J. Partial Differential Equations
Vol. 3, *No.* 1(1990), 13—20

INITIAL-BOUNDARY VALUE PROBLEM FOR A DEGENERATE QUASILINEAR PARABOLIC EQUATION OF ORDER $2m$[①]

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(Received July 27, 1987; revised June 20, 1989)

Abstract In this paper we consider the initial-boundary value problem for the higher-order degenerate quasilinear parabolic equation

$$\frac{\partial u(x,t)}{\partial t}+\sum_{|\alpha|\leqslant M}(-1)^{|\alpha|}D^{\alpha}A_{\alpha}(x,t,\delta u,D^{m}u)=0$$

Under some structural conditions for $A_{\alpha}(x,t,\delta u,D^{m}u)$, existence and uniqueness theorem are proved by applying variational operator theory and Galërkin method.

Key Words Higher-order degenerate equation; semibounded-variational operator; Galërkin method.

Classifications 35K35; 35K65.

1. Introduction

Let Ω be a bounded domain in R^{n}, $Q=\Omega\times(0,T]$. Consider the following initial—boundary value problem for the parabolic equation of order $2m$:

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}+\displaystyle\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^{\alpha}A_{\alpha}(x,t,\delta u,D^{m}u)=0, & (x,t)\in Q\\ \delta u=(u,Du,\cdots,D^{m-1}u) & \\ \delta u=0, \quad (x,t)\in\partial\Omega\times(0,T], & \\ u(x,0)=u_{0}(x), \quad x\in\Omega, u_{0}(x)\in L^{2}(\Omega) & \end{cases}\tag{1}$$

When $A_{t}(u)=\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^{\alpha}A_{\alpha}(x,t,\delta u,D^{m}u)$ for each t in $[0,T]$ is a regular elliptic operator in the Sobolev space $W^{m,p}(\Omega)$, the problem (1) had been considered by [1]—[3]. In this paper we discuss initial-boundary value problem (1) for a weak degenerate equation. This is the generalization of a result obtained by the writers for the equation of second order (see [4]).

First we introduce the fundamental space V which denotes the completion of $\mathring{C}^{m}(\Omega)=\{\varphi(x,\cdot)\in C^{m}(\bar{\Omega})$: which vanish on a neighborhood of $\partial\Omega\}$ with respect to norm

$$\|\varphi(x,\cdot)\|_{V}=\Big\{\sum_{|\alpha|=m}\|\lambda(x)D^{\alpha}\varphi(x,\cdot)\|^{p}_{L^{p}(\Omega)}\Big\}^{1/p}$$

where $\lambda(x)\in L^{p}(\Omega)$, $\lambda^{-1}(x)\in L^{s}(\Omega)$, $s>n>1$, $\dfrac{1}{s}+\dfrac{1}{p}=\dfrac{1}{q}<1$, $2\leqslant p\leqslant\dfrac{ns}{s-n}$.

① This work is supported by the National Natural Science Fund of China.

Now we introduce $L^p(0,T;V)$ which is a space of functions defined on $[0,T]$ with values in V such that $\left\{\int_0^T \| u(x,t) \|_V^p dt\right\}^{1/p} < +\infty$.

Lemma 1.1 *V is a separable and reflexive Banach space (cf. [5]).*

Lemma 1.2 *V is continuously embedded in $\dot{W}^{m,q}$, and $\dot{W}^{m,q}$ is compactly embedded in $\dot{W}^{m-1,\sigma}$, where σ satisfies $p<\sigma<\frac{nq}{n-q}$, and $q=\frac{ps}{p+s}$.*

Proof It is sufficient to point out that the following inequality can be established for function $u(x,\cdot)\in \dot{C}^m(\Omega)$

$$\| D^m u \|_{L^q} = \left\{\int_\Omega |\lambda^{-1}(x)\lambda(x)D^m u|^q dx\right\}^{1/q} \leqslant \| \lambda^{-1}(x) \|_{L^s} \| \lambda(x) D^m u \|_{L^p}$$

By the interpolation inequality, $\left\{\sum_{|\alpha|=m} \| D^\alpha u \|_{L^q}^q\right\}^{1/q}$ would be an equivalent norm on $\dot{W}^{m,q}$, then the first conclusion of the lemma is obtained. By the compact imbedding theorem it is easy to show that $\dot{W}^{m,q}$ is compactly embedded in $\dot{W}^{m-1,\sigma}$ if σ satisfied $p<\sigma<\frac{nq}{n-q}$.

