J. Partial differential Equations
Vol. 4 No. 1(1991),77−96

NONTRIVIAL SOLUTIONS FOR SOME SEMILINEAR ELLIPTIC EQUATIONS WITH CRITICAL SOBOLEV EXPONENTS①

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(Received July 9, 1989; revised June 4, 1990)

Abstract Let Ω be a bounded domain in $R^n(n\geqslant 4)$ with smooth boundary $\partial\Omega$. We discuss the existence of nontrivial solutions of the Dirichlet problem

$$\begin{cases} -\triangle u = a(x)|u|^{4/(n-2)}u + \lambda u + g(x,u), & x\in\Omega \\ u=0, & x\in\partial\Omega \end{cases}$$

where $a(x)$ is a smooth function which is nonnegative on $\overline{\Omega}$ and positive somewhere, $\lambda>0$ and $\lambda\notin\sigma(-\triangle)$. We weaken the conditions on $a(x)$ that are generally assumed in other papers dealing with this problem.

Key Words Semilinear elliptic equation; Sobolev exponent; Critical value; Critical point; (P · S) condition.

Classifications 35J20; 35J60; 35D05.

1. Introduction

Let $\Omega\subset R^n(n\geqslant 4)$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper, we are concerned with the problem of finding u satisfying the following semilinear elliptic problem

$$(P1)\qquad \begin{cases} -\triangle u = a(x)|u|^{4/(n-2)}u + \lambda u + g(x,u), & x\in\Omega \\ u=0, & x\in\partial\Omega \end{cases}$$

where λ is a positive constant, $a(x)$ is a smooth function on Ω which is nonnegative and positive somewhere, $g(x,u)$ is a lower−order perturbation of $|u|^{(n+2)/(n-2)}$ in the sence that

$$\lim_{u\to\infty}\frac{g(x,u)}{|u|^{(n+2)/(n-2)}}=0 \text{ and } g(x,0)=0$$

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① The project supported by Natural Science Foundation of Zhejiang Province.

The important results concerning the problem (P1) have been obtained by H. Brezis and L. Nirenberg [1], they showed that (P1) possesses a positive solution for $a(x)=1, 0<\lambda<\lambda_1$ and $g(x,u)=0$. There has been some progress in this direction due to D. Fortunato, A. Capozzi and G. Palmieri [2], J. F. Escobar[3], C. F. Wang and R. Y. Xue [4], W. D. Lu and C. J. He [5]. In [3] the author showed that, for $0<\lambda<\lambda_1, g(x,u)=0$ and $a(x)$ satisfying some technical restriction, (P1) possesses a positive solution. For $a(x)\geqslant\delta>0$ (δ is a positive constant) and $g(x,u)$ satisfying other conditions, W. D. Lu and C. J. He [5] have proved that there is a constant λ_j^* such that (P1) has at least one nontrivial solution for any $\lambda\in(\lambda_j^*,\lambda_j)(j=1,2,\cdots)$. In this paper, we follow the method developed by H. Brezis and L. Nirenberg [1], weaken the conditions on $a(x)$ that are generally assumed in other papers dealing with these problems with critical Sobolev exponents, extend the results in [1], [2] and [3].

2. Some Preliminaries

Define

$$I(u)=\frac{1}{2}\int_\Omega|\nabla u|^2dx-\frac{\lambda}{2}\int_\Omega u^2dx$$
$$-\frac{n-2}{2n}\int_\Omega a(x)|u|^{2n/(n-2)}dx-\int_\Omega G(x,u)dx,\quad u\in H_0^1(\Omega)\qquad(2.1)$$

$$G(x,u)=\int_0^u g(x,u)du$$

It is well known that the solutions of (P1) correspond to critical points of the functional $I(u)$.

Let $\|\cdot\|$, $|\cdot|_p$ denote respectively the norms in $H_0^1(\Omega)$ and $L^p(\Omega)$ $(1\leqslant p<+\infty)$ and let

$$S=\inf\{\|u\|^2:|u|^2_{2n/(n-2)}=1,\quad u\in H_0^1(\Omega)\}$$

denote the best constant for the embedding $H_0^1(\Omega)\hookrightarrow L^{2n/(n-2)}(\Omega)$. We denote by $0<\lambda_1<\lambda_2\leqslant\lambda_3\leqslant\cdots$ the sequence of eigenvalues of the eigenvalue problem

$$\begin{cases}-\triangle u=\lambda u, & \text{in }\Omega\\ u=0, & \text{on }\partial\Omega\end{cases}$$

and $\lambda_0=0$.

We are now in a position to collect the various hypotheses to be placed on the nonlinearity $g(x,u)$ and nonnegative function $a(x)$ in this paper.

(G1) $g(x,u)\in C(\Omega\times R^1,R^1)$;

(G2) $\lim\limits_{u\to\infty}\dfrac{g(x,u)}{|u|^{(n+2)/(n-2)}}=0$, uniformly in $x\in\Omega$

(G3) $\lim\limits_{u\to\infty}\dfrac{g(x,u)}{u}=0$, uniformly in $x\in\Omega$.

