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THE CONDITIONS FOR SOME LINEAR PARTIAL DIFFERENTIAL EQUATIONS TO BE SOLVABLE IN $\mathscr{S}'$

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Abstract Using Bargmann's transformation and some basic results of theory of analytic functions with several complex variables, we have disscussed two classes of LPDOs in this paper. We prove that each operator of one class of them is surjective both from $\mathscr{S}'$ to $\mathscr{S}'$ and from L^2 to L^2, but not injective, and each operator of another class is injective from $\mathscr{S}'$ to $\mathscr{S}'$ but not surjective. And in the latter case, the necessary and suffcient conditions for the corresponding equations to be solvable in $\mathscr{S}'$ are given.

Key Words Solvability; Hermite expansion; Bargmann Space.
Classifications 35A; 35D; 35G.

0. Introduction

Let $P(x,D_x)$ be a linear partial differential operator with polynomial coefficients, and let $\mathscr{S}'$ be the space of temperate distributions on R^n, we consider the equation

$$P(x,D_x)u = f \tag{0. 1}$$

If (0. 1) has a solution $u\in\mathscr{S}'$ for given $f\in\mathscr{S}'$, we call (0. 1) solvable in $\mathscr{S}'$. $P(x,D_x)$ will be called a solvable operator if $P(x,D_x)$ is a mapping from $\mathscr{S}'$ onto itself.

The problem on solvability in $\mathscr{S}'$ was paid great attention to long ago. Hörmander and Lojasiewicz first proved the existence of fundamental solutions in $\mathscr{S}'$ for equations with constant coefficients respectively in [1] and [2]. Since then, several mathematicians have simplified or improved the proofs (see [3], [4] and [5]).

It may be worth to point out that the problems on the local solvability and the hypoellipticity of left (right) invariant differential operators on nilpotent Lie groups, by means of their unitary representations, can be reduced to ones on solvability and uniqueness of solutions of (0. 1) in $\mathscr{S}'$, as be shown in some recent works (see Section2 and Section 4 of Chapter 2 in [6]). Therefore it seems to be reasonable that study of solvability and uniqueness in $\mathscr{S}'$ will be paid much attention to. Just because of this background, we made a systematic study on the problem for a class of LPDOs in [7]. In this paper, other two classes of LPDOs are discussed. We find that each operator of the first class is surjective both from $\mathscr{S}'$ to $\mathscr{S}'$ and from L^2 to L^2, but not injective. And each operator of another class is injective from $\mathscr{S}'$ into $\mathscr{S}'$, but not surjective. In the latter case, the necessary and sufficient conditions for the corresponding equations to be solvable in $\mathscr{S}'$ are given.

The main tools used in this paper are Bargmann's transformation and some basic results of theory of analytic functions with several complex variables.

1. A Class of Unsolvable Operators

Let

$$E_j = \frac{1}{\sqrt{2}}(x_j - \partial_j), \quad j = 1, 2, \cdots, n$$

and

$$E^\alpha = \prod_{j=1}^{n} E_j^{\alpha_j}, \quad \alpha = (\alpha_1, \cdots, \alpha_n) \in I_+^n$$

where I_+ is the set of nonnegative integers and $I_+^n = \overbrace{I_+ \times \cdots \times I_+}^{n}$.

Let

$$B(x, \partial_x) = b(E) = \sum_{|\alpha| = m} b_\alpha E^\alpha \tag{1.1}$$

with complex constants b_α.

It is clear that $b(E)$ is a linear partial differential operator whose coefficients are polynomials of x with the principal part of constant coefficient. We shall prove that $b(E)$ is not solvable in $\mathscr{S}'$ and give necessary and sufficient conditions for the equation

$$b(E)u = f, \qquad f \in \mathscr{S}' \tag{1.2}$$

to have a solution in $\mathscr{S}'$.

Since the Fourier transformation makes $B(E)$ be turned into $\sum_{|\alpha| \leqslant m} (i)^{|\alpha|} b_\alpha E^\alpha$ which is still of the same kind as $b(E)$, it is no use for solving our problem. Therefore we shall introduce the following Bargmann transformation instead of the Fourier transformation.

Let $\mathfrak{Z}$ be the space of holomorphic functions on C^n. For given real number k, set

$$\mathfrak{Z}^k = \left\{ f : f \in \mathfrak{Z}, |f|_k = \left(\int |f(z)|^2 (1 + |z|^2)^k du(z) \right)^{1/2} < +\infty \right\}$$

where $du(z) = \pi^{-n} e^{-|z|^2} d^n z$ with $d^n z = \prod_{j=1}^{n} d\xi_j d\eta_j$, where $z = \xi + i\eta$. Put $\mathfrak{Z}^{+\infty} = \bigcup_k \mathfrak{Z}^k$ and $\mathfrak{Z}^{-\infty} = \bigcap_k \mathfrak{Z}^k$.

Let

$$A(z, x) = \pi^{-n/4} \exp\left[-\frac{1}{2}(z^2 + x^2) + \sqrt{2}\, z \cdot x \right], \qquad \forall\, z \in C^n, x \in R^n$$

where $z^2 = \sum_{j=1}^{n} z_j^2$ and $z \cdot x = \sum_{j=1}^{n} z_j x_j$.