Lemma 1.3 *If $u(x,t)\in L^p(0,T;V)$ and $u'(x,t)=\frac{\partial u(x,t)}{\partial t}\in L^{p'}(0,T;V^*)$, then $u(x,t)\in C^0(0,T;L^2(\Omega))$ (cf. [3]).*

Definition *A function $u(x,t)\in L^p(0,T;V)\cap C^0(0,T;L^2(\Omega))$ is called a generalized solution of problem (1) if $u'(x,t)\in L^{p'}(0,T;V^*)$, $u(x,0)=u_0(x)$, and $u(x,t)$ satisfies*

$$\int_0^T \langle u',v\rangle dt + \int_0^T A_t(u,v)dt = 0, \quad \forall\, v(x,t)\in L^p(0,T;V)$$

where

$$\langle u',v\rangle = \int_\Omega u'(x,\cdot)v(x,\cdot)dx$$

$$A_t(u,v) = \sum_{|\alpha|\leqslant m}\int_\Omega A_\alpha(x,t,\delta u,D^m u)D^\alpha v(x,\cdot)dx \tag{2}$$

Furthermore we assume that $A_\alpha(x,t,\zeta,\xi)$ are Caratheodory functions and satisfy the following structural conditions:

Structural condition I

$$\sum_{|\alpha|=m}|A_\alpha(x,t,\zeta,\xi)\eta_\alpha| \leqslant a_0 \sum_{\substack{|\alpha|=|\beta|=m\\ |l|\leqslant m-1}} |\lambda(x)\eta_\alpha|(|\lambda(x)\xi_\beta|^{p-1}+|\zeta_l|^{\sigma/p'}+|a_1(x)|)$$

$$\sum_{|\alpha|\leqslant m-1}|A_\alpha(x,t,\zeta,\xi)| \leqslant b_0 \sum_{\substack{|\beta|=m\\ |l|\leqslant m-1}} (|\lambda(x)\xi_\beta|^{p/\sigma'}+|\zeta_l|^{\sigma/\sigma'}+|b_1(x)|)$$

where $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{\sigma}+\frac{1}{\sigma'}=1$, $a_1(x)\in L^{p'}(\Omega)$, $b_1(x)\in L^{\sigma'}(\Omega)$. The condition $\lambda^{-1}(x)\in L^s(\Omega)$ implies that the equation in (1) is weakly degenerate.

Remark There is a relation between the growth factors p and σ:

$$\frac{\sigma}{\sigma'} > \frac{\sigma}{p'} > \frac{p}{\sigma'} > p-1$$

Structural condition Ⅱ

$$\sum_{|\alpha|=m} A_\alpha(x,t,\zeta,\xi)\xi_\alpha \geqslant \mu_1 \sum_{|\beta|=m} |\lambda(x)\xi_\beta|^p - \nu_1 \sum_{|l|\leqslant m-1} |\zeta_l|^r - |a_2(x,t)|$$

$$\sum_{|\alpha|\leqslant m-1} A_\alpha(x,t,\zeta,\xi)\zeta_\alpha \geqslant -\mu_2 \sum_{|l|\leqslant m-1} |\zeta_l|^r - \nu_2 \sum_{\substack{|\beta|=m\\|l|\leqslant m-1}} |\zeta_l|(|\lambda(x)\xi_\beta|^{p/r'} + |b_2(x,t)|)$$

where $\mu_1>0$, μ_2, ν_1, $\nu_2\geqslant 0$ are constants, $1<r<q$, $a_2(x,t)\in C^0(0,T;L^1(\Omega))$, $b_2(x,t)\in C^0(0,T;L^{r'}(\Omega))$.