(G4) There are positive constants M and $\theta\in(0,\frac{1}{2})$ such that, for $|u|\geqslant M$,

$$G(x,u)\leqslant\left(\theta-\frac{n-2}{2n}\right)a(x)|u|^{2n/(n-2)}$$
$$+\theta g(x,u)u+(\theta-\frac{1}{2})(\lambda-\beta_0)u^2$$

where β_0 is some constant such that $\beta_0\in[0,\lambda_1)$.

(G5) $\lim\limits_{u\to\infty}\dfrac{g(x,u)}{|u|^{(n-2)/(n+2)}}=0$, uniformly in $x\in\Omega$.

(A1) There is an interior maximum point x_0 of $a(x)$ at which all partial derivatives of $a(x)$ of order less than or equal to α vanish, where $\alpha=1$ if $n=4$; $\alpha=2$ if $n\geqslant 5$.

(A2) There is a maximum point $x_0\in\partial\Omega$ of $a(x)$ at which all partial derivatives of $a(x)$ of order less than or equal to α vanish, where $\alpha=1$ if $n=4$; $\alpha=6$ if $n=5$; $\alpha=4$ if $n=6$; $\alpha=3$ if $n=7$ or 8; $\alpha=2$ if $n\geqslant 9$.

Remark 1 It is obvious that (G2) is implied by (G5).

In this paper we need the following

Theorem 2.1 *Let E be a Banach space, $E=V\oplus X$, where V is a finite dimensional subspace of E. $I(u)\in C^1(E,R^1)$ satisfies the following*

(I_1) *There are two positive constants $\rho,\alpha>0$ such that*

$$I|_{X\cap\partial B_\rho}\geqslant\alpha>0;$$

(I_2) *There is an element*

$$v\in X,\quad \|v\|=1$$

and a positive constant $R>\rho$ such that

$$I|_{\partial Q}\leqslant 0$$

where $Q=(\bar{B}_R\cap V)\oplus\{tv\,|\,0\leqslant t\leqslant R\}$;

(I_3) $C_0=\inf\limits_{h\in\Gamma}\max\limits_{u\in Q}I(h(u))$, $\Gamma=\{h\in C^1(Q,E):h(u)=u,u\in\partial Q\}$;

Then there exists a sequence $\{u_j\}\subset E$ such that

$$I'(u_j) \to 0, I(u_j) \to C_0 \text{ as } j \to +\infty$$

We shall give a proof of it in the Appendix.

In the sequel, C denotes various positive constants and $C(\varepsilon)$ denotes various positive constants which depend on ε. $V_i (i=1,2,\cdots)$ denotes the eigenspace corresponding to λ_i,

$$\sigma(-\triangle) = \{\lambda \in R^1 : \lambda \text{ is an eigenvalue of the eigenvalue problem } -\triangle u = \lambda u \text{ in } \Omega, u = 0 \text{ on } \partial\Omega\}$$

3. Existence of Nontrivial Solutions

Our first result is the following

Theorem 3. 1 *Assume that*

$$\lambda \notin \sigma(-\triangle), g(x,u)u \geqslant 0 \text{ for all } x \in \Omega, u \in R^1$$

and assume that $a(x)$ satisfies (A1) (or (A2)), and $g(x,u)$ satisfies (G1), (G2), (G3) and (G4) (or (G5)). Then (P1) possesses at least one nontrivial solution.

Without loss of generality, we may assume $\lambda_k < \lambda < \lambda_{k+1}$ for some nonnegative integer k. Let $\psi(x)$ be a piecewise nonincreasing function of $|x|$ which satisfies

$$\psi(x) = 1, \text{ for } |x| \leqslant \rho_0$$
$$\psi(x) = 0, \text{ for } |x| \geqslant 2\rho_0$$
$$|\nabla\psi| < \rho_0^{-1} \text{ for } \rho_0 < |x| < 2\rho_0$$

and

$$u_\varepsilon(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{(n-2)/2}$$

We now construct a test function φ on Ω as follows:

$$\varphi(x) = \begin{cases} u_\varepsilon(x-x_1) & \text{for } |x-x_1| \leqslant \rho_0 \\ \varepsilon_0(G(x-x_1) - a(x-x_1)\psi(x-x_1)) & \\ \quad \text{for } \rho_0 < |x-x_1| < 2\rho_0 & \\ \varepsilon_0 G(x-x_1) & \text{for } x \in \Omega \cap \{x : |x-x_1| \geqslant 2\rho_0\} \end{cases} \quad \text{if } n = 4 \quad (3.1)$$

$$\varphi(x) = \psi(x-x_1)u_\varepsilon(x-x_1) \quad \text{if } n \geqslant 5 \quad (3.2)$$

where x_1 is a point in Ω such that $x_1 = x_0$ if $a(x)$ satisfies (A1); $\rho_0 = \frac{1}{4}|x_1 - x_0|$,

$B_{4\rho_0}(x_1) \subset \Omega$ if $a(x)$ satisfies (A2), $G(x-x_1)$ is the positive solution of

$$(\triangle + \hat{\lambda})G = 0 \text{ on } \Omega - \{x_1\}, \quad \hat{\lambda} = \min(\lambda, 1/2\lambda_1)$$

satisfying $G(x-x_1)=0$ on $\partial\Omega$. When $n=4$, $G(x-x_1)$ behaves like $|x-x_1|^{-2}$ near x_1 and has an expansion for $|x-x_1|$ small,

$$G(x-x_1) = |x-x_1|^{-2} + \frac{1}{2}\hat{\lambda}\ln|x-x_1|^{-1} + \alpha(x-x_1)$$