Define the Bargmann transformation $T : \mathscr{S}'(R^n) \to \mathfrak{Z}^{+\infty}$ as follows:

$$Tf = \int_{R^n} f(x) A(z, x) dx, \quad f \in \mathscr{S}' \tag{1.3}$$

where the integration is formal, the real meaning is that the distribution f acts on $A(z, \cdot)$. In view of [8], T produces the following topological isomorphic:

$$\mathscr{S}' \to \mathfrak{Z}^{+\infty}, \quad \text{and} \quad L^2 \to \mathfrak{Z}^0 \quad \text{and} \quad \mathscr{S} \to \mathfrak{Z}^{-\infty} \tag{1.4}$$

In particular

$$T\restriction_{L^2}: L^2 \to \mathfrak{Z}^0$$

is a unitary mapping

$$(f,g) = \int \bar{f} g dx = (Tf, Tg)$$

$$= \int \overline{Tf} Tg d\mu(z), \quad f \in \mathscr{S}', \quad g \in \mathscr{S} \tag{1.5}$$

For given $p \in R$, we introduce the space

$$\mathfrak{Z}_p = \{F(z) \in \mathfrak{Z};\ \| F \|_p = \sup_{C^n}[(1+|z|^2)^{p/2} e^{-|z|^2/2} |F(z)|]\}$$

By [8], we have

$$\mathfrak{Z}^{+\infty} = \bigcup_k \mathfrak{Z}^k = \bigcup_p \mathfrak{Z}_p \tag{1.6}$$

For the moment, we only simply mention these basic properties of T and $\mathfrak{Z}^{+\infty}$ etc. The others will be replenished if necessary.

Theorem 1.1 *The equation* (1.2) *has a solution in* $\mathscr{S}'$ *if and only if there exist* $C>0$ *and* $p \in R$ *such that*

$$\left| \partial_z^\alpha \left(\frac{F}{b} \right)(z) \right| \leqslant C(1+|z|^2)^{p/2} e^{|z|^2/2} \tag{1.7}$$

provided $|\alpha| \leqslant 2$ *and* $z \in \Omega = \{z; z \in C^n, b(z) \neq 0\}$, *where*

$$F(z) = (Tf)(z), \quad b(z) = \sum_{|\alpha| \leqslant m} b_\alpha z^\alpha$$

Proof It is easy to verify

$$TE_j u = z_j Tu, \quad \forall\, u \in \mathscr{S}', j = 1, 2, \cdots, n$$

Writing

$$Tu = U$$

we see that (1.2) is equivalent to

$$b(z)U(z) = F(z), \quad U(z) \in \mathfrak{Z}^{+\infty} \tag{1.2'}$$

Hence it suffices to show that (1.2') has a solution $U \in \mathfrak{Z}^{+\infty}$ if and only if (1.7) holds.

Let $U \in \mathfrak{Z}^{+\infty}$ be a solution of (1.2'). Then (1.6) yields $U \in \mathfrak{Z}_{p'}$ with some $p' \in R$. Hence

$$|U(z)|(1+|z|^2)^{p'/2} e^{-|z|^2/2} \leqslant \| U \|_{p'}$$

that is

$$|U(z)| \leqslant (1+|z|^2)^{-p'/2} e^{|z|^2/2} \| U \|_{p'}$$

$$= C_0 (1+|z|^2)^{-p'/2} e^{|z|^2/2}$$

From the equation (1.2), we see

$$U(z) = \frac{F(z)}{b(z)}, \quad \forall\, z \in \Omega$$

Thus

$$\left| \frac{F(z)}{b(z)} \right| \leqslant C_0 (1+|z|^2)^{-p'/2} e^{|z|^2/2}, \quad \forall\, z \in \Omega$$

Let $U = Tu$ and $u \in \mathscr{S}'$, then a direct calculation gives

$$\partial_z^\alpha U = \partial_z^\alpha (Tu) = T({}^t E^\alpha u)$$

where ${}^t E^\alpha = \prod_{j=1}^{n} {}^t E_j^{\alpha_j}$ with $E_j = \frac{1}{\sqrt{2}}(x_j + \partial_{x_j})$.

Note that ${}^t E^\alpha u \in \mathscr{S}'$ therefore $\partial_z^\alpha U \in 3^{+\infty}$ for any $\alpha \in I_+^n$ and $U \in 3^{+\infty}$.

For any $\alpha \in I_+^n$, by the same argument as in the case $\alpha = 0$, there are $C_\alpha > 0$ and $p'_\alpha \in I_+^n$ such that

$$\left| \partial_z^\alpha \left(\frac{F}{b}(z) \right) \right| \leqslant C_\alpha (1 + |z|^2)^{-p'_\alpha/2} e^{|z|^2/2} \tag{1.8}$$

for any $z \in \Omega$. Putting $C = \max\limits_{|\alpha| \leqslant 2} C_\alpha$ and $p = \max\limits_{|\alpha| \leqslant 2} (-p'_\alpha)$, we then obtain (1.7) immediately.

To prove the sufficiency, we are going to show that $F(z)/b(z)$ can be extended from Ω to $\mathbf{C}^n$ so as to be a member of $3^{+\infty}$. We begin with the following lemma.

Lemma 1.1 *Let $g(t)$, $p(t)$ be analytic functions of real variable $t \in \mathbf{R}$ with complex value, and let $p(t)$ be a (nonzero) polynomial of order m, $t_1, \cdots, t_k$ being its all real zeros. Suppose that there is a constant $C > 0$ independent of t, such that*

$$\left| \frac{g(t)}{p(t)} \right| \leqslant C$$

for each $t \neq t_j$, $j = 1, 2, \cdots, k$. Then $g(t)/p(t)$ can be extended to an analytic function in $\mathbf{R}$.

Proof We make the decomposition

$$p(t) = (t - t_1)^{l_1} \cdots (t - t_k)^{l_k} q(t)$$

where the polynomial $q(t)$ does not have real zero.