Structural condition Ⅲ

$$H_0: \sum_{|\alpha|=|\beta|=m} \frac{\partial A_\alpha(x,t,\zeta,\xi)}{\partial \xi_\beta}\eta_\alpha\eta_\beta \geqslant K_0 \sum_{|\alpha|=|\beta|=m} |\lambda(x)\xi_\beta|^{p-1}|\lambda(x)\eta_\alpha|^2$$

$$H_1: \left|\sum_{\substack{|\alpha|=m\\|l|\leqslant m-1}} \frac{\partial A_\alpha(x,t,\zeta,\xi)}{\partial \xi_l}\eta_\alpha\right| \leqslant K_1 \sum_{\substack{|\alpha|=m\\|l|\leqslant m-1}} |\lambda(x)\eta_\alpha|(|\zeta_l|^{\sigma/p'-1}+1)$$

$$H_2: \left|\sum_{\substack{|\alpha|\leqslant m-1\\|\beta|=m}} \frac{\partial A_\alpha(x,t,\zeta,\xi)}{\partial \xi_\beta}\eta_\beta\right|$$

$$\leqslant K_2 \sum_{\substack{|\beta|=m\\|l|\leqslant m-1}} |\lambda(x)\eta_\beta|(|\lambda(x)\xi_\beta|^{p/p'-p/\sigma} + |\zeta_l|^{\sigma/p'-1} + 1)$$

$$H_3: \left|\sum_{\substack{|\alpha|\leqslant m-1\\|l|\leqslant m-1}} \frac{\partial A_\alpha(x,t,\zeta,\xi)}{\partial \xi_l}\right| \leqslant K_3 \sum_{\substack{|\beta|=m\\|l|\leqslant m-1}} (|\lambda(x)\xi_\beta|^{p/\sigma'-p/\sigma} + |\zeta_l|^{\sigma/\sigma'-1} + 1)$$

where $K_0>0$, K_1, K_2, $K_3\geqslant 0$ are constants.

Example Consider the following equation of order 4 in $Q=\Omega\times(0,1]$

$$\frac{\partial u(x,t)}{\partial t} + D^2(\sqrt{xy}(u_{xx}+u_{xy}+u_{yy}) + \sqrt[4]{xy}(|u_x|^{7/6}+|u_y|^{7/6}+\sqrt{xyt})) = 0$$

where $\Omega=\{0<x<1, 0<y<1\}$.

For $p=2$, $n=2$, $s=3$ and $\lambda(x,y)=\sqrt[4]{xy}$, we have

$$\lambda^{-1}(x,y)\in L^s(\Omega),\ q=6/5,\ \sigma=5/2,\ \frac{\sigma}{p'}=5/4,$$

$$A(x,t,Du,D^2u) = \sqrt{xy}D^2u + \sqrt[4]{xy}|Du|^{7/6} + \sqrt{xyt}$$

satisfies all the structural conditions.

Remark Our work may be more easily done by the following hypotheses instead of the structural conditions Ⅰ and Ⅱ

$$|A_\alpha(x,t,\zeta,\xi)| \leqslant a_0\Big(\sum_{|\beta|=m}|\lambda(x)\xi_\beta|^{p-1} + \sum_{|l|\leqslant m-1}|\zeta_l|^{p-1} + |a_1(x)|\Big),\quad |\alpha|\leqslant m$$

and

$$\sum_{|\alpha|\leqslant m} A_\alpha(x,t,\zeta,\xi)\eta_\alpha \geqslant \mu_1\Big(\sum_{|\beta|=m}|\lambda(x)\xi_\beta|^p + \sum_{|l|\leqslant m-1}|\zeta_l|^p\Big) - a_2(x,t)$$

These hypotheses are simpler but stronger because

$$\frac{\sigma}{\sigma'} > \frac{\sigma}{p'} > \frac{p}{\sigma'} > p-1 \quad\text{and}\quad r<q<p$$

2. Some Properties of Operator A_t

Lemma 2.1 *Under the structural condition* Ⅰ, *a bounded operator* A_t *can be defined by*

