

Figure 2

Now let's analysis the singularities of u_3. Denote by $\widehat{GDEF}=C_0\bigcap\{t=1\}$ as shown in Figure 2. Obviously, it is sufficient for us to consider the singularities of u_3 on $\{t=1\}$. By the choice of $\Sigma_i (i=1,2,3,4)$ we know the singularities of u_3 on $\widehat{GD}$ are the same as those on $\widehat{EF}$. So we may only to consider the singularities of u_3 on $\widehat{GD}$ and $\widehat{DE}$. The method we will use is the socalled "Jump", i. e. for any q belonging to $\{(t,x_1,x_2) \mid t=1, x_2>0\}$, we consider the singularities of $u_3(p)$ with respect to δ as δ is small enough, where δ is $\mathrm{dist}(q, \widehat{GDEF})$.

Definition 1 *Suppose that $u(p)$ and $v(p)$ are two functions defined in a neighborhood of p_0. If $u(p)-v(p)$ is smooth at p_0, then the singularities of $u(p)$ are the same as $v(p)$ at $p=p_0$. Denote this by $u(p)\underset{p_0}{\sim}v(p)$.*

By the above definition we obviously have

Lemma 1 *Suppose that the planes σ_1 and σ_2 intersect in $\mathbf{R}^3$, and one of wedges W caused by the intersection is divided into two pyramids W_1 and W_2 by the third plane σ_3. If $(E*\chi_{C_q^-\cap W})(q)$ is smooth with respect to q for any q belonging to a neighborhood U of p, where p is an arbitrary point of $\mathbf{R}^3$, then we have*

$$(E*\chi_{C_q^-\cap W_1})(q) \underset{p}{\sim} -(E*\chi_{C_q^-\cap W_2})(q) \quad , \quad \forall\, q\in U$$

2. The Singularities of u_3 on $\widehat{GD}$

First, let's introduce some notations (see Figure 2):

$\mathscr{A}' = \Sigma_2^+ \cap \Sigma_3^-$; $\mathscr{B}' = \Sigma_1^- \cap \Sigma_4^+$;

$\Delta_1 = \Sigma_3^- \cap \Sigma_1^+ \cap \Sigma_4^+$, i. e. the pyramid $O\text{-}BNC$;

$\Delta_2 = \Sigma_1^- \cap \Sigma_4^- \cap \Sigma_2^+$, i. e. the pyramid $O\text{-}B'NA'$;

$\Delta_3 = \Sigma_3^+ \cap \Sigma_2^+ \cap \Sigma_1^-$, i. e. the pyramid $O\text{-}B_1'M_1C$;

$\Delta_4 = \Sigma_3^- \cap \Sigma_2^- \cap \Sigma_4^+$, i. e. the pyramid $O\text{-}A'B_1M_1$;

$\Delta_2' = \Sigma_1^- \cap \Sigma_2^+$; $\Delta_2'' = \Delta_2' \setminus \Delta_2 = \Sigma_1^- \cap \Sigma_2^+ \cap \Sigma_4^+$, i. e. the pyramid $O\text{-}A'NB_1'$;

$\Delta_4' = \Sigma_3^- \cap \Sigma_4^+$; $\Delta_4'' = \Delta_4' \setminus \Delta_4 = \Sigma_3^- \cap \Sigma_4^+ \cap \Sigma_2^+$, i. e. the pyramid $O\text{-}A'M_1B$.

We have the following conclusion.

Proposition 1 *$\mathscr{A}' = (\mathscr{A}' \cap \mathscr{B}') \cup \mathscr{A} \cup \Delta_1 \cup \Delta_2$, and the sets of this expression have not any common internal points; $\mathscr{B}' = (\mathscr{A}' \cap \mathscr{B}') \cup \mathscr{B} \cup \Delta_3 \cup \Delta_4$, and the sets of this expression have the same property as above.*

Proof

$$\begin{aligned}
&(\mathscr{A}' \cap \mathscr{B}') \cup \mathscr{A} \cup \Delta_1 \cup \Delta_2 \\
&= (\Sigma_2^+ \cap \Sigma_3^- \cap \Sigma_1^- \cap \Sigma_4^+) \cup (\Sigma_1^+ \cap \Sigma_2^+ \cap \Sigma_3^- \cap \Sigma_4^-) \\
&\quad \cup (\Sigma_3^- \cap \Sigma_1^+ \cap \Sigma_4^+ \cap \Sigma_2^+) \cup (\Sigma_3^- \cap \Sigma_1^+ \cap \Sigma_4^+ \cap \Sigma_2^-) \\
&\quad \cup (\Sigma_1^- \cap \Sigma_4^- \cap \Sigma_2^+ \cap \Sigma_3^+) \cup (\Sigma_1^- \cap \Sigma_4^- \cap \Sigma_2^+ \cap \Sigma_3^-) \\
&= \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{V} \cup \mathrm{VI}
\end{aligned}$$

Clearly $\mathrm{I} \cup \mathrm{VI} = \Sigma_1^- \cap \Sigma_2^+ \cap \Sigma_3^-$ and $\mathrm{II} \cup \mathrm{III} = \Sigma_1^+ \cap \Sigma_2^+ \cap \Sigma_3^-$.