Expanding $g(t)$ into the power series of $t - t_1$, then by hypotheses, we know that $|g(t)/p(t)|$ is bounded in the set $\{t \in \mathbf{R}, 0 < t - t_1 < \delta\}$ for sufficient small $\delta > 0$. Hence there is an analytic function $g_1(t)$ satisfying

$$\frac{g(t)}{p(t)} = \frac{g_1(t)}{(t - t_2)^{l_2} \cdots (t - t_k)^{l_k} q(t)}$$

for each $t \neq t_j$, $j = 2, \cdots, k$.

Expanding g_1 into the power series of $t - t_2$, we can find an analytic g_2 satisfying

$$\frac{g(t)}{p(t)} = \frac{g_2(t)}{(t - t_3)^{l_3} \cdots (t - t_k)^{l_k} q(t)}$$

for each $t \neq t_j$, $j = 3, \cdots, k$.

Repeating the above procedure, finally we can obtain an analytic function g_k satisfying $g(t)/p(t) = g_k(t)/q(t)$ for all $t \in \mathbf{R}$. Since the polynomial $q(t)$ has not any real zero, the right side of above equality defines an analytic extension of $g(t)/p(t)$ on $\mathbf{R}$.

Before giving the proof of the sufficiency of Theorem 1.1, we need to study the set $\Omega = \{z; b(z) \neq 0, z \in C^n\}$. Let $\partial\Omega$ be the boundary of Ω. Since $b(z)$ is a given polynomial and $b \neq 0$, by the uniqueness theorem of analytic functions, the boundary $\partial\Omega$ is just the set of the zeros of $b(z)$ in C^n, hence it is also the complementary set of Ω relative to C^n. Moreover, Ω and $\partial\Omega$ are unbounded subsets of C^n, and Ω is not a convex set. These make the proof not easy. We will overcome the difficulty by means of Lemma 1.1.

Proof of sufficiency For $r > 0$, we introduce the subset of Ω as follows.

$$\Omega_r = \{z: b(z) \neq 0, |z| < r\}$$

Let

$$\phi(z', z'', t) = (\frac{F}{b})(z' + t(z'' - z'))$$

where $z', z'' \in \Omega_r$ and $t \in \boldsymbol{R}$. Since $b(z' + t(z'' - z'))$ takes the value $b(z') \neq 0$ at $t = 0$, $b(z' + t(z'' - z'))$ is not a vanishing polynomial of t. Hence, by the condition (1.7) with $\alpha = 0$ and Lemma 1.1. $\phi(z', z'', t)$ is analytic for $t \in \boldsymbol{R}$. Consequently.

$$\begin{aligned}(\frac{F}{b})(z'') - (\frac{F}{b})(z') &= \phi(z', z'', 1) - \phi(z', z'', 0) \\ &= \int_0^1 \frac{\partial \phi}{\partial t}(z', z'', t)dt \\ &= \int_0^1 \sum_{j=1}^{n} \frac{\partial}{\partial z_j}(\frac{F}{b})(z' + t(z'' - z')) \cdot (z''_j - z'_j)dt\end{aligned}$$

For fixed $z', z'' \in \Omega_r$, $b(z' + t(z'' - z'))$, as a polynomial of t, has possibly (at most m) zeros in $(0,1)$. But it is easy to show that these zeros do not destroy the integrability of the above integration. In fact, if $t \in (0,1)$ is not one of these zeros, then the condition (1.7) (taking $|\alpha| = 1$) implies the estimate

$$\begin{aligned}&\left|\sum_{j=1}^{n} \frac{\partial}{\partial z_j}(\frac{F}{b})(z' + t(z'' - z')) \cdot (z''_j - z'_j)\right| \\ &\leqslant nc(1 + |z' + t(z'' - z')|^2)^{p/2} e^{|z' + t(z'' - z')|^2/2} |z'' - z'| \\ &\leqslant nc(1 + r^2)^{p/2} e^{r^2/2} |z'' - z'| \\ &= C_1(r) |z'' - z'|\end{aligned}$$

Hence

$$|F(z'')/b(z'') - F(z')/b(z')| \leqslant C_1(r)|z'' - z'|, \quad \forall z', z'' \in \Omega_r$$

Now let z^0 be any zero of $b(z)$, then the proceeding argument implies $z^0 \in \partial\Omega$. Taking $r = 2|z^0|$, according to above inequality and Cauchy criterion of limit existence, we see that

$$\lim_{\substack{z' \in \Omega \\ z' \to z^0}} (\frac{F}{b})(z') = \lim_{\substack{z' \in \Omega \\ z' \to z^0}} (\frac{F}{b})(z')$$

exists. Hence we can define

$$U(z) = \begin{cases} F(z)/b(z), & \text{if} \quad b(z) \neq 0, \\ \lim\limits_{\substack{z' \in \Omega \\ z' \to z}} F(z')/b(z'), & \text{if} \quad b(z) = 0 \end{cases}$$

It is easy to know that $U(z)$ is analytic in Ω, satisfies the condition (1.7) and is continuous on C^n. Next we want to verify that $\frac{\partial}{\partial x_j}U$ and $\frac{\partial}{\partial y_j}U$ are continuous in C^n and satisfy the C-R condition of analytic functions. In fact, taking $|\alpha| = 2$ in (1.7), then by the same argument as above, we can prove that $\lim\limits_{\substack{z' \to z \\ z' \in \Omega}} \frac{\partial}{\partial z_j}(\frac{F}{b})(z')$ exist $(j = 1, 2, \cdots, n)$, and that the functions are continuous in C^n which are defined by

$$U_j(z) = \begin{cases} \dfrac{\partial U}{\partial z_j}(z) & \text{if} \quad b(z) \neq 0 (\text{i. e. } z \in \Omega) \\ \lim\limits_{\substack{z' \to z \\ z' \in \Omega}} \dfrac{\partial U}{\partial z_j}(z) & \text{if} \quad b(z) = 0 (\text{i. e. } z \in \partial\Omega) \end{cases}$$