$A_t(u,v)$ *from* (2), *such that* $u(x,\cdot)\in V\xrightarrow{A_t}A_t(u)\in V^*$ *and*

$$A_t(u,v) = \langle A_t(u),v\rangle,\quad \forall\, v(x,\cdot)\in V \tag{3}$$

Proof By the structural condition I, we have

$$\left|\sum_{|\alpha|=m}\int_\Omega A_\alpha(x,t,\delta u,D^m u)D^\alpha v dx\right|$$

$$\leqslant a_0 \sum_{\substack{|\alpha|=|\beta|=m\\|l|\leqslant m-1}}\int_\Omega |\lambda(x)D^\alpha v|(|\lambda(x)D^\beta u|^{p-1}+|D^l u|^{\sigma/p'}+|a_1(x)|)dx$$

$$\leqslant a_0\sum \|\lambda(x)D^\alpha v\|_{L^p}(\|\lambda(x)D^\beta u\|_{L^p}^{p-1}+\|D^l u\|_{L^\sigma}^{\sigma/p'}+\|a_1(x)\|_{L^{p'}})$$

$$\left|\sum_{|\alpha|\leqslant m-1}\int_\Omega A_\alpha(x,t,\delta u,D^m u)D^\alpha v dx\right|$$

$$\leqslant b_0 \sum_{\substack{|\beta|=m\\|l|\leqslant m-1}}\int_\Omega |D^l v|(|\lambda(x)D^\beta u|^{p/\sigma'}+|D^l u|^{\sigma/\sigma'}+|b_1(x)|)dx$$

$$\leqslant b_0\sum \|D^l v\|_{L^\sigma}(\|\lambda(x)D^\beta u\|_{L^p}^{p/\sigma'}+\|D^l u\|_{L^\sigma}^{\sigma/\sigma'}+\|b_1(x)\|_{L^{\sigma'}})$$

By Lemma 1.2 $V\hookrightarrow \mathring{W}^{m-1,\sigma}$, we obtain

$$|A_t(u,v)|\leqslant K(u)\|v(x,\cdot)\|_V \tag{4}$$

hence $A_t(u,v)$ is bounded linear functional with respect to $v(x,\cdot)$, and an operator A_t can be defined as (3). Further, if $\|u_n(x,\cdot)\|_V\leqslant M$, by (3) and (4) we have $|\langle A_t(u_n),v\rangle|\leqslant M_0\|v(x,\cdot)\|_V$, thus $\|A_t(u_n)\|_{V^*}\leqslant M_0$, i.e. A_t is bounded.

Corollary 2.1 *Let* $\theta\in(0,1]$, $u(x,\cdot),v(x,\cdot),\omega(x,\cdot)\in V$, *then*

$$\lim_{\theta\to 0^+}\langle A_t(u+\theta\omega),v\rangle = \langle A_t(u),v\rangle \tag{5}$$

Proof By the estimation established in Lemma 2.1 and by $\frac{\sigma}{\sigma'}>\frac{\sigma}{p'}>\frac{p}{\sigma'}>p-1$ we have

$$|\langle A_t(u+\theta\omega),v\rangle|$$

$$\leqslant a_0\sum_{\substack{|\alpha|=|\beta|=m\\|l|\leqslant m-1}}\|\lambda(x)D^\alpha v\|_{L^p}\cdot 2^{\sigma/\sigma'}(\|\lambda(x)D^\beta u\|_{L^p}^{p-1}+\|\lambda(x)D^\beta\omega\|_{L^p}^{p-1}$$

$$+\|D^l u\|_{L^\sigma}^{\sigma/p'}+\|D^l\omega\|_{L^\sigma}^{\sigma/p'}+\|a_1(x)\|_{L^{p'}})$$

$$+b_0\sum_{\substack{|\beta|=m\\|l|\leqslant m-1}}\|D^l v\|_{L^\sigma}\cdot 2^{\sigma/\sigma'}(\|\lambda(x)D^\beta u\|_{L^p}^{p/\sigma'}+\|\lambda(x)D^\beta\omega\|_{L^p}^{p/\sigma'}$$

$$+\|D^l u\|_{L^\sigma}^{\sigma/\sigma'}+\|D^l\omega\|_{L^\sigma}^{\sigma/\sigma'}+\|b_1(x)\|_{L^{\sigma'}})$$

each term of right side is independent of θ, hence using dominated convergence theorem (note that $A_\alpha(x,t,\zeta,\xi)$ are Caratheodory functions) we have

$$\lim_{\theta\to 0^+}\langle A_t(u+\theta\omega),v\rangle$$

$$=\int_\Omega \lim_{\theta\to 0^+}\sum_{|\alpha|\leqslant m}A_\alpha(x,t,\delta u+\theta\delta\omega,D^m u+\theta D^m\omega)D^\alpha v dx = \langle A_t(u),v\rangle$$

Lemma 2.2 *Under the structural condition* II, *the operator* A_t *satisfies*

$$\langle A_t(u),u\rangle\geqslant C_0\|u(x,\cdot)\|_V^p - N(t),\quad \textit{for all } u(x,t)\in L^p(0,T;V)$$

where $C_0>0$ is constant and $N(t)\in C^0[0,T]$.