For any point $(t, x_1, x_2) \in \mathrm{IV}$, we have

$$\begin{cases}
(t, x_1, x_2) \in \Sigma_1^+ \Rightarrow x_1 + x_2 + \sqrt{2}\,t \geqslant 0 \\
(t, x_1, x_2) \in \Sigma_2^- \Rightarrow -x_1 + x_2 + \sqrt{2}\,t < 0 \\
(t, x_1, x_2) \in \Sigma_3^- \Rightarrow x_1 - x_2 + \sqrt{2}\,t < 0 \\
(t, x_1, x_2) \in \Sigma_4^+ \Rightarrow -x_1 - x_2 + \sqrt{2}\,t \geqslant 0
\end{cases}$$

These equations have not any solutions, i. e. IV is empty.

By the same way, we have $\mathrm{V} = \Sigma_1^- \cap \Sigma_2^+ \cap \Sigma_3^+ \cap \Sigma_4^-$ is also empty.

Hence

$$(\mathscr{A}' \cap \mathscr{B}') \cup \mathscr{A} \cup \Delta_1 \cup \Delta_2 = (\mathrm{I} \cup \mathrm{VI}) \cup (\mathrm{II} \cup \mathrm{III}) = \mathscr{A}'$$

Similarly, we can conclude:

$$\mathscr{B}' = (\mathscr{A}' \cap \mathscr{B}') \cup \mathscr{B} \cup \Delta_3 \cup \Delta_4$$

It is obvious that the sets of each of these expressions have not any common internal points.

For any $p_0 = (1, \cos\alpha, \sin\alpha) \in \widehat{GD}$ where $\alpha \in (0, \pi/4)$ as shown in Figure 2, we consider the singularities of $u_3(p)$ expressed by (1. 8) at $p = p_0$. Given $\varepsilon > 0$ small enough, for any $\delta \in (-\varepsilon, \varepsilon)$, we define

$$\psi(\delta) = (E * \chi_{C_\gamma^- \cap \mathscr{A}} - E * \chi_{C_\gamma^- \cap \mathscr{B}})(p) \tag{2.1}$$

i. e. $u_3(p)$, where $p=(1,(1+\delta)\cos\alpha,(1+\delta)\sin\alpha)$. For this function, we have

Proposition 2

$$\psi(\delta) \underset{\delta=0}{\sim} \psi_1(\delta) = 2(E*\chi_{C_r^-\cap\Delta_3}(p) - E*\chi_{C_r^-\cap\Delta_1}(p))$$

Proof By Proposition 1, we get

$$\begin{aligned}\psi(\delta) &= (E*\chi_{C_r^-\cap\mathscr{A}} - E*\chi_{C_r^-\cap\mathscr{B}})(p)\\ &= (E*\chi_{C_r^-\cap\Delta_3} + E*\chi_{C_r^-\cap\Delta_4} - E*\chi_{C_r^-\cap\Delta_1} - E*\chi_{C_r^-\cap\Delta_2})(p) +\\ &\quad + E*\chi_{C_r^-\cap\mathscr{A}'}(p) - E*\chi_{C_r^-\cap\mathscr{B}'}(p)\\ &= \psi_2(\delta) + \psi_3(\delta) - \psi_4(\delta)\end{aligned} \tag{2.2}$$

Noting the positions of p, $\mathscr{A}'$ and $\mathscr{B}'$, we know $\psi_3(\delta)$ and $\psi_4(\delta)$ are smooth with respect to δ at $\delta=0$. So, we get

$$\begin{aligned}\psi(\delta) \underset{\delta=0}{\sim} \psi_2(\delta) &= (E*\chi_{C_r^-\cap\Delta_3} + E*\chi_{C_r^-\cap\Delta_4} - E*\chi_{C_r^-\cap\Delta_1} -\\ &\quad - E*\chi_{C_r^-\cap\Delta_2})(p)\end{aligned} \tag{2.3}$$

By the choice of Δ_2', Δ_2'', Δ_4', Δ_4'' and Lemma 1, we have

$$E*\chi_{C_r^-\cap\Delta_2}(p) \underset{\delta=0}{\sim} -E*\chi_{C_r^-\cap\Delta_2'}(p)$$

and

$$E*\chi_{C_r^-\cap\Delta_4}(p) \underset{\delta=0}{\sim} -E*\chi_{C_r^-\cap\Delta_4'}(p)$$

Combining (2. 3) with this we have

$$\begin{aligned}\psi(\delta) \underset{\delta=0}{\sim} \psi_2^{(1)}(\delta) &= (E*\chi_{C_r^-\cap\Delta_3} - E*\chi_{C_r^-\cap\Delta_1})(p) +\\ &\quad + (E*\chi_{C_r^-\cap\Delta_2'} - E*\chi_{C_r^-\cap\Delta_4'})(p)\end{aligned} \tag{2.4}$$

Now, we come to simplify $\psi_2^{(1)}(\delta)$

$$\begin{aligned}\psi_2^{(1)}(\delta) &= E*\chi_{C_r^-\cap\Delta_3}(p) - E*\chi_{C_r^-\cap\Delta_1}(p) + E*\chi_{C_r^-\cap\Delta_4}(p) -\\ &\quad - E*\chi_{C_r^-\cap\Delta_2}(p) + E*\chi_{C_r^-\cap\Delta_2'}(p) - E*\chi_{C_r^-\cap\Delta_4'}(p)\\ &= E*\chi_{C_r^-\cap\mathscr{A}}(p) + E*\chi_{C_r^-\cap\mathscr{B}}(p) - E*\chi_{C_r^-\cap\mathscr{A}'}(p) +\\ &\quad + E*\chi_{C_r^-\cap\mathscr{B}'}(p) + E*\chi_{C_r^-\cap\Delta_2'}(p) - E*\chi_{C_r^-\cap\Delta_4'}(p)\end{aligned}$$