Let $z = \xi + i\eta, \xi, \eta \in R^n$, we shall verify that

$$\frac{\partial U}{\partial \xi_j} = U_j, \qquad j = 1, 2, \cdots, n$$

For $z \in \Omega$, since U is analytic in Ω, it follows from the C-R condition of analytic functions that

$$\frac{\partial U}{\partial \xi_j} = \frac{\partial U}{\partial z_j} = U_j, \quad j = 1, 2, \cdots, n$$

For $z \in \partial\Omega$ (i. e. , as above, $z \in C^n \backslash \Omega$), let

$$\Psi_j(z', s, t) = U(z' + tse_j)$$

where $z' \in \Omega$; $s \in (-1/2, 1/2)$, $t \in R$, and e_j is the j-th coordinate vector in R^n. By a similar argument as above, we can prove that Ψ_j are analytic in t for fixed z' and s, and that $z' + tse_j \in \Omega$ for any real t, except the possible real zeros $t_1, t_2, \cdots, t_k (k \leqslant m)$ of $b(z' + tse_j)$ (as a polynomial of t), thus we have

$$\lim_{s \to 0} (U(z + se_j) - U(z))/s = \lim_{s \to 0} \lim_{\substack{z' \to z \\ z' \in \Omega}} (U(z' + se_j) - U(z'))/s$$

$$= \lim_{s \to 0} \lim_{\substack{z' \to z \\ z' \in \Omega}} (\Psi_j(z', s, 1) - \Psi_j(z', s, 0))/s$$

$$= \lim_{s \to 0} \lim_{\substack{z' \to z \\ z' \in \Omega}} \frac{1}{s} \int_0^1 \frac{\partial \Psi_j}{\partial t}(z' + tse_j) dt$$

$$= \lim_{s \to 0} \lim_{\substack{z' \to z \\ z' \in \Omega}} \int_0^1 \frac{\partial U}{\partial z_j}(z' + tse_j) dt$$

$$= \lim_{s \to 0} \lim_{\substack{z' \to z \\ z' \in \Omega}} \int_0^1 U_j(z' + tse_j) dt = U_j(z)$$

where the final equality follows by the continuity of U_j in C^n. So that, we get

$$\frac{\partial U}{\partial \xi_j} = U_j$$

in C^n, $j = 1, 2, \cdots, n$. Similarly, we also have

$$\frac{\partial U}{\partial \eta_j} = \sqrt{-1} U_j, \qquad j = 1, 2, \cdots, n$$

Hence

$$\frac{\partial U}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial U}{\partial \xi_j} + \sqrt{-1} \frac{\partial U}{\partial \eta_j} \right)$$

$$= \frac{1}{2}(U_j - U_j) = 0, \qquad j = 1, 2, \cdots, n$$

This shows that U satisfies the C-R equation of analytic functions in C^n. Therefore U is a holomophic function on C^n, namely $U \in \mathfrak{Z}$. Taking $\alpha = 0 \in I_+^n$ in the condition (1.7), we see that $U \in \mathfrak{Z}^{+\infty}$.

It is obvious that

$$b(z)U(z) = F(z)$$

in the open set Ω, hence it also holds on C^n by the uniqueness theorem of analytic function. Therefore the equation (1.2) has a solution $u = T^{-1}U \in \mathscr{S}'$.

Corollary 1.1 *If the equation* (1.2) *is solvable in* $\mathscr{S}'$, *then the zero of the polynomial* $b(z)$ *in* C^n *is just the zero of* $F(z) = (Tf)(z)$.

Corollary 1.2 *The LPDO* (1.1) *is injective from* $\mathscr{S}'$ *into itself, but not surjective.*

Proof Corollary 1.1 implies that (1.1) is not surjective. The injectivity follows from the uniqueness theorem of analytic functions. In fact, suppose that $u \in \mathscr{S}'$ satisfies $b(E)u = 0$, then $U = Tu$ satisfies $b(z)U(z) = 0$ and $U \in \mathfrak{Z}^{+\infty}$. Because $b \neq 0$ in the open set Ω, we get $U = 0$ in Ω. Hence the above uniqueness theorem implies that U vanishes on C^n.

Remark 1.1 Some results of theory of analytic functions with several complex variables used above (the C-R condition, the uniqueness theorem etc.) refer to R. Nara-Simham's *Analysis on Real and Complex Manifolds*.

Remark 1.2 From the proof of Theorem 1.1, it is easy to get a necessary condition (1.8) which appears to be stronger than (1.7). But from the proof of the sufficiency, to guarantee the solvability it suffices that (1.8) holds for $|\alpha| \leqslant 2$. This shows that if (1.8) holds for $|\alpha| \leqslant 2$, so does it for all $\alpha \in I_+^n$.

If (1.8) holds only for $|\alpha| \leqslant 1$ (or weaker $|\alpha| = 0$), can do it hold for all $\alpha \in I_+^n$? It seems to be a quite difficult problem.

2. A Class of Solvable Operators in $\mathscr{S}'$

Let

$$'E_j = \frac{1}{\sqrt{2}}(x_j + \partial_j)$$

with $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1, \cdots, n$, and let

$$'E^\alpha = \prod_{j=1}^{n} {'E_j^{\alpha_j}}$$

and

$$b('E) = \sum_{|\alpha| \leqslant m} b_\alpha {'E^\alpha} \tag{2.1}$$

where $b_\alpha (|\alpha| \leqslant m)$ are complex constants and $b_\beta \neq 0$ for some β with $|\beta| = m$.