Proof Since $1<r<q$ in structural condition Ⅱ and V is continuously embedded in $\dot{W}^{m,q}$, hence for $|l|\leqslant m-1$ we have

$$\| D^l u \|_{L^r}^r \leqslant \varepsilon_1 \| D^l u \|_{L^q}^q + N_1 \leqslant \varepsilon \| u \|_V^p + N_1$$

Thus

$$\int_\Omega \sum_{|\alpha|=m} A_\alpha(x,t,\delta u, D^m u) D^\alpha u dx$$

$$\geqslant \mu_1 \int_\Omega \sum_{|\alpha|=m} |\lambda(x) D^\alpha u|^p dx - \nu_1 \sum_{|l|\leqslant m-1} \| D^l u \|_{L^r}^r - \| a_2(x,t) \|_{L^1}$$

$$\geqslant \mu_1 \| u \|_V^p - \varepsilon \| u \|_V^p - (N_2 + \| a_2(x,t) \|_{L^1})$$

$$\int_\Omega \sum_{|\alpha|\leqslant m-1} A_\alpha(x,t,\delta u, D^m u) D^\alpha u dx$$

$$\geqslant -\mu_2 \sum_{|l|\leqslant m-1} \| D^l u \|_{L^r}^r - \nu_2 \sum_{\substack{|\beta|=m\\|l|\leqslant m-1}} \int_\Omega |D^l u| (|\lambda(x) D^\beta u|^{p/r'} + |b_2(x,t)|) dx$$

$$\geqslant -\varepsilon_1 \| u \|_V^p - c_1 \| D^l u \|_{L^r}^r - c_2 \| b_2(x,t) \|_{L^{r'}}^{r'}$$

$$\geqslant -\varepsilon \| u \|_V^p - (N_3 + c_2 \| b_2(x,t) \|_{L^{r'}}^{r'})$$

Let $\varepsilon=\mu_1/4$ and $N(t)=N_2+N_3+\| a_2(x,t) \|_{L^1}+c_2 \| b_2(x,t) \|_{L^{r'}}^{r'}$, we can obtain

$$\langle A_t(u), u\rangle \geqslant \frac{\mu_1}{2} \| u \|_V^p - N(t)$$

Lemma 2.3 *Under the structural condition* Ⅲ, *A_t is a semibounded-variational operator, i.e. $\forall\ M>0$ and $\forall\ u(x,\cdot)$, $v(x,\cdot)\in V(M)=\{\omega(x,\cdot)\in V: \| \omega(x,\cdot) \|_V \leqslant M\}$, A_t satisfies*

$$\langle A_t(u)-A_t(v), u-v\rangle \geqslant -k(M)(\| u-v \|_{W^{m-1,q}}^p + \| u-v \|_{W^{m-1,q}}^2)$$

where constant $k(M)\geqslant 0$ is independent of $u(x,\cdot)$ and $v(x,\cdot)$.

Proof Let $\omega(x,\cdot)$ denote $v+\theta(u-v)$, $\theta\in(0,1]$, by structural condition Ⅲ we have

$$\langle A_t(u)-A_t(v), u-v\rangle$$

$$= \sum_{\substack{|\alpha|\leqslant m\\|\beta|\leqslant m}} \int_\Omega \int_0^1 \frac{\partial A_\alpha(x,t,\delta\omega, D^m\omega)}{\partial \tau_\beta} D^\beta(u-v) d\theta D^\alpha(u-v) dx$$

$$= \sum_{\substack{|\alpha|=m\\|\beta|=m}} + \sum_{\substack{|\alpha|=m\\|\beta|\leqslant m-1}} + \sum_{\substack{|\alpha|\leqslant m-1\\|\beta|=m}} + \sum_{\substack{|\alpha|\leqslant m-1\\|\beta|\leqslant m-1}} = I_0+I_1+I_2+I_3$$