Here $\alpha(x-x_1)$ is a function on Ω which behaves like $|x-x_1|^2\ln|x-x_1|$ near x_1. In order to have the function $\varphi(x)$ in (3. 1) to be continuous across $\partial B_{\rho_0}(x_1)$, we require that ε_0 should satisfy

$$\varepsilon_0(\rho_0^{-2} + \frac{1}{2}\hat{\lambda}\ln\rho_0^{-1}) = \frac{\varepsilon}{\varepsilon^2 + \rho_0^2} \tag{3. 3}$$

Through the proof of Theorem 3. 1 we set $\rho_0 = \varepsilon^{1/4}$ if $n=4$ (In this case, by (3. 3) we have $\frac{\varepsilon}{2} \leqslant \varepsilon_0 \leqslant 2\varepsilon$ for ε small); $\rho_0 = \varepsilon^{4/13}$ if n=5; $\rho_0 = \varepsilon^{8/17}$ if $n=6$; $\rho_0 = \varepsilon^{4/7}$ if $n=7$ or 8; $\rho_0 = \varepsilon^{35/36}$ if $n \geqslant 9$.

Lemma 3. 2 *If* ε_0, ρ_0 *and* $\varphi(x)$ *are defined as before*, P *is a projection on* $\bigoplus_{i \leqslant k} V_i$, *then for* ε small, we have

$$|\varphi|_1 \leqslant C\varepsilon^{(n-2)/2}, \; |(I-P)\varphi|_1 \leqslant C\varepsilon^{(n-2)/2}, \; \|P\varphi\|^2 \leqslant C\varepsilon^{(n-2)}$$

$$|\varphi|_{(n+2)/(n-2)}^{(n+2)/(n-2)} \leqslant C\varepsilon^{(n-2)/2}$$

$$|(I-P)\varphi|_{(n+2)/(n-2)}^{(n+2)/(n-2)} \leqslant C\varepsilon^{(n-2)/2}$$

$$|P\varphi|_{2n/(n-2)}^{2n/(n-2)} \leqslant C\varepsilon^n$$

$$\int_\Omega a(x)|\varphi|^{2n/(n-2)}dx \geqslant \frac{a(x_0)}{2}\omega_{n-1}\int_0^1 \frac{r^{(n-1)}}{(1+r^2)^n}dr$$

$$\int_\Omega a(x)|(I-P)\varphi|^{2n/(n-2)}dx \geqslant \frac{a(x_0)}{2}\omega_{n-1}\int_0^1 \frac{r^{n-1}}{(1+r^2)^n}dr$$

The proof is essentially the same as Lemma 1. 1 in [1] or Lemma 2 in [4], we shall omit it.

Arguing as Lemma 2. 2 in [2] (using Lemma 3. 2) we obtain the following

Lemma 3. 3 *Let*

$$\hat{u} \in M_-, \quad t \in R^1, \quad u = \hat{u} + t(I-P)\varphi$$

Then we have

$$\int_\Omega a(x)|u|^{2n/(n-2)}dx \geqslant \int_\Omega a(x)|t(I-P)\varphi|^{2n/(n-2)}dx$$
$$+\frac{1}{2}\int_\Omega a(x)|\hat{u}|^{2n/(n-2)}dx - K_1 t^{2n/(n-2)}\varepsilon^{n(n-2)/(n+2)}$$

for ε small, where $M_- = \bigoplus_{i\leqslant k} V_i$ (If $k=0$ we set $M_- = \varnothing$), K_1 is a positive constant independing on t and ε.

Lemma 3. 4 *If $a(x)$ satisfies (A1) (or (A2)), ρ_0, ε_0 and $\varphi(x)$ are defined as before, then we have, for ε small,*

$$\frac{\int_\Omega |\nabla\varphi|^2 dx - \hat{\lambda}\int_\Omega |\varphi|^2 dx}{\left(\int_\Omega a(x)|\varphi|^{2n/(n-2)}dx\right)^{(n-2)/n}} \leqslant (\max_{\bar{\Omega}} a(x))^{(2-n)/n} S - C\hat{\lambda}\varepsilon^2, \quad \text{if } n \geqslant 5$$

$$\frac{\int_\Omega |\nabla\varphi|^2 dx - \hat{\lambda}\int_\Omega |\varphi|^2 dx}{\left(\int_\Omega a(x)|\varphi|^{2n/(n-2)}dx\right)^{(n-2)/n}} \leqslant (\max_{\bar{\Omega}} a(x))^{(2-n)/n} S - C\hat{\lambda}\varepsilon\ln\varepsilon^{-1}, \quad \text{if } n = 4$$

The proof of Lemma 3. 4 is contained in [3].