Let $p'=(1,(1+\delta)\cos\alpha,-(1+\delta)\sin\alpha)$ is the symmetric point of p about $\{x_2=0\}$. Since $\mathscr{B}$, $\mathscr{B}'$ and Δ_4' are the symmetric regions of $\mathscr{A}$, $\mathscr{A}'$ and Δ_2' respectively.

$$\begin{aligned}\psi_2^{(1)}(\delta) &= (E*\chi_{C_r^-\cap\mathscr{A}}(p) - E*\chi_{C_r^-\cap\mathscr{A}}(p')) + (E*\chi_{C_r^-\cap\mathscr{A}'}(p') -\\ &\quad - E*\chi_{C_r^-\cap\mathscr{A}'}(p)) + (E*\chi_{C_r^-\cap\Delta_2'}(p) - E*\chi_{C_r^-\cap\Delta_4'}(p))\end{aligned} \tag{2.5}$$

Furthermore, we have

$$\mathscr{A} = (\Sigma_3^- \cap \Sigma_2^+ \cap \Sigma_5^+) \cup (\Sigma_1^+ \cap \Sigma_4^- \cap \Sigma_5^-) = \mathscr{A}_1 \cup \mathscr{A}_2$$

where

$\mathscr{A}_1 = \Sigma_3^- \cap \Sigma_2^+ \cap \Sigma_5^+$, i. e. the pyramid $O\text{-}MBB'$,

$\mathscr{A}_2 = \Sigma_1^+ \cap \Sigma_4^- \cap \Sigma_5^-$, i. e. $O\text{-}NBB'$;

$$\mathscr{A}' = (\Sigma_3^- \cap \Sigma_2^+ \cap \Sigma_5^+) \cup (\Sigma_3^- \cap \Sigma_2^+ \cap \Sigma_5^-) = \mathscr{A}_1 \cup \mathscr{A}_2'$$

where

$$\mathscr{A}_2' = \Sigma_3^- \cap \Sigma_2^+ \cap \Sigma_5^-, \text{ i. e. } O\text{-}M_1BB';$$

$$\Delta_2' = \Sigma_1^- \cap \Sigma_2^+ = (\Sigma_2^+ \cap \Sigma_5^-) \backslash (\Sigma_1^+ \cap \Sigma_5^-)$$

$$= (\mathscr{A}_2 \cup (\Sigma_2^+ \cap \Sigma_5^- \cap \Sigma_3^+)) \setminus (\mathscr{A}_2 \cup (\Sigma_1^+ \cap \Sigma_5^- \cap \Sigma_4^+))$$

Combining (2.5) with this, we have

$$\psi_2^{(1)}(\delta) = E * \chi_{C_p^- \cap (\Sigma_2^+ \cap \Sigma_5^- \cap \Sigma_3^+)}(p) - E * \chi_{C_p^- \cap (\Sigma_2^+ \cap \Sigma_5^- \cap \Sigma_3^+)}(p') + $$
$$+ E * \chi_{C_p^- \cap (\Sigma_1^+ \cap \Sigma_5^- \cap \Sigma_4^+)}(p') - E * \chi_{C_p^- \cap (\Sigma_1^+ \cap \Sigma_5^- \cap \Sigma_4^+)}(p)$$

Since $\Sigma_1^+ \cap \Sigma_5^- \cap \Sigma_4^+$ and $\Sigma_2^+ \cap \Sigma_5^- \cap \Sigma_3^+$ are $\{(t, x_1, x_2) \mid x_1 + x_2 + \sqrt{2}t \geqslant 0, -x_1 - x_2 + \sqrt{2}t \geqslant 0, x_1 < 0\}$ and $\{(t, x_1, x_2) \mid -x_1 + x_2 + \sqrt{2}t \geqslant 0, x_1 - x_2 + \sqrt{2}t \geqslant 0, x_1 < 0\}$ respectively, the symmetric region of $\Sigma_1^+ \cap \Sigma_5^- \cap \Sigma_4^+$ is $\Sigma_2^+ \cap \Sigma_5^- \cap \Sigma_3^+$ about $\{x_2 = 0\}$.

Therefore

$$\psi_2^{(1)}(\delta) = 2(E * \chi_{C_p^- \cap (\Sigma_1^+ \cap \Sigma_5^- \cap \Sigma_4^+)}(p') - E * \chi_{C_p^- \cap (\Sigma_1^+ \cap \Sigma_5^- \cap \Sigma_4^+)}(p))$$

Because $\Sigma_1^+ \cap \Sigma_5^- \cap \Sigma_4^+ = O\text{-}BB_1'N = (O\text{-}BCB_1') \cup (O\text{-}BCN) = (O\text{-}BCB_1') \cup \Delta_1$, and there are not any common internal points for $O\text{-}BCB_1'$ and Δ_1, we have

$$\psi_2^{(1)}(\delta) = 2(E * \chi_{C_p^- \cap (O\text{-}BCB_1')}(p') - E * \chi_{C_p^- \cap (O\text{-}BCB_1')}(p) + $$
$$+ E * \chi_{C_p^- \cap \Delta_1}(p') - E * \chi_{C_p^- \cap \Delta_1}(p))$$

As $O\text{-}BCB_1'$ is symmetric about $\{x_2 = 0\}$,

$$E * \chi_{C_p^- \cap (O\text{-}BCB_1')}(p') = E * \chi_{C_p^- \cap (O\text{-}BCB_1')}(p)$$

Hence

$$\psi_2^{(1)}(\delta) = \psi_1(\delta) = 2(E * \chi_{C_p^- \cap \Delta_1}(p') - E * \chi_{C_p^- \cap \Delta_1}(p))$$

Combining this with (2.4), we immediately get the conclusion.