In this section, we shall prove that $b('E)$ is a solvable operator on $\mathscr{S}'$, that is, for any $f \in \mathscr{S}'$, there exists $u \in \mathscr{S}'$ satisfying

$$b('E)u = f \tag{2.2}$$

Using the Bargmann transformation T mentioned in Section 1 and writing $U(z) = Tu$, we know that (2.2) is equivalent to

$$b(\partial_z)U = F \tag{2. 2'}$$

where

$$b(\partial_z) = \sum_{|\alpha|\leqslant m} b_\alpha \partial_z^\gamma$$

with $z=(z_1,\cdots,z_n)\in C^n$, $\partial_{z_j}=\frac{1}{2}\left(\frac{\partial}{\partial \xi_j}-i\frac{\partial}{\partial \eta_j}\right)$, while $z_j=\xi_j+i\eta_j$, $\xi_j,\eta_j\in \boldsymbol{R}$ and $F(z)=(Tf)(z)$. Then, it suffices to show that the equation (2. 2′) has a solution $U\in 3^{+\infty}$ for any $F\in 3^{+\infty}$.

We first do some preparatory works.

Proposition 2. 1 *The equation* (2. 2′) *is solvable in* $3^{+\infty}$, *if and only if there are* $C>0$ *and* $p\in R$ *such that*

$$|(F,v)_\mu| \leqslant C\|\bar{b}(z)v(z)\|_p, \quad \forall\ v\in 3^{+\infty} \tag{2. 3}$$

where

$$\bar{b}(z) = \sum_{|\beta|\leqslant m} \bar{b}_\beta z^\beta$$

with $\bar{b}$ *the conjugate complex number of* b.

Proof Write

$$U = Tu, \quad u\in \mathscr{S}'$$

and

$$V = Tv, \quad v\in \mathscr{S}$$

Note that

$$T(b({}^tE)u) = b(\partial_z)Tu = b(\partial_z)U$$

and that

$$T(\bar{b}(E)v) = \bar{b}(z)Tv = \bar{b}(z)V$$

thus using (1. 5), we have

$$\begin{aligned}(b(\partial_z)U,V)_\mu &= (T(b({}^tE)u,Tv)_\mu = (b({}^tE)u,v) = (\bar{b}(E)^*u,v)\\ &= (u,\bar{b}(E)v) = (Tu,T(\bar{b}(E)v))_\mu = (U,\bar{b}(z)V)_\mu\end{aligned}$$

that is

$$(b(\partial_z)U,V)_\mu = (U,\bar{b}(z)V)_\mu \tag{2. 4}$$

Now, suppose that (2. 2′) has a solution $U\in 3^{+\infty}=\bigcup_k 3^k$. We assume $U\in 3^{p'}$ with some $p'\in R$, then

$$|u|_{p'}^2 = \int |U(z)|^2(1+|z|^2)^{p'}d\mu(z) < +\infty$$

and hence for all $V\in 3^{-\infty}$, we get

$$\begin{aligned}|(F,V)_\mu| &\leqslant |(b(\partial_z)U,V)_\mu| = |(U,\bar{b}(z)V(z))_\mu|\\ &\leqslant |U|_{p'}|\bar{b}(z)V(z)|_{-p'}\end{aligned}$$

Consequently

$$\bar{b}(z)V(z)\in 3^{-\infty}$$

By [8], in $3^{-\infty}$ there exists the seminorm sequence $\{\|\cdot\|_p\}_{p\in R}$ being equivalent to the seminorm sequence $\{|\cdot|_p\}_{p\in R}$, where $\|\cdot\|_p$ is the seminorm of 3_p defined in the preceding section. Thus there is $p=p(p')\in \boldsymbol{R}$ such that

$$|W|_{-p'} \leqslant C'\|W\|_p, \quad \forall\ W\in 3^{-\infty}$$

where C' is a positive constant only dependent on p'. Hence we get

$$|(F,V)_\mu| \leqslant C'|U|_{p'} \| \bar{b}(z)V(z) \|,$$
$$= C \| \bar{b}(z)V(z) \|, \quad \forall \in 3^{-\infty}$$

The necessity is proved.

Conversely, suppose that the estimate (2. 3) holds, and let

$$3_b^{-\infty} = \{\bar{b}(z)V(z), V(z) \in 3^{-\infty}\} \subset 3^{-\infty}$$

then (2. 3) shows that $(F,V)_\mu$ determines a continuous linear functional on the subspace $3_b^{-\infty}$ of $3^{-\infty}$. Thus, by the functional extension theorem in the Frechet space $3^{-\infty}$, there is $U \in (3^{-\infty})' = 3^{+\infty}$ satisfying

$$(F,V)_\mu = (U, \bar{b}(z))_\mu, \qquad \forall V \in 3^{-\infty}$$

Hence combining with

$$(U, \bar{b}(z)V(z))_\mu = (b(\partial_z)U, V)_\mu$$

obtained in proof of the necessity, we have

$$(b(\partial_z)U, V)_\mu = (F,V)_\mu, \quad \forall V \in 3^{-\infty}$$

Consequently

$$b(\partial_z)U = F$$

This shows that the equation (2. 2') has a solution $U \in 3^{+\infty}$.

The following lemma is a direct extension of Ehrenpreis lemma.