The hypotheses H_j can be used to estimate I_j respectively ($j=0,1,2,3$). First, it is easy to prove that when $p\geqslant 2$, $I(y)=\int_0^1 |y+\theta|^{p-2} d\theta \geqslant \frac{1}{p-1}\cdot\frac{1}{2^{p-1}}$, $\forall\ y\in R$, hence by H_0 we have

$$I_0 = \sum_{\substack{|\alpha|=m\\|\beta|=m}} \int_\Omega \int_0^1 \frac{\partial A_\alpha(x,t,\delta\omega, D^m\omega)}{\partial \tau_\beta} D^\beta(u-v) d\theta D^\alpha(u-v) dx$$

$$\geqslant K_0 \sum_{\substack{|\alpha|=m\\|\beta|=m}} \int_\Omega \int_0^1 |\lambda(x)D^\alpha v + \lambda(x)\theta D^\alpha(u-v)|^{p-2} d\theta |\lambda(x)D^\beta(u-v)|^2 dx$$

$$\geqslant \frac{K_0}{(p-1)2^{p-1}} \int_\Omega \sum_{\substack{|\alpha|=m\\|\beta|=m}} |\lambda(x)D^\alpha(u-v)|^{p-2} |\lambda(x)D^\beta(u-v)|^2 dx$$

$$= c_0 \| u-v \|_V^p$$

Next, for all $u,v \in V(M)$, it is obvious that

$$\int_\Omega \int_0^1 |\lambda(x)D^m\omega|^p d\theta dx \leqslant c(M), \quad \int_\Omega \int_0^1 |D^l \omega|^\sigma d\theta dx \leqslant c(M)$$

hence by H_1 we have

$$|I_1| = \left| \sum_{\substack{|\alpha|=m\\|\beta|\leqslant m-1}} \int_\Omega \int_0^1 \frac{\partial A_\alpha(x,t,\delta\omega,D^m\omega)}{\partial \tau_\beta} D^\beta(u-v) d\theta D^\alpha(u-v) dx \right|$$

$$\leqslant K_1 \sum_{\substack{|\alpha|=m\\|l|\leqslant m-1}} \int_\Omega \int_0^1 |\lambda(x)D^\alpha(u-v)| |D^l(u-v)| (|D^l\omega|^{\sigma/p'-1}+1) d\theta dx$$

$$\leqslant \sum_{\substack{|\alpha|=m\\|l|\leqslant m-1}} \int_\Omega \int_0^1 [\varepsilon|\lambda(x)D^\alpha(u-v)|^p + K(\varepsilon)|D^l(u-v)|^{p'}(|D^l\omega|^{\sigma-p'}+1) d\theta dx$$

$$\leqslant \varepsilon \|u-v\|_V^p + \sum_{|l|\leqslant m-1} K(\varepsilon) \left\{ \int_\Omega \int_0^1 |D^l(u-v)|^\sigma d\theta dx \right\}^{p'/\sigma}$$

$$\cdot \left\{ \left[\int_\Omega \int_0^1 |D^l\omega|^\sigma d\theta dx \right]^{(\sigma-p')/\sigma} + 1 \right\}$$

$$\leqslant \varepsilon \|u-v\|_V^p + c_1(\varepsilon, M) \sum_{|l|\leqslant m-1} \| D^l(u-v) \|_{L^\sigma}^{p'}$$

by H_2 we have

$$|I_2| = \left| \sum_{\substack{|\beta|=m\\|\alpha|\leqslant m-1}} \int_\Omega \int_0^1 \frac{\partial A_\alpha(x,t,\delta\omega,D^m\omega)}{\partial \tau_\beta} D^\beta(u-v) d\theta D^\alpha(u-v) dx \right|$$

$$\leqslant K_2 \sum_{\substack{|\beta|=m\\|l|\leqslant m-1}} \int_\Omega \int_0^1 |\lambda(x)D^\beta(u-v)| |D^l(u-v)|$$

$$\cdot (|\lambda(x)D^\beta\omega|^{p/p'-p/\sigma} + |D^l\omega|^{\sigma/p'-1} + 1) d\theta dx$$