Now, we come to consider the singularities of $\psi_5(\delta) = \frac{1}{2}\psi_1(\delta) = E * \chi_{C_p^- \cap \Delta_3}(p) - E * \chi_{C_p^- \cap \Delta_1}(p)$ at $\delta = 0$.

Since $C_p^- \cap \Delta_1$ and $C_p^- \cap \Delta_3$ are all empty as $0 < \delta < \varepsilon$, we immediately have

Proposition 3 $\psi_5(\delta) = 0$ *as* $0 < \delta < \varepsilon$.

When $-\varepsilon < \delta < 0$, we have the following conclusion

Proposition 4 *There is a constant* $C > 0$ *such that* $\psi_5(\delta) \leqslant -C(-\delta)^{5/2}$ *as* $-\varepsilon < \delta < 0$.

Proof Denote by $p = (1, (1+\delta)\cos\alpha, (1+\delta)\sin\alpha)$ where $0 < \alpha < \frac{\pi}{4}$, then

$$\psi_5(\delta) = (E * \chi_{C_p^- \cap \Delta_3})(p) - (E * \chi_{C_p^- \cap \Delta_1})(p)$$

$$= \iiint\limits_{C_p^- \cap \Delta_3} \frac{(2\pi)^{-1} dt dx_1 dx_2}{[(1-t)^2 - ((1+\delta)\cos\alpha - x_1)^2 - ((1+\delta)\sin\alpha - x_2)^2]^{1/2}}$$

$$- \iiint\limits_{C_p^- \cap \Delta_1} \frac{(2\pi)^{-1} dt dx_1 dx_2}{[(1-t)^2 - ((1+\delta)\cos\alpha - x_1)^2 - ((1+\delta)\sin\alpha - x_2)^2]^{1/2}}$$

Considering the transformations

$$\begin{cases} t' = t \\ x_1' = x_1 - (1+\delta)\cos\alpha \\ x_2' = x_2 - (1+\delta)\sin\alpha \end{cases} \quad \text{and} \quad \begin{cases} t'' = (x_1'^2 + x_2'^2)^{1/2} + t' \\ x_1'' = (2(x_1'^2 + x_2'^2))^{1/2} \\ x_2'' = \operatorname{tg}^{-1}(x_2'/x_1') - \pi/2 \end{cases}$$

we have $\left|\dfrac{\partial(t,x_1,x_2)}{\partial(t',x_1',x_2')}\right| = x_1'/2.$

Let $p_0=(0,(1+\delta)\cos\alpha,(1+\delta)\sin\alpha)$ and $p_{1+\delta}=(1+\delta,(1+\delta)\cos\alpha,(1+\delta)\sin\alpha)$ are two points in the space $O\text{-}tx_1x_2$, then by noting the positions of Δ_1 and Δ_2, we get

$$C_P^- \cap \Delta_1 \neq \varnothing, C_P^- \cap \Delta_3 \neq \varnothing \text{ as } P \in \overline{p_{1+\delta}p}$$

and

$$C_P^- \cap \Delta_1 = \varnothing, C_P^- \cap \Delta_3 = \varnothing \text{ as } P \in \overset{\circ}{\overline{p_0p_{1+\delta}}}$$

where the notation " $\overset{\circ}{\overline{\quad}}$ " represents the line without two end points.

By the above transformations, we get

$$\psi_5(\delta) = \frac{(2\pi)^{-1}}{2}\left[\int_{1+\delta}^{1} dt'' \iint_{\partial C^-_{p_{t''}}\cap\Delta_3} \frac{x_1' dx_1' dx_2'}{*} - \int_{1+\delta}^{1} dt'' \iint_{\partial C^-_{p_{t''}}\cap\Delta_1} \frac{x_1' dx_1' dx_2'}{*}\right] \tag{2.6}$$

where (in the space (t,x_1,x_2)) $p_{t''}=(t'',(1+\delta)\cos\alpha,(1+\delta)\sin\alpha)$, $\partial C^-_{p_{t''}}=\{(t,x_1,x_2)\mid (t''-t)^2=((1+\delta)\cos\alpha-x_1)^2+((1+\delta)\sin\alpha-x_2)^2\}$ and $*=((1-t)^2-((1+\delta)\cos\alpha-x_1)^2-((1+\delta)\sin\alpha-x_2)^2)^{1/2}$.

Considering the following transformation:

$$\begin{cases} t'' = t' & (2.7)\\ x_1'' = \dfrac{\sqrt{2}}{2}x_1'x_2' & (2.8)\\ x_2'' = t' - \dfrac{\sqrt{2}}{2}x_1' & (2.9)\end{cases}$$

we have $\left|\dfrac{\partial(t',x_1',x_2')}{\partial(t'',x_1'',x_2'')}\right| = 2/x_1'$, and $x_2''=t'=t$.