Lemma 2. 1 *Let a and $z^{(j)}$, $(j=1,2,\cdots,m)$ be given $m+1$ points on the complex plane $\mathbf{C}$; and let R be a positive number, then there exists a circle $L_r(a)$ with centre a and radius $r \leqslant R$ such that for every $z \in L_r(a)$*

$$\min_{1\leqslant j\leqslant m} \{|z - z^{(j)}|\} \geqslant \frac{R}{2(m+1)} \tag{2. 5}$$

Proof Without loss of generality, we assume that a is the origin and that $0 < r_1 \leqslant r_2 \leqslant \cdots \leqslant r_k \leqslant R < r_{k+1} \leqslant \cdots \leqslant r_m$ with $r_j = |z^{(j)}|$, $j=1,2,\cdots,m$. Then $r_1,\cdots,r_k$ divide the interval $[0,R]$ into at most $k+1$ subintervals, so that there is at least an interval with the length $\geqslant \frac{R}{k+1}$. We denote it by $[r_l, r_{l+1}]$ and let $r = (r_l + r_{l+1})/2$, then the circle with center a and radius r is just desired.

Lemma 2. 2 *Let $q(w) = q_0 w^{m-1} + \cdots + q_m$ be a polynomial of $w \in \mathbf{C}$ with $q_0 \neq 0$, and let R be a given positive number and $a \in \mathbf{C}$, then there is a circle $L_r(a)$ with centre a and radius $r \leqslant R$ such that*

$$|q(w)| \leqslant \left(\frac{R}{2(m+1)}\right)^m |q_0|, \quad \forall w \in L_r(a) \tag{2. 6}$$

Proof Denote the m zeros of q by $z^{(j)}$, $j=1,2,\cdots,m$. Then

$$q(w) = q_0(w - z^{(1)})\cdots(w - z^{(m)})$$

Moreover, by Lemma 2. 1, there is a circle $L_r(a)$ such that

$$|w - z^{(j)}| \leqslant \frac{R}{2(m+1)}, \qquad j = 1,2,\cdots,m$$

if $w \in L_r(a)$. Therefore (2. 6) is obtained.

Proposition 2. 2 *Let $b(z) = \sum_{|\beta|\leqslant m} b_\beta z^\beta$ be a polynomial of order m in $z=(z_1,\cdots,z_n)$*

$\in C^n$, *then for each* $p \in R$, *there is a constant* C *only dependent on* p *such that*

$$|V(z)|_{p-m} \leqslant C \| b(z)V(z) \|_p \tag{2.7}$$

for all $V \in 3^{-\infty}$.

Proof First, let $z=(z_1, w), w=(z_2, \cdots, z_n) \in C^{n-1}$, then $b(z)$ can be written as

$$b(z) = \sum_{j=0}^{m_1} B_j(w) z_1^{m_1 - j} \tag{2.8}$$

where $B_j(w)$ are polynomials of $w \in C^{n-1}$, $m_1 \leqslant m$. We shall prove that for $p \in R$, there is $C_1(p) > 0$ such that

$$\| B_0(w)V(z) \|_{p-m_1} \leqslant C_1(p) \| b(z)V(z) \|_p, \quad \forall V \in 3^{-\infty} \tag{2.9}$$

In fact, set

$$C_\rho = \{z: |z| \leqslant \rho, \quad z \in C^n\}$$

and $z=(z_1, w) \in C_\rho$, then when $B_0(w) \neq 0$, by taking $R = \dfrac{1}{1+\rho}$ and $a = z_1$ in Lemma 2.2, we get from (2.6) and (2.8) that

$$|b(z', w)| \geqslant [2(m_1+1)(1+\rho)]^{-m_1} |B_0(w)| \tag{2.10}$$

where $L_r(z_1)$ is the circle with radius $r = \dfrac{1}{1+\rho}$ and centre z_1 on the plane C. It is obvious that (2.10) still holds for $B_0(w) = 0$. Let $V \in 3^{-\infty}$, then we have

$$|b(z_1', w)V(z_1', w)| \geqslant [2(m_1+1)(1+\rho)]^{-m_1} |B_0(w)| |V(z_1', w)|$$

when $z_1' \in L_r(z_1)$. Hence we obtain

$$\sup_{|z| < \rho + (1+\rho)^{-1}} |b(z)V(z)|$$
$$\geqslant [2(m_1+1)(1+\rho)]^{-m_1} |B_0(w)| |V(z_1', w)| \tag{2.11}$$

where $z=(z_1, w) \in C_\rho$. Fixing w temporarily, then $V(s, w)$ can be regarded as an analytic function of $s \in C$. Thus, using the maximum principle on the circular domain $|s - z_1| \leqslant r$, we get

$$|V(z, w) \leqslant |V(s^*, w)|$$

where s^* is some point on $L_r(z_1)$. Set $z_1' = s^*$ in (2.11) we obtain

$$\sup_{|z| < \rho + (1+\rho)^{-1}} |b(z)V(z)| \geqslant [2(m_1+1)(1+\rho)]^{-m_1}$$
$$\cdot \sup_{|z| \leqslant \rho} \{ |B_0(w)| |V(z_1, w)| \}$$

Since the inequality holds for all $z=(z, w) \in C_\rho$, we have

$$\sup_{|z| < \rho + (1+\rho)^{-1}} |b(z)V(z)| \geqslant [2(m_1+1)(1+\rho)]^{-m_1} \cdot \sup_{|z| \leqslant \rho} \{ |B_0(w)| \cdot |V(z_1, w)| \}$$