Lemma 3. 5 *If $a(x)$ satisfies (A1) (or (A2)), ε_0 and ρ_0 and $\varphi(x)$ are defined as before, then we have, for ε small,*

$$\left[\frac{\int_\Omega |\nabla(I-P)\varphi|^2 dx - \lambda\int_\Omega |(I-P)\varphi|^2 dx}{\left(\int_\Omega a(x)|(I-P)\varphi|^{2n/(n-2)}dx - K_1\varepsilon^{n(n-2)/(n+2)}\right)^{(n-2)/n}}\right]^{\frac{n}{2}}$$
$$\leqslant (\max_{\bar{\Omega}} a(x))^{(n-2)/2} S^{n/2} - C\varepsilon^2, \text{ if } n \geqslant 5 \tag{3.4}$$

$$\left[\frac{\int_\Omega |\nabla(I-P)\varphi|^2 dx - \lambda\int_\Omega |(I-P)\varphi|^2 dx}{\left(\int_\Omega a(x)|(I-P)\varphi|^{2n/(n-2)}dx - K_1\varepsilon^{n(n-2)/(n+2)}\right)^{(n-2)/n}}\right]^{n/2}$$
$$\leqslant (\max_{\bar{\Omega}} a(x))^{(2-n)/2} S^{n/2} - C\varepsilon\ln\varepsilon^{-1}, \text{ if } n = 4 \tag{3.5}$$

where P and $I-P$ denote respectively the projections of $H_0^1(\Omega)$ onto M_- and M_+.

Proof From Lemma 3. 2, Lemma 3. 3 and Lemma 3. 4 we deduce that, for $n \geqslant$

5 and ε small,

$$\frac{\int_\Omega |\nabla(I-P)\varphi|^2dx-\lambda\int_\Omega |(I-P)\varphi|^2dx}{\left(\int_\Omega a(x)|(I-P)\varphi|^{2n/(n-2)}dx-K_1\varepsilon^{n(n-2)/(n+2)}\right)^{(n-2)/n}}$$

$$\leqslant\frac{\int_\Omega|\nabla\varphi|^2dx-\hat{\lambda}\int_\Omega|\varphi|^2dx-\int_\Omega|\nabla P\varphi|^2dx+\lambda\int_\Omega|P\varphi|^2dx}{\left(\int_\Omega a(x)|\varphi|^{2n/(n-2)}dx-C\int_\Omega a(x)|\varphi|^{(n+2)/(n-2)}|P\varphi|dx-C\varepsilon^{n(n-2)/(n+2)}\right)^{(n-2)/n}}$$

$$\leqslant\frac{\int_\Omega|\nabla\varphi|^2dx-\hat{\lambda}\int_\Omega|\varphi|^2dx-\int_\Omega|\nabla P\varphi|^2dx+\lambda\int_\Omega|P\varphi|^2dx}{\left(\int_\Omega a(x)|\varphi|^{2n/(n-2)}dx-C|P\varphi|_\infty\int_\Omega|\varphi|^{(n+2)/(n-2)}dx-C\varepsilon^{n(n-2)/(n+2)}\right)^{(n-2)/n}}$$

$$\leqslant\frac{\int_\Omega|\nabla\varphi|^2dx-\hat{\lambda}\int_\Omega|\varphi|^2dx+C\varepsilon^{n-2}}{\left(\int_\Omega a(x)|\varphi|^{2n/(n-2)}dx-C\varepsilon^{n-2}-C\varepsilon^{n(n-2)/(n+2)}\right)^{(n-2)/n}}$$

$$\leqslant\left[\frac{\int_\Omega|\nabla\varphi|^2dx-\int_\Omega|\varphi|^2dx}{\left(\int_\Omega a(x)|\varphi|^{2n/(n-2)}dx\right)^{(n-2)/n}}+\frac{C\varepsilon^{n-2}}{\left(\int_\Omega a(x)|\varphi|^{2n/(n-2)}dx\right)^{(n-2)/n}}\right]\cdot$$

$$\left[\frac{\int_\Omega a(x)|\varphi|^{2n/(n-2)}dx}{\int_\Omega a(x)|\varphi|^{2n/(n-2)}dx-C\varepsilon^{n-2}-C\varepsilon^{n(n-2)/(n+2)}}\right]^{\cdot}$$

$$\leqslant\left[(\max_\Omega a(x))^{(2-n)/n}S-C\hat{\lambda}\varepsilon^2+C\varepsilon^{n-2}\right]\cdot\left[1+\frac{C\varepsilon^{n-2}+C\varepsilon^{n(n-2)/(n+2)}}{\int_\Omega a(x)|\varphi|^{2n/(n-2)}dx}\right]^{(n-2)/n}$$

$$\leqslant\left((\max_\Omega a(x))^{(2-n)/n}S-C\varepsilon^2\right)\cdot\left[1+C\varepsilon^{n(n-2)/(n+2)}\right]$$

$$\leqslant(\max_\Omega a(x))^{(2-n)/n}S-C\varepsilon^2$$

Hence

$$\left[\frac{\int_\Omega |\nabla (I-P)\varphi|^2 dx - \lambda\int_\Omega |(I-P)\varphi|^2 dx}{\left(\int_\Omega a(x)|(I-P)\varphi|^{2n/(n-2)}dx - K_1\varepsilon^{n(n-2)/(n+2)}\right)^{(n-2)/n}}\right]^{n/2}$$

$$\leqslant [(\max_\Omega a(x))^{(2-n)/n} S - C\varepsilon^2]^{n/2} \leqslant (\max_\Omega a(x))^{(2-n)/2} S^{n/2} - C\varepsilon^2.$$