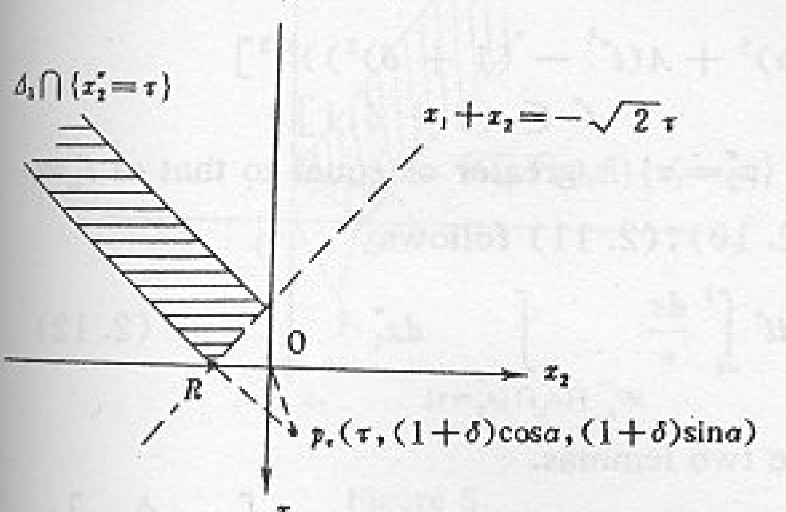

Figure 3

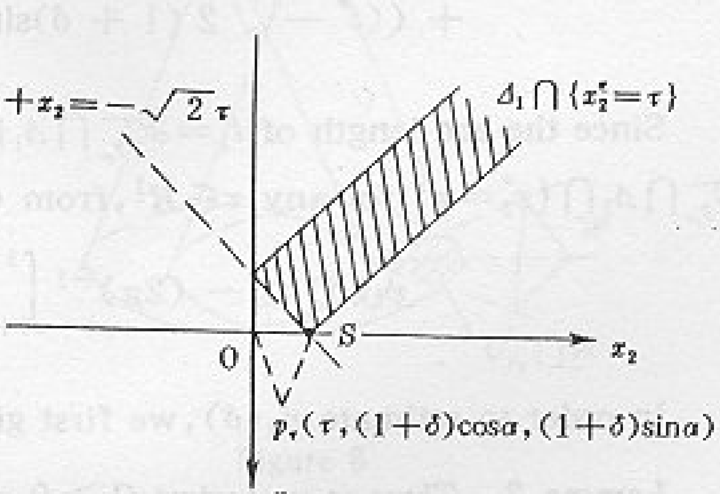

Figure 4

Combining this with (2.6), we get

$$\psi_5(\delta) = (2\pi)^{-1}\int_{1+\delta}^{1} dt'' \int_{-\infty}^{\infty} \frac{d\tau}{*} \int_{\partial C^-_{p_{t''}}\cap\Delta_3\cap\{x_2''=\tau\}} dx_1''$$

$$-(2\pi)^{-1}\int_{1+\delta}^{1}dt''\int_{-\infty}^{\infty}\frac{d\tau}{*}\int_{\partial C^-_{p_{t''}}\cap\Delta_1\cap\{x''_2=\tau\}}dx''_1 \qquad (2.10)$$

where $*=((1-\tau)^2-(t''-\tau)^2)^{1/2}$.

From (2.8) it follows that the meaning of dx''_1 in $\int_{\partial C^-_{p_{t''}}\cap\Delta_3\cap\{x''_2=\tau\}}dx''_1$ is the length element of arc$\partial C^-_{p_{t''}}\cap\Delta_3\cap\{x''_2=\tau\}$ and dx''_1 of $\int_{\partial C^-_{p_{t''}}\cap\Delta_1\cap\{x''_2=\tau\}}dx''_1$ is also the length element. Without loss of generality, we assume the length is zero when $\partial C^-_{p_{t''}}\cap\Delta_3\cap\{x''_2=\tau\}$ (or $\partial C^-_{p_{t''}}\cap\Delta_1\cap\{x''_2=\tau\}$) is empty in (2.10).

We consider $\partial C^-_{p_{t''}}\cap\Delta_3\cap\{x''_2=\tau\}$ and $\partial C^-_{p_{t''}}\cap\Delta_1\cap\{x''_2=\tau\}$ (The above figures are sections on $\{x''_2=\tau\}$.), and know that the shortest distance from $p_\tau=(\tau,(1+\delta)\cos\alpha,(1+\delta)\sin\alpha)$ (in (t,x_1,x_2)) to $\Delta_3\cap\{x''_2=\tau\}$ is the length of $\overline{p_\tau R}$, as $0<\tau\leqslant(1+\delta)\sin\left(\frac{\pi}{4}-\alpha\right)$, and that from p_τ to $\Delta_1\cap\{x''_2=\tau\}$ is that of $\overline{p_\tau S}$, where $R=(\tau,0,-\sqrt{2}\tau)$ and $S=(\tau,0,\sqrt{2}\tau)$. By simple computation, we get

$$\text{the length of }\overline{p_\tau S}<t''-\tau<\text{the length of }\overline{p_\tau R},\qquad \text{as } a<\tau<b \qquad (2.11)$$

where

$$a=\frac{1}{2}[-t''-\sqrt{2}(1+\delta)\sin\alpha+((t''+\sqrt{2}(1+\delta)\sin\alpha)^2+4(t''^2-(1+\delta)^2))^{1/2}]$$

$$b=\frac{1}{2}[-t''+\sqrt{2}(1+\delta)\sin\alpha+((t''-\sqrt{2}(1+\delta)\sin\alpha)^2+4(t''^2-(1+\delta)^2))^{1/2}]$$

$$t''\in(1+\delta,1]$$

Since the arc length of $l_1^\tau=\partial C^-_{p_{t''}}\cap\Delta_1\cap\{x''_2=\tau\}$ is greater or equal to that of $l_3^\tau=\partial C^-_{p_{t''}}\cap\Delta_3\cap\{x''_2=\tau\}$ for any $\tau\in\mathbf{R}^1$, from (2.10), (2.11) follows

$$\psi_5(\delta)\leqslant-(2\pi)^{-1}\int_{1+\delta}^{1}dt''\int_a^b\frac{d\tau}{*}\int_{\partial C^-_{p_{t''}}\cap\Delta_1\cap\{x''_2=\tau\}}dx''_1 \qquad (2.12)$$

In order to estimate $\psi_5(\delta)$, we first give two lemmas.