Hence, using the maximum principle of analytic functions, we get

$$\sup_{|z| < \rho + (1+\rho)^{-1}} |b(z)V(z)| \geqslant [2(m_1+1)(1+\rho)]^{-m_1} \sup_{|z| = \rho} |B_0(w)V(z)|$$

and consequently

$$\sup_{|z| = \rho + (1+\rho)^{-1}} |b(z)V(z)(1+|z|^2)^{p/2} e^{-|z|^2/2}|$$
$$= [1 + (\rho + (1+\rho)^{-1})^2]^{p/2} e^{-(\rho + (1+\rho)^{-1})^2/2} \cdot \sup_{|z| = \rho + (1+\rho)^{-1}} |b(z)V(z)|$$
$$\geqslant [1 + (\rho + (1+\rho)^{-1})^2]^{p/2} e^{-(\rho + (1+\rho)^{-1})^2/2}$$

$$\cdot [2(m_1+1)(1+\rho)]^{-m_1} \cdot \sup_{|z|=\rho} |B_0(w)V(z)|$$

Now let

$$g(\rho) = 2(m_1+1)^{-m_1}(1+\rho)^{-m_1}[1+(\rho+(1+\rho)^{-1})^2]^{p/2}$$
$$\cdot (1+\rho^2)^{(m_1-p)/2} e^{(\rho^2-(\rho+(1+\rho)^{-1})^2)/2}$$

then it is easy to verify that $g(\rho)$ has a positive infimum C_0. Therefore we know

$$\sup_{|z|=\rho+(1+\rho)^{-1}} |b(z)V(z)(1+|z|^2)^{p/2}e^{-|z|^2/2}|$$
$$\geqslant g(\rho) \sup_{|z|=\rho} \{|B_0(w)V(z)|(1+|z|^2)^{(p-m_1)/2}e^{-|z|^2/2}\}$$
$$\geqslant C_0 \sup_{|z|=\rho} |B_0(w)V(z)(1+|z|^2)^{(p-m_1)/2}e^{-|z|^2/2}|$$

Hence, we obtain the estimate

$$\| b(z)V(z) \|_p \geqslant \sup_{0<\rho<+\infty} \sup_{|z|=\rho+(1+\rho)^{-1}} \{(1+|z|^2)^{p/2}e^{-|z|^2/2}|b(z)V(z)|\}$$
$$\geqslant C_0 \sup_{0<\rho<+\infty} \sup_{|z|=\rho} \{(1+|z|^2)^{(p-m_1)/2}e^{-|z|^2/2}|b(z)V(z)|\}$$
$$= C_0 \| B_0(w)V(z) \|_{p-m}$$

that is

$$\| b(z)V(z) \|_p \geqslant C \| B_0(w)V(z) \|_{p-m_1}, \quad \forall V \in 3^{-\infty}$$

Secondly, write $B_0(w)$ as

$$B_0(w) = \sum_{j=0}^{m_2} B_j^{(1)}(w_1) z_2^{m_2-j}$$

with $w_1 = (z_3, \cdots, z_n) \in C^{n-2}$, where $B_j^{(1)}(w_1)$ are polynomials of w_1. Then according to the preceding proof we get

$$\| b(z)V(z) \|_p \geqslant C \| B_0(w)V(z) \|_{p-m_1}$$
$$\geqslant C_1 \| B_0^{(1)}(w_1)V(z) \|_{p-m_1-m_2}$$

A finite repetition of above procedure finally leads to

$$\| b(z)V(z) \|_p \geqslant C' \| V(z) \|_{p-(m_1+\cdots+m_n)}$$

Note that $m_1+\cdots+m_n \leqslant m$, thus we obtain

$$\| V(x) \|_{p-(m_1+\cdots+m_n)} \geqslant \| V \|_{p-m}$$

Hence (2.7) is proved.

Theorem 2.1 *For any $f \in \mathscr{S}'$, the equation (2.2) has a solution in $\mathscr{S}'$.*

Proof As stated above and according to Proposition 2.1, it suffices to verify the estimate (2.3) holds.

Let $F \in 3'$, p being a real number. Let l be such a number that $l+p>n$, then the integral $\int (1+|z|^2)^{-(p+1)} d^n z$ converges. Let $V \in 3^{-\infty}$, then we have the estimate

$$|(F,V)_n| \leqslant |F|_p |V|_{-p}$$
$$= |F|_p \Big[\int (1+|z|^2)^{-p} |V(z)|^2 dz \Big]^{1/2}$$
$$= \pi^{-n/2} |F|_p \Big\{ \int (1+|z|^2)^{l/2} e^{-|z|^2/2} |V(z)|^2 \cdot (1+|z|^2)^{-(l+p)} d^n z \Big\}^{1/2}$$

$$\leqslant C \| V \|_l$$

where

$$C = \pi^{-n/2} |F|_p \left(\int (1 + |z|^2)^{-(l+p)} d^n z \right)^{1/2}$$

Hence, by substituting $b(z)$ by $\bar{b}(z)$ in Proposition 2.2, we get

$$|(F,V)_\mu| \leqslant C \| V \|_l \leqslant C' \| \bar{b}(z) V(z) \|_{l+m}$$

This is just (2.3).

Corollary 2.1 *The LPDO (2.1) is a surjective mapping from $\mathscr{S}'$ to itself, but not an injective one.*

Proof Theorem 2.1 shows that (2.1) is surjective from $\mathscr{S}'$ to $\mathscr{S}'$. To prove that (2.1) is not injective, it suffices to show that the equation (2.2′) has a nontrivial solution in $\mathscr{S}'$ when $F \equiv 0$. Set

$$U(z) = e^{z^0 z}, \quad \forall\, z = (z_1, \cdots, z_n) \in C^n$$

where $z^0 \in C^n$ is an arbitrary zero of $b(z)$ and

$$z^0 z = \sum_{j=1}^{n} z_j^0 z_j$$

then it is easy to see that $U(z) \in 3^{-\infty}$ and that

$$b(\partial_z) U(z) = b(z^0) e^{z^0 z} = 0$$

Thus, the homogeneous equation relative to (2.2′) has really a nontrivial solution in $3^{+\infty}$.