Similarly, we have

$$\left[\frac{\int_\Omega |\nabla (I-P)\varphi|^2 dx - \lambda\int_\Omega |(I-P)\varphi|^2 dx}{\left(\int_\Omega a(x)|(I-P)\varphi|^{2n/(n-2)}dx - K_1\varepsilon^{n(n-2)/(n+2)}\right)^{(n-2)/n}}\right]^{n/2}$$

$$\leqslant (\max_\Omega a(x))^{(2-n)/2} S^{n/2} - C\varepsilon \ln \varepsilon^{-1}, \text{ if } n=4$$

Lemma 3.6 *Assume that there are some constant* $C_0 \in (0, \frac{1}{n}(\max_\Omega a(x))^{(2-n)/2} S^{n/2})$ *and some sequence* $\{u_j\} \subset H_0^1(\Omega)$ *such that, as* $j \to +\infty$,

$$I'(u_j) \to 0, \qquad I(u_j) \to C_0$$

We also assume that (*G*1), (*G*2), (*G*3) *and* (*G*4) *hold. Then* (*P*1) *possesses a nontrivial solution.*

Proof

$$I'(u_j)v = \int_\Omega \nabla u_j \nabla v dx - \lambda \int_\Omega u_j v dx - \int_\Omega a(x)|u_j|^{4/(n-2)} u_j v dx$$
$$- \int_\Omega g(x,u_j) v dx = o(1)\|v\|, \quad \text{for any } v \in H_0^1(\Omega) \qquad (3.6)$$

$$I(u_j) = \frac{1}{2}\int_\Omega |\nabla u_j|^2 dx - \frac{\lambda}{2}\int_\Omega |u_j|^2 dx$$
$$- \frac{n-2}{2n}\int_\Omega a(x)|u_j|^{2n/(n-2)}dx - \int_\Omega G(x,u_j)dx$$
$$= C_0 + o(1) \qquad (3.7)$$

Choosing $v=u_j$ in (3.6) and combining it with (3.7) we deduce

$$(\frac{1}{2} - \theta)\int_\Omega |\nabla u_j|^2 dx \leqslant C_0 + o(1)\|u_j\| + \int_\Omega G(x,u_j)dx$$

$$- \theta\int_\Omega g(x,u_j)u_j dx + (\frac{n-2}{2n} - \theta)\int_\Omega a(x)|u_j|^{2n/(n-2)} dx$$

$$- (\theta - \frac{1}{2})\lambda\int_\Omega |u_j|^2 dx + o(1)$$

$$\leqslant C_0 + o(1) + o(1)\|u_j\| + (\frac{1}{2} - \theta)\beta_0\int_\Omega u_j^2 dx$$

$$+ \int_{\{|u_j|\leqslant M\}} [G(x,u_j) - \theta g(x,u_j)u_j - (\theta - \frac{1}{2})(\lambda - \beta_0)|u_j|^2$$

$$- (\frac{n-2}{2n} - \theta)a(x)|u_j|^{2n/(n-2)}]dx$$

Set

$$C_1 = \max_{\substack{x\in\bar\Omega \\ |u|\leqslant M}} \{|G(x,u)| + \theta|g(x,u)u|$$

$$+ |(\frac{1}{2} - \theta)(\lambda - \beta_0)|u^2 + |\frac{n-2}{2n} - \theta|a(x)|u|^{2n/(n-2)}\}$$

Then

$$(\frac{1}{2} - \theta)(1 - \frac{\beta_0}{\lambda_1})\int_\Omega |\nabla u_j|^2 dx \leqslant C_0 + o(1) + o(1)\|u_j\| + C_1|\Omega| \quad (3.8)$$

It follows from (3. 8) that $\{u_j\}$ is a bounded sequence in $H_0^1(\Omega)$.

Since $H_0^1(\Omega)$ is a Hilbert space, there is an element $u\in H_0^1(\Omega)$ and a subsequence (still denote by $\{u_j\}$) such that

$u_j \rightharpoonup u$ weakly in $H_0^1(\Omega)$,

$u_j \to u$ strongly in $L^q(\Omega)$ for $1\leqslant q < \frac{2n}{n-2}$,

$u_j \to u$ a. e. on Ω,

$a(x)|u_j|^{4/(n-2)}u_j \rightharpoonup a(x)|u|^{4/(n-2)}u$ weakly in $(L^{2n/(n-2)}(\Omega))^*$,

$g(x,u_j) \rightharpoonup g(x,u)$ weakly in $(L^{2n/(n-2)}(\Omega))^*$

Let $j\to+\infty$ in (3. 6), then we obtain

$$\int_\Omega \nabla u\nabla v dx - \lambda\int_\Omega uv dx - \int_\Omega a(x)|u|^{4/(n-2)}uv dx - \int_\Omega g(x,u)v dx = 0$$

for any $v\in H_0^1(\Omega)$. This implies that u is a solution of (P1). Now we prove $u\not\equiv 0$. Indeed suppose that $u\equiv 0$, we have, as $j\to+\infty$,

$$\int_\Omega g(x,u_j)u_j dx\to 0,\ \int_\Omega G(x,u_j)dx\to 0,\ \int_\Omega |u_j|^2 dx\to 0$$