$$\leqslant \varepsilon \|u-v\|_V^p + \sum_{\substack{|\beta|=m\\|l|\leqslant m-1}} K(\varepsilon) \left\{ \int_\Omega \int_0^1 |D^l(u-v)|^\sigma d\theta dx \right\}^{p'/\sigma}$$

$$\cdot \left\{ \left[\int_\Omega \int_0^1 |\lambda(x)D^\beta\omega|^p d\theta dx \right]^{(\sigma-p')/\sigma} + \left[\int_\Omega \int_0^1 |D^l\omega|^\sigma d\theta dx \right]^{(\sigma-p')/\sigma} + 1 \right\}$$

$$\leqslant \varepsilon \|u-v\|_V^p + c_2(\varepsilon, M) \sum_{|l|\leqslant m-1} \| D^l(u-v) \|_{L^\sigma}^{p'}$$

analogously, by H_3 we have

$$|I_3| \leqslant \varepsilon \|u-v\|_V^p + c_3(\varepsilon, M) \sum_{|l|\leqslant m-1} \| D^l(u-v) \|_{L^2}^2$$

Let $\varepsilon = c_0/6$, $\max\{c_1(\varepsilon,M), c_2(\varepsilon,M), c_3(\varepsilon,M)\} = \frac{1}{2}k(M)$, by these estimations we obtain for all $u,v \in V(M)$

$$\langle A_t(u)-A_t(v),u-v\rangle \geqslant -k(M)(\|u-v\|^{p'}_{W^{m-1,p}}+\|u-v\|^{2}_{W^{m-1,p}})$$

3. Existence of Solution

In this section the solution of problem (1) will be structured by Galërkin method. Since V is a separable and reflexive Banach space, we can find a "basis" $\{\omega_i(x)\}$ in V such that $\forall\ k$, $\omega_1,\cdots,\omega_k$ are linearly independent and the linear combinations $\sum_{\text{finite}} C_j\omega_j(x), C_j\in R$, are dense in V. Let us define an "approximate solution" of problem (1) by $u_k(x,t)=\sum_{j=1}^{k}d_j^{(k)}(t)\omega_j(x)$, where $d_j^{(k)}(t)$ are chosen such that they satisfy the following initial value problem of system of k nonlinear differential equations

$$\begin{cases}\int_\Omega u_k'(x,t)\omega_j(x)dx+\langle A_t(u_k),\omega_j\rangle=0\\ u_k(x,0)=u_{0k}(x), \qquad j=1,2,\cdots,k\end{cases} \tag{6}$$

here $u_{0k}(x)=\sum_{j=1}^{k}C_j^{(k)}\omega_j(x)\overset{L^2(\Omega)}{\Rightarrow}u_0(x)$ strongly.

By Lemma 4 in Section 3 of [2], problem (6) admits a solution $\{d_j^{(k)}(t)\}$.

Multiplying (6) by $d_j^{(k)}(t)$ and summing over j and integrating with respect to t from 0 to τ, we obtain

$$\int_0^\tau\int_\Omega u_k'(x,t)u_k(x,t)dxdt+\int_0^\tau\langle A_t(u_k),u_k\rangle dt=0,\quad \forall\ \tau\in(0,T] \tag{7}$$

By Lemma 2. 2 we have

$$\frac{1}{2}\|u_k(x,\tau)\|^2_{L^2}-\frac{1}{2}\|u_k(x,0)\|^2_{L^2}+c_0\int_0^\tau\|u_k(x,t)\|^p_V dt-\int_0^\tau N(t)dt\leqslant 0$$

Note $\|u_k(x,0)\|^2_{L^2}\leqslant M_0\|u_0(x)\|^2_{L^2}$, hence

$$\max_{t\in[0,T]}\|u_k(x,t)\|_{L^2(\Omega)}\leqslant M,\qquad \int_0^\tau\|u_k(x,t)\|^p_V dt\leqslant M \tag{8}$$

Therefore $\{u_k(x,t)\}$ is a bounded set in $L^p(0,T;V)$ and we may extract a subsequence (we also denote the subsequence by $\{u_k\}$) which converges to $u(x,t)$ weakly in $L^p(0,T;V)$. By Lemma 2. 1 A_t is bounded, thus $\{A_t(u_k)\}$ is a bounded set in $L^{p'}(0,T;V^*)$ and we may also extract a subsequence (we also denote it by $\{A_t(u_k)\}$) such that it converges to $\Pi(x,t)$ weakly in $L^{p'}(0,T;V^*)$.