Theorem 2.2 *For each $f \in L^2(R^n)$, the equation (2.2) has a solution $u \in L^2(R^n)$.*

Proof We first formulate the prior estimate

$$\| \varphi \|_{L^2} \leqslant C \| \bar{b}(E) \varphi \|_{L^2}, \qquad \forall\, \varphi \in \mathscr{S}(R^n) \tag{2.12}$$

In fact, let $T\varphi = \Psi(z)$, then $\Psi \in 3^{-\infty}$. Hence, by (2.4) we have

$$(\bar{b}(z), \bar{b}(z))_\mu = (\Psi, b(\partial_z)[\bar{b}(z)\Psi(z)])_\mu$$

$$= \left(\Psi, \sum_{|\alpha| \leqslant m} \frac{\bar{b}^{(\alpha)}(z)}{\alpha!} b^{(\alpha)}(\partial_z) \Psi \right)_\mu$$

$$= \sum_{|\alpha| \leqslant m} \frac{1}{\alpha!} (\Psi, \bar{b}^{(\alpha)}(z) b^{(\alpha)}(\partial_z) \Psi)_\mu$$

$$= \sum_{|\alpha| \leqslant m} \frac{1}{\alpha!} (b^{(\alpha)}(\partial_z) \Psi, b^{(\alpha)}(\partial_z) \Psi)_\mu$$

that is

$$|\bar{b}(z) \Psi(z)|_0^2 = \left| \sum_{|\alpha| \leqslant m} \frac{1}{\alpha!} b^{(\alpha)}(\partial_z) \right|_0^2$$

where $b^\alpha(z) = \partial_z b(z)$. Since $b(z)$ is a polynomial of order m, there is $\alpha' \in I_+^n$ with $|\alpha'| = m$ such that $b_{\alpha'} \neq 0$. Because of

$$\frac{b^{(\alpha')}(z)}{\alpha'!} = b_{\alpha'}$$

we have

$$\frac{b^{(\alpha')}(\partial_z)\Psi}{\alpha'!} = b_{\alpha'} \Psi$$

Hence we conclude

$$|b_a||\Psi|_0 \leqslant |\Psi(z)\bar{b}(z)|_0$$

that is

$$|\Psi|_0 \leqslant C|\bar{b}(z)\Psi(z)|_0 \tag{2.13}$$

By(1. 5), we see that

$$|\Psi|_0 = |T\varphi|_0 = \|\varphi\|_{L^2}$$

and

$$|\bar{b}(z)\Psi(z)|_0 = |\bar{b}(z)T\varphi|_0 = |T(\bar{b}(E)\varphi)|_0 = \|\bar{b}(E)\varphi\|_{L^2}$$

Substituting them into (2. 13), then (2. 12) is obtained immediately.

Now let $f\in L^2(R^n)$, then for every $\varphi\in\mathscr{S}'(R^n)$ we have

$$|(f,\varphi)| \leqslant \|f\|_{L^2}\|\varphi\|_{L^2} \leqslant C\|f\|_{L^2}\|\bar{b}(E)\varphi\|_{L^2}$$

By this inequality and a standard argument, we can find $u\in L^2$ satisfying

$$(f,\varphi) = (u,\bar{b}(E)\varphi) = (u,b({}^tE)^*\varphi) = (b({}^tE)u,\varphi), \quad \forall\varphi\in\mathscr{S}$$

Hence the fact that $\mathscr{S}$ is dense in L^2 yields

$$b({}^tE)u = f$$

so $u\in L^2$ is a solution of the equation (2. 2). The proof is completed.

Finally, we give a simple example to illustrate the application of our results to theory of right invariant differential operators on the Heisenberg group.

Let

$$q = (q_1,\cdots,q_n)\in R^n, \quad p = (p_1,\cdots,p_n)\in R^n$$

and $t\in R$, then (t,q,p) can be regarded as a local coordinate of the Heisenberg group H^n. Write

$$T = \frac{\partial}{\partial t}$$

$$L_j = \frac{\partial}{\partial q_j} - \frac{1}{2}p_j\frac{\partial}{\partial t}$$

$$M_j = \frac{\partial}{\partial p_j} + \frac{1}{2}q_j\frac{\partial}{\partial t}, \quad j = 1,2,\cdots,n$$

We consider the right invariant differential operator of the form

$$K = \sum_{j=1}^{n}(L_j + iM_jT)^2 \quad \text{with } i = \sqrt{-1} \tag{2.14}$$

It is easy to know that K is an operator of non-principal type. We shall show that K is hypoelliptic.

In view of [6], a unitary representation of K in $L^2(R^n)$ is

$$\sigma_K(\pm\lambda)(x,D) = \sum_{j=1}^{n}(|\lambda|^{1/2}x_j - |\lambda|^{1/2}\frac{\partial}{\partial x_j})^2, \quad \lambda\in R\backslash 0 \tag{2.15}$$

It is obvious that the principal part of $\sigma_K(\pm 1)(x,D)$ is $\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}$, which is an elliptic operator. If we can prove that (2. 15) is an injective mapping from $\mathscr{S}'(R^n)$ into itself, the hypoellipticity of K follows directly from Proposition 4. 7 of [6]. We take the similarity transformation

$$x_j = |\lambda|^{1/2}y_j, \quad j = 1,2,\cdots,n$$

such that (2. 15) is changed into the operator $2|\lambda|^2\sum_{j=1}^{n}E_j^2$. By Corollary 1. 2, the latter is an injective mapping from $\mathscr{S}'$ to itself and so is from $\mathscr{S}$ to $\mathscr{S}$. Therefore K is really a hypoelliptic operator on H^n.

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