Lemma 2 *There is a constant $C_1>0$ such that $b-a\geqslant C_1(-\delta)$ as $t''\in\left[1+\frac{\delta}{2},1\right]$.*

By using Mid-value Theorem, we can immediately get this Lemma.

Lemma 3 *There is a constant $C_2>0$ such that*

$$\int_{\partial C^-_{p_{t''}}\cap\Delta_1\cap\{x''_2=\tau\}}dx''_1\geqslant C_2(-\delta)$$

as $t\in\left[1+\frac{\delta}{2},1\right]$ and $\tau\in(a,b-\frac{1}{2}(b-a)]$.

Proof On $\{x_2''=\tau\}$, $\partial C^-_{p_r''}\cap\Delta_1\cap\{x_2''=\tau\}$ is shown in Figure 5. We easily know that the length of $\overline{SS_1}$ is equal to the distance from S to $\partial C^-_{p_r''}\cap\Delta_1\cap\{x_2''=\tau\}$, where S_1 is on the stretched line of $\overline{p_rS}$.

Since

$$\begin{aligned}&(\text{the length of }\overline{SS_1})=(t''-\tau)-(\text{the length of }\overline{p_rS})\\&=(t''-\tau)-((1+\delta)^2\cos^2\alpha+((1+\delta)\sin\alpha-\sqrt{2}\,\tau)^2)^{1/2}\\&\geqslant C_3((t''-\tau)^2-(1+\delta)^2\cos^2\alpha-((1+\delta)\sin\alpha-\sqrt{2}\,\tau)^2)\\&=C_3(b-\tau)\{\tau+\frac{1}{2}[t''-\sqrt{2}(1+\delta)\sin\alpha\\&\quad+((t''-\sqrt{2}(1+\delta)\sin\alpha)^2+4(t''^2-(1+\delta)^2))^{1/2}]\}\\&\geqslant C_4(b-\tau)\end{aligned}$$

by using Lemma 2 we have

$$\text{the length of }\overline{SS_1}\geqslant C_5(-\delta)$$

as $\tau\in(a,b-\frac{1}{2}(b-a)]$, where C_2, C_4, C_5 are all positive constants.

Therefore there is a constant $C_2>0$ such that

$$\int_{\partial C^-_{p_r''}\cap\Delta_1\cap\{x_2''=\tau\}}dx_1''\geqslant C_2(-\delta)$$

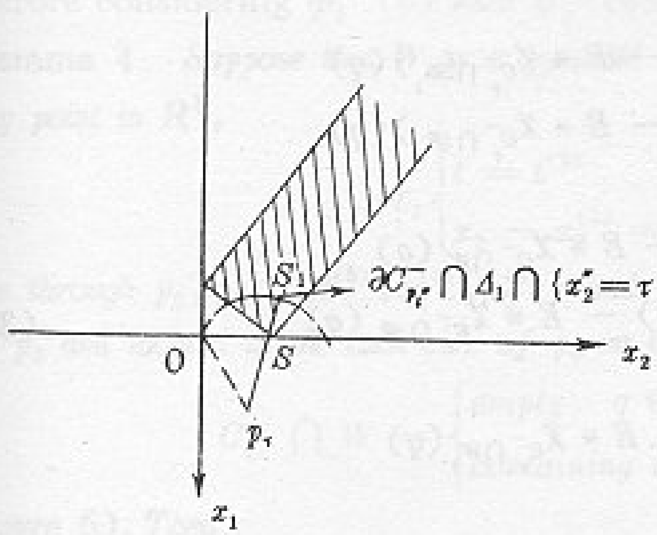

Figure 5

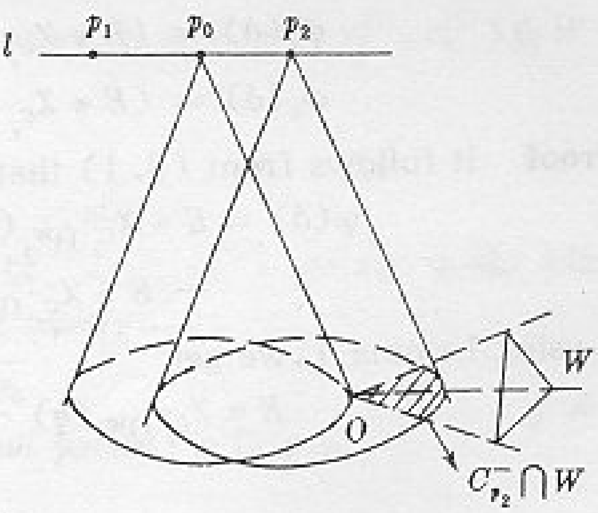

Figure 6

Using the expression of $*$, Lemma 2 and Lemma 3, we immediately get this Proposition from (2.12).