Then

$$I(u_j)=\frac{1}{2}\int_\Omega |\nabla u_j|^2 dx-\frac{n-2}{2n}\int_\Omega a(x)|u_j|^{2n/(n-2)}dx+o(1)$$
$$=C_0+o(1) \quad (3.9)$$
$$I'(u_j)u_j=\int_\Omega |\nabla u_j|^2 dx-\int_\Omega a(x)|u_j|^{2n/(n-2)}dx+o(1)=o(1) \quad (3.10)$$

Without loss of generality, we may assume that $\lim_{j\to\infty}\int_\Omega |\nabla u_j|^2 dx=r$ for some nonnegative constant r. It follows from (3.9) and (3.10) that

$$\lim_{j\to\infty}\int_\Omega a(x)|u_j|^{2n/(n-2)}dx=r \quad \text{and} \quad C_0=\frac{1}{n}r \quad (3.11)$$

On the other hand, we have

$$\int_\Omega |\nabla u_j|^2 dx\geqslant S\left(\int_\Omega |u_j|^{2n/(n-2)}dx\right)^{(n-2)/n}$$
$$\geqslant (\max_\Omega a(x))^{(2-n)/n}\left(\int_\Omega a(x)|u_j|^{2n/(n-2)}dx\right)^{(n-2)/n}S$$

Then we obtain that

$$r\geqslant S\,(\max_\Omega a(x))^{(2-n)/n}r^{(n-2)/n} \quad (3.12)$$

(3.11) and (3.12) imply that

$$C_0=0 \quad \text{or} \quad C_0\geqslant\frac{1}{n}(\max_\Omega a(x))^{(2-n)/2}S^{n/2}$$

which is a contradiction to $C_0\in(0,\frac{1}{n}(\max_\Omega a(x))^{(2-n)/2}S^{n/2})$. Thus $u\not\equiv 0$.

Lemma 3.7 *Assume that there are some constant*

$$C_0 \in (0, \frac{1}{n}(\max a(x))^{(2-n)/2} S^{n/2})$$

and some sequence $\{u_j\} \subset H_0^1(\Omega)$ *such that, as* $j \to +\infty$,

$$I'(u_j) \to 0, \quad I(u_j) \to C_0$$

We also assume that (G1), (G3) and (G5) hold, $\lambda \notin \sigma(-\triangle)$. *Then (P1) possesses a nontrivial solution.*

Proof Set

$$M_- = \bigoplus_{i \leq k} V_i \quad (\text{If } k = 0 \text{ we set } M_- = \varnothing), \ M_+ = \bigoplus_{i \geq k+1} V_i$$

By $I'(u_j) \to 0$ and $I(u_j) \to C_0$, we know that

$$I(u_j) = \frac{1}{2}\int_\Omega |\nabla u_j|^2 dx - \frac{\lambda}{2}\int_\Omega |u_j|^2 dx - \frac{n-2}{2n}\int_\Omega a(x)|u_j|^{2n/(n-2)} dx - \int_\Omega G(x, u_j) dx = C_0 + o(1) \tag{3.13}$$

$$I'(u_j)v = \int_\Omega \nabla u_j \nabla v dx - \lambda \int_\Omega u_j v dx - \int_\Omega a(x)|u_j|^{4/(n-2)} u_j v dx - \int_\Omega g(x, u_j) v dx = o(1)\|v\|, \quad v \in H_0^1(\Omega) \tag{3.14}$$

Taking $I(u_j) - \frac{1}{2}I'(u_j)u_j$ we obtain

$$\frac{1}{n}\int_\Omega a(x)|u_j|^{2n/(n-2)} dx \leqslant C_0 + o(1)\|u_j\| + \int_\Omega G(x, u_j) dx - \frac{1}{2}\int_\Omega g(x, u_j) dx + o(1) \tag{3.15}$$

Since

$$M_- \oplus M_+ = H_0^1(\Omega), M_- \cap M_+ = \{0\}$$

there are two sequences $\{u_j^-\}$ and $\{u_j^+\}$ such that

$$u_j = u_j^- + u_j^+, \quad u_j^- \in M_-, \quad u_j^+ \in M^+$$

Choosing $v=u_j^-$ or $v=u_j^+$ in (3.14), we have

$$(\frac{\lambda}{\lambda_k}-1)\int_\Omega |\nabla u_j^-|^2dx \leqslant -\int_\Omega a(x)|u_j|^{4/(n-2)}u_ju_j^-dx$$
$$-\int_\Omega g(x,u_j)u_j^-dx + o(1)\|u_j^-\| \qquad (3.16)$$

$$(1-\frac{\lambda}{\lambda_{k+1}})\int_\Omega |\nabla u_j^+|^2dx \leqslant \int_\Omega a(x)|u_j|^{4/(n-2)}u_ju_j^+dx$$
$$+\int_\Omega g(x,u_j)u_j^+dx + o(1)\|u_j^+\| \qquad (3.17)$$

(3.17) and (3.16) imply

$$\int_\Omega |\nabla u_j|^2dx = \int_\Omega |\nabla u_j^-|^2dx + \int_\Omega |\nabla u_j^+|^2dx$$
$$\leqslant C\int_\Omega a(x)|u_j|^{2n/(n-2)}dx + C\int_\Omega g(x,u_j)u_jdx$$
$$+2C\int_\Omega a(x)|u_j|^{(n+2)/(n-2)}|u_j^-|dx$$
$$+2C\int_\Omega |g(x,u_j)u_j^-|dx + o(1)\|u_j\| \qquad (3.18)$$