Analogously to [4] we can prove that $u(x,t)$ has generalized derivative $u'(x,t)\in L^{p'}(0,T;V^*)$ and $u'(x,t)=-\Pi(x,t)$, $u(x,0)=u_0(x)$. It remains to prove $\Pi(x,t)=A_t(u)$.

Let us consider the subspace $L^p(0,T;V(M))$, here the closed ball $V(M)=\{\omega(x,\cdot)\in V:\|\omega(x,\cdot)\|_V\leqslant M\}$ and M is the constant in (8). By Lemma 2. 3 for $\forall\ v(x,t)\in L^p(0,T;V(M))$ we have

$$\int_0^T\langle A_t(u_k)-A_t(v),u_k-v\rangle dt$$
$$\geqslant -k(M)(\|u_k-v\|^{p'}_{W^{m-1,p}}+\|u_k-v\|^{2}_{W^{m-1,p}}) \tag{9}$$

Combining this with (7), the left side of (9) becomes

$$-\frac{1}{2}\| u_k(x,T)\|^2_{L^2}+\frac{1}{2}\| u_k(x,0)\|^2_{L^2}-\int_0^T\langle A_t(u_k),v\rangle dt-\int_0^T\langle A_t(v),u_k-v\rangle dt$$

For the right side of (9), noting that V is compactly embedded in $\mathring{W}^{m-1,p}$, thus let $k\to\infty$ in (9), we obtain

$$\int_0^T\langle \Pi-A_t(v),u-v\rangle dt\geqslant -k(M)(\| u-v\|^{p'}_{W^{m-1,p}}+\| u-v\|^2_{W^{m-1,p}})\quad(10)$$

Now $\forall\ \omega(x,t)\in L^p(0,T;V(M))$, taking $v(x,t)=u(x,t)\pm\theta\omega(x,t)$ in (10), $\theta\in(0,1)$, we have

$$\mp\theta\int_0^T\langle \Pi-A_t(u\pm\theta\omega),\omega\rangle dt\geqslant -k(M)(\theta^{p'}\|\omega\|^{p'}_{W^{m-1,p}}+\theta^2\|\omega\|^2_{W^{m-1,p}})$$

Dividing it by θ (note $p'>1$), then let $\theta\to+0$ and we obtain, $\mp\int_0^T\langle \Pi-A_t(u),\omega\rangle dt\geqslant 0$, i.e., $\int_0^T\langle \Pi-A_t(u),\omega\rangle dt=0$, hence $\Pi=A_t(u)=-u'(x,t)$.

Thus we have $u(x,t)\in L^p(0,T;V)\bigcap C^0(0,T;L^2(\Omega))$, $u'(x,t)+A_t(u)=0$, and $u(x,0)=u_0(x)$, hence $u(x,t)$ is the solution of the problem (1) and we obtain

Theorem 3.1 *Assume that structural conditions* Ⅰ, Ⅱ *and* Ⅲ *hold, then the initial boundary value problem* (1) *has a solution in* $L^p(0,T;V)\bigcap C^0(0,T;L^2(\Omega))$.

Furthermore, if we have following condition Ⅲ′ instead of Ⅲ

$$\sum_{\substack{|\alpha|\leqslant m\\|\beta|\leqslant m}}\frac{\partial A_\alpha(x,t,\zeta,\xi)}{\partial\tau_\beta}\eta_\alpha\eta_\beta\geqslant k_0\sum_{\substack{|\alpha|=m\\|\beta|=m}}|\lambda(x)\xi_\beta|^{p-2}|\lambda(x)\eta_\alpha|^2$$

then by the estimation for I_0 in Lemma 2.3 we have

$$\langle A_t(u)-A_t(v),u-v\rangle\geqslant c_0\| u-v\|^p_V$$

hence A_t is a strongly monotonic operator and we can obtain the following uniqueness result.

Theorem 3.2 *Assume that structural conditions* Ⅰ, Ⅱ *and* Ⅲ′ *hold, then the initial boundary value problem* (1) *has a unique solution in* $L^p(0,T;V)\bigcap C^0(0,T;V)$.

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