According to Proposition 3 and Proposition 4 we have the third derivative of $\psi_5(\delta)=E*\chi_{C^-_p\cap\Delta_1}(p)-E*\chi_{C^-_p\cap\Delta_1}(p)$ does not exist at $\delta=0$. And, using Proposition 2 we conclude

Theorem 2 *The total differential of $u_3(p)$ does not exist on $\widehat{GD}$.*

3. The Singularities of u_3 on $\widehat{DE}$

First, let's introduce some notations (see Figure 2):

$W_1 = \Sigma_2^+ \cap \Sigma_3^- \cap \Sigma_5^+$, i. e. $O\text{-}BB'M$

$V_1^{(1)} = \Sigma_2^+ \cap \Sigma_5^+$; $\quad W_1' = V_1^{(1)} \backslash W_1 = \Sigma_2^+ \cap \Sigma_5^+ \cap \Sigma_3^+$, i. e. $O\text{-}BB_1'M$

$V_2^{(1)} = \Sigma_3^+ \cap \Sigma_5^+$; $\quad W_1'' = V_2^{(1)} \backslash W_1' = \Sigma_3^+ \cap \Sigma_5^+ \cap \Sigma_2^-$, i. e. $O\text{-}B_1'B_1M$

$W_2 = \Sigma_1^+ \cap \Sigma_4^- \cap \Sigma_5^-$, i. e. $O\text{-}BB'N$

$V_1^{(2)} = \Sigma_1^+ \cap \Sigma_5^-$; $\quad W_2' = V_1^{(2)} \backslash W_2 = \Sigma_1^+ \cap \Sigma_5^- \cap \Sigma_4^+$, i. e. $O\text{-}B_1'BN$

$V_2^{(2)} = \Sigma_5^- \cap \Sigma_4^+$; $\quad W_2'' = V_2^{(2)} \backslash W_2 = \Sigma_5^- \cap \Sigma_4^+ \cap \Sigma_1^-$, i. e. $O\text{-}B_1'B_1N$

$\mathscr{B}_1 = \Sigma_1^- \cap \Sigma_4^+ \cap \Sigma_5^+$, i. e. $O\text{-}N_1B_1B_1'$

$\mathscr{B}_2 = \Sigma_2^- \cap \Sigma_3^+ \cap \Sigma_5^-$, i. e. $O\text{-}M_1B_1B_1'$

Obviously, we have

$$\mathscr{A} = W_1 \cup W_2, \quad \mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \tag{3.1}$$

and there are not any internal points of $W_1 \cap W_2$ and $\mathscr{B}_1 \cap \mathscr{B}_2$.

For any point $q_0 = (1, \cos\beta, \sin\beta)$, where $\beta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$, shown in Figure 2, we come to study the singularities of $u_3(q)$ at $q = q_0$. For any $\delta \in (-\varepsilon, \varepsilon)$ where $\varepsilon > 0$ small enough, we define

$$\varphi(\delta) = (E * \chi_{C_q^- \cap \mathscr{A}} - E * \chi_{C_q^- \cap \mathscr{B}})(q) \tag{3.2}$$

i. e. $u_3(q)$, where $q = (1, (1+\delta)\cos\beta, (1+\delta)\sin\beta)$. Then, we have

Proposition 5 $\quad \varphi(\delta) \underset{\delta=0}{\sim} \varphi_1(\delta) + \varphi_2(\delta)$

where

$$\varphi_1(\delta) = (E * \chi_{C_q^- \cap W_1'} - E * \chi_{C_q^- \cap \mathscr{B}_1})(q)$$

$$\varphi_2(\delta) = (E * \chi_{C_q^- \cap W_2'} - E * \chi_{C_q^- \cap \mathscr{B}_2})(q)$$

Proof It follows from (3.1) that

$$\varphi(\delta) = E * \chi_{C_q^- \cap W_1}(q) + E * \chi_{C_q^- \cap W_2}(q) - E * \chi_{C_q^- \cap \mathscr{B}_1}(q) - E * \chi_{C_q^- \cap \mathscr{B}_2}(q) \tag{3.3}$$

By using Lemma 1, we get

$$E * \chi_{C_q^- \cap W_1}(q) \underset{\delta=0}{\sim} - E * \chi_{C_q^- \cap W_1'}(q)$$

and

$$E * \chi_{C_q^- \cap W_1''}(q) \underset{\delta=0}{\sim} - E * \chi_{C_q^- \cap W_1'}(q)$$

So

$$E * \chi_{C_q^- \cap W_1}(q) \underset{\delta=0}{\sim} E * \chi_{C_q^- \cap W_1''}(q) \tag{3.4}$$

In the same way we have

$$E * \chi_{C_q^- \cap W_2}(q) \underset{\delta=0}{\sim} E * \chi_{C_q^- \cap W_2''}(q) \tag{3.5}$$

Combining (3.4) and (3.5) with (3.3), we get

$$\varphi(\delta) \underset{\delta=0}{\sim} (E * \chi_{C_q^- \cap W_1''} - E * \chi_{C_q^- \cap \mathscr{B}_1})(q) + (E * \chi_{C_q^- \cap W_2''} - E * \chi_{C_q^- \cap \mathscr{B}_2})(q)$$
$$= \varphi_1(\delta) + \varphi_2(\delta)$$

Now, we consider the singularities of $\varphi_1(\delta)$ *at* $\delta=0$. *First, we have*

Proposition 6 $\varphi_1(\delta)\underset{\delta=0}{\sim}-(\varphi_1^{(1)}(\delta)+\varphi_1^{(2)}(\delta))$

where

$$\varphi_1^{(1)}(\delta)=E*\chi_{C_q^-\cap W_3'}(q),\quad \varphi_1^{(2)}(\delta)=E*\chi_{C_q^-\cap W_4'}(q)$$
$$W_3'=\Sigma_2^-\cap\Sigma_6^+\cap\Sigma_1^-,\quad \text{i. e. } O\text{-}M_1N_1B_1'$$
$$W_4'=\Sigma_6^-\cap\Sigma_3^+\cap\Sigma_4^+,\quad \text{i. e. } O\text{-}M_1N_1B_1$$