On the other hand, by (G5) there is a constant $C(\varepsilon)$ depending on ε such that

$$|G(x,u_j)| \leqslant \varepsilon|u_j|^{2n/(n+2)} + C(\varepsilon)$$
$$|g(x,u_j)u_j| \leqslant \varepsilon|u_j|^{2n/(n+2)} + C(\varepsilon) \qquad (3.19)$$
$$|g(x,u_j)| \leqslant \varepsilon|u_j|^{(n-2)/(n+2)} + C(\varepsilon)$$

Combining (3.15) with (3.19) we get

$$\int_\Omega a(x)|u_j|^{2n/(n-2)}dx \leqslant C + C(\varepsilon) + o(1)\|u_j\|$$
$$+C\varepsilon\int_\Omega |u_j|^{2n/(n+2)}dx \qquad (3.20)$$

Making use of (3.18), (3.19), (3.20) and Hölder inequality, we have

$$\int_\Omega |\nabla u_j|^2dx \leqslant C + C(\varepsilon) + o(1)\|u_j\| + C\varepsilon\int_\Omega |u_j|^{2n/(n+2)}dx$$

$$+ C(\max_{\Omega} a(x))^{(n-2)/(2n)} \cdot \left(\int_{\Omega} a(x)|u_j|^{2n/(n-2)}dx\right)^{(n+2)/(2n)} \| u_j^- \|$$

$$+ \varepsilon \| u_j \|^2 + C(\varepsilon)\int_{\Omega} |g(x,u_j)|^2 dx$$

$$\leqslant C + C(\varepsilon) + o(1) \| u_j \|^2$$

$$+ C[C + C(\varepsilon) + o(1) \| u_j \| + C\varepsilon \| u_j \|^{2n/(n+2)}]^{(n+2)/(2n)} \| u_j^- \|$$

$$+ \varepsilon \| u_j \|^2 + C(\varepsilon)\int_{\Omega} [\varepsilon|u_j|^{(n-2)/(n+2)} + C(\varepsilon)]^2 dx$$

$$\leqslant C + C(\varepsilon) + o(1) \| u_j \|^2 + C\varepsilon \| u_j \| \cdot \| u_j^- \|$$

$$+ \varepsilon \| u_j \|^2 + C(\varepsilon) \| u_j \|^{2(n-2)/(n+2)}$$

$$\leqslant C + C(\varepsilon) + o(1) \| u_j \|^2 + C\varepsilon \| u_j \|^2 + C(\varepsilon) \| u_j \|^{2(n-2)/(n+2)}$$

Let ε be small enough such that $1 - C\varepsilon \geqslant 1/2$. Then

$$\frac{1}{2} \| u_j \|^2 \leqslant (1 - C\varepsilon)\int_{\Omega} |\nabla u_j|^2 dx$$

$$\leqslant C + C(\varepsilon) + o(1) \| u_j \|^2 + C(\varepsilon) \| u_j \|^{2(n-2)/(n+2)} \quad (3.21)$$

(3.21) implies that $\{u_j\}$ is bounded in $H_0^1(\Omega)$. Argued as Lemma 3.6, the proof can be completed easily.

Proof of Theorem 3.1 We make ε small enough such that (3.4), (3.5) and (3.22) hold.

$$\int_{\Omega} a(x)|(I-P)\varphi|^{2n/(n-2)}dx - K_1 \varepsilon^{n(n-2)/(n+2)}$$

$$\geqslant \frac{1}{2}\int_{\Omega} a(x)|(I-P)\varphi|^{2n/(n-2)}dx \quad (3.22)$$

Let

$$E = H_0^1(\Omega), X = \bigoplus_{i \geqslant k+1} V_i, V = \bigoplus_{i \leqslant k} V_i$$

and

$$v = d(I-P)\varphi \text{ with } d^{-1} = \| (I-P)\varphi \|$$

in Theorem 2.1. By Theorem 2.1, Lemma 3.6 and Lemma 3.7 we know that it is only needed to prove that $I(u)$ defined by (2.1) satisfies the conditions in Theorem 2.1 and that C_0 defined in Theorem 2.1 satisfies

$$C_0 < \frac{1}{n}(\max_{\Omega} a(x))^{(2-n)/2} S^{n/2}$$

It is obvious that $g(x,u)u \geqslant 0$ implies $G(x,u) \geqslant 0$ for all $x \in \Omega, u \in R^1$.