Proof Denote by

$W_3=\Sigma_6^+\cap\Sigma_1^+\cap\Sigma_2^-$, i. e. $O\text{-}MN_1B_1$

$W_4=\Sigma_6^-\cap\Sigma_4^-\cap\Sigma_3^+$, i. e. $O\text{-}MN_1B_1$

$V_3=\Sigma_2^-\cap\Sigma_6^+;W_3'=V_3\backslash W_3=\Sigma_2^-\cap\Sigma_6^+\cap\Sigma_1^-$, i. e. $O\text{-}M_1N_1B_1'$

$V_4=\Sigma_6^-\cap\Sigma_3^+;W_4'=V_4\backslash W_4=\Sigma_6^-\cap\Sigma_3^+\cap\Sigma_4^+$, i. e. $O\text{-}M_1N_1B_1$.

Then, it follows

$$\mathscr{B}_1\subset W_1',W_1''\backslash\mathscr{B}_1=W_3\cup W_4$$

and $W_3\cap W_4$ does not exist any internal points. So

$$\varphi_1(\delta)=E*\chi_{C_q^-\cap W_3}(q)+E*\chi_{C_q^-\cap W_4}(q)\tag{3. 6}$$

By using Lemma 1, we have

$$E*\chi_{C_q^-\cap W_3}(q)\underset{\delta=0}{\sim}-E*\chi_{C_q^-\cap W_3'}(q)=-\varphi_1^{(1)}(\delta)$$

and

$$E*\chi_{C_q^-\cap W_4}(q)\underset{\delta=0}{\sim}-E*\chi_{C_q^-\cap W_4'}(q)=-\varphi_1^{(2)}(\delta)$$

Combining (3. 6) with this we immediately get this conclusion.

Before considering $\varphi_1^{(1)}(\delta)$ and $\varphi_1^{(2)}(\delta)$, we give a lemma.

Lemma 4 *Suppose that* W *is a pyramid with vertex* O, $p_0=(t^{(0)},x_1^{(0)},x_2^{(0)})\notin W$ *is an arbitrary point in* $\mathbf{R}^3$,

$$l:\begin{cases}t=t^{(0)}\\ x_2-x_2^{(0)}=k(x_1-x_1^{(0)})\end{cases}$$

is a line through p_0, $p_1=(t^{(0)},x_1^{(1)},x_2^{(1)})$ *and* $p_2=(t^{(0)},x_1^{(2)},x_2^{(2)})$ *are two points which are close to* p_0 *and located in the both side of* p_0 *on* l, *and satisfy*

$$C_q^-\cap W=\begin{cases}\text{empty} & q\in\overline{p_1p_0}\\ \text{containing internal points} & q\in\overline{p_2p_0}\end{cases}\qquad q\neq p_0$$

(*see Figure* 6). *Then*

1) *There are constants* $C_1,C_2>0$ *such that*

$C_1\delta^{5/2}\leqslant E*\chi_{C_q^-\cap W}(q)\leqslant C_2\delta^{5/2}$ *as* $q\in\overline{p_2p_0}$, $q\neq p_0$ *and* $\delta=\operatorname{dist}(p_0,q)$

2) $E*\chi_{C_q^-\cap W}(q)\notin C^{5/2+\eta}(p_0)$ *for any* $\eta>0$.

The proof of this lemma can be found in [4].

By the above lemma, we immediately get

Proposition 7

1) $\varphi_1^{(1)}(\delta)=\varphi_1^{(2)}(\delta)=0$ *as* $0<\delta<\varepsilon$.

2) *There are constants* $C_1,C_2>0$ *such that*

$$C_1(-\delta)^{5/2}\leqslant\varphi_1^{(1)}(\delta)\leqslant C_2(-\delta)^{5/2}$$

and

$$C_1(-\delta)^{5/2} \leqslant \varphi_1^{(2)}(\delta) \leqslant C_2(-\delta)^{5/2}$$

as $-\varepsilon < \delta < 0$.

We have the same conclusion as $\varphi_2(\delta)$.

Using Propositions 5, 6, 7 we get

Theorem 3 *There are singularities of* $u_3(p)$ *on* $\widehat{DE}$, *and* $u_3(p)$ *does not belong to* $C^{5/2+\eta}$ *for any* $\eta > 0$.

Summing up Theorems 2, 3 we immediately get the main theorem—Theorem 1 given in Section 1.

Acknowledgement The author is profoundly grateful to Professors Chen Shuxing and Hong Jiaxing for their advice and encouragement while preparing this paper.

References

[1] Chen Shuxing, Reflection and interaction of progressing waves for semilinear wave equations. *Research Report, Preprint Series, No.* **19**, (1986) (Nankai Institute of Mathematics).

[2] Bony J. M., Interaction des singularitiés pour les équations aux dérivées partielles non lineaires. *Sem. Goulaouic-Meyer-Schwartz*, No. **22** (1979-1980).

[3] Bony J. M., Interaction des singularities équations aux dérivées partielles non lineaires. *Sem. Goulaouic-Meyer-Schwartz*, No. **22** (1979—1980).

[3] Bony J. M., Interaction des singularitiés pour les équations aux dérivées partielles non linéaires. *Sem. Goulaouic-Meyer-Schwartz*, No. **2** (1980—1981).

[4] Rauch J. & Reed M., Singularities produced by the nonlinear interaction of three progressing waves; Examples. *Comm. P. D. E.*, 7 (1982), 1117—1133.