First, we prove that $I(u)$ satisfies condition (I_1) in Theorem 2. 1. From (G2) and (G3) we deduce that

$$|G(x,u)| \leqslant \frac{1}{4}(\lambda_{k+1} - \lambda)|u|^2 + C_2|u|^{2n/(n-2)}$$

for some positive constant C_2. Hence, for any $u \in X$

$$I(u) \geqslant \frac{1}{2}(1 - \frac{\lambda}{\lambda_{k+1}})\int_{\Omega} |\nabla u|^2 dx - \frac{n-2}{2n}\int_{\Omega} a(x)|u|^{2n/(n-2)}dx - \int_{\Omega} G(x,u)dx$$

$$\geqslant \frac{1}{2}(1 - \frac{\lambda}{\lambda_{k+1}}) \| u \|^2 - \frac{n-2}{2n}\int_{\Omega} a(x)|u|^{2n/(n-2)}dx$$

$$- \frac{1}{4}(1 - \frac{\lambda}{\lambda_{k+1}}) \| u \|^2 - C \| u \|^{2n/(n-2)}$$

$$\geqslant \frac{1}{4}(1 - \frac{\lambda}{\lambda_{k+1}}) \| u \|^2 - C \| u \|^{2n/(n-2)}$$

So there are positive constants ρ and α such that

$$I|_{\partial B_\rho \cap X} \geqslant \alpha > 0$$

Secondly, we shall prove that there is a constant $R > \rho$ such that

$$I|_{\partial Q} \leqslant 0$$

where

$$Q = (\bar{B}_R \cap V) \oplus \{t\bar{v} | 0 \leqslant t \leqslant R\}$$

It is obvious that there are two constants R_1 and $R, R \geqslant R_1 > \rho$, such that

$$\frac{t^2}{2}\int_{\Omega} |\nabla v|^2 dx - \frac{\lambda}{2}t^2\int_{\Omega} |v|^2 dx - \frac{1}{2}\frac{n-2}{2n}\int_{\Omega} a(x)|tv|^{2n/(n-2)}dx \leqslant 0, \text{ for } t \geqslant R_1 \tag{3.23}$$

$$\frac{1}{2}(1 - \frac{\lambda}{\lambda_k})R^2 + \frac{1}{2}R_1^2\Big[\int_{\Omega} |\nabla v|^2 dx + \lambda\int_{\Omega} |v|^2 dx\Big] \leqslant 0 \tag{3.24}$$

(i) If $u \in (\bar{B}_R \cap V) \oplus \{Rv\}$, then $u = \bar{u} + Rv$ for $\bar{u} \in \bar{B}_R \cap V$,

$$I(u)\leqslant\frac{R^2}{2}\int_\Omega|\nabla v|^2dx-\frac{\lambda}{2}R^2\int_\Omega|v|^2dx-\frac{n-2}{2n}\int_\Omega a(x)|\bar{u}+Rv|^{2n/(n-2)}dx$$

But

$$\int_\Omega a(x)|\bar{u}+Rv|^{2n/(n-2)}dx$$

$$\geqslant\int_\Omega a(x)|Rv|^{2n/(n-2)}dx-K_1R^{2n/(n-2)}d^{2n/(n-2)}\varepsilon^{n(n-2)/(n+2)}$$

$$+\frac{1}{2}\int_\Omega a(x)|\bar{u}|^{2n/(n-2)}dx$$

$$\geqslant\frac{1}{2}R^{2n/(n-2)}\int_\Omega a(x)|v|^{2n/(n-2)}dx$$

Hence

$$I(u)\leqslant\frac{R^2}{2}\int_\Omega|\nabla v|^2dx-\frac{\lambda}{2}R^2\int_\Omega|v|^2dx$$

$$-\frac{1}{2}\cdot\frac{n-2}{2n}\int_\Omega a(x)|Rv|^{2n/(n-2)}dx\leqslant 0$$

(ii) If $u=\bar{u}+tv$ with

$$\|\bar{u}\|=R,\bar{u}\in V,R_1\leqslant t\leqslant R$$

arguing as (i) we have $I(u)\leqslant 0$.

(iii) If $u\in\overline{B}_R\cap V$, then

$$I(u)=\frac{1}{2}\int_\Omega|\nabla u|^2dx-\frac{\lambda}{2}\int_\Omega|u|^2dx-\frac{n-2}{2n}\int_\Omega a(x)|u|^{2n/(n-2)}dx-\int_\Omega G(x,u)dx$$

$$\leqslant\frac{1}{2}(\lambda_k-\lambda)\int_\Omega|u|^2dx\leqslant 0$$

(iv) If $u=\bar{u}+tv$ with $\bar{u}\in\partial\overline{B}_R\cap V,0\leqslant t\leqslant R_1$, then

$$I(u)\leqslant\frac{1}{2}(1-\frac{\lambda}{\lambda_k})\int_\Omega|\nabla\bar{u}|^2dx+\frac{1}{2}\int_\Omega|\nabla tv|^2dx+\frac{\lambda}{2}\int_\Omega|tv|^2dx-\int_\Omega G(x,u)dx$$

$$\leqslant \frac{1}{2}(1 - \frac{\lambda}{\lambda_k})R^2 + \frac{1}{2}R_1^2\left[\int_\Omega |\nabla v|^2 dx + \lambda \int_\Omega |v|^2 dx\right] \leqslant 0$$

From (i), (ii), (iii) and (iv) we deduce that $I|_{\partial Q} \leqslant 0$.

Finally, we prove $C_0 < \frac{1}{n}(\max_{\bar{\Omega}} a(x))^{(2-n)/2} S^{n/2}$. In fact,

$$C_0 = \inf_{h \in \Gamma} \max_{u \in Q} I(h(u)) \leqslant \max_{u \in Q} I(u)$$

On the other hand, for $u = \bar{u} + tv \in Q, \bar{u} \in V$ and $0 \leqslant t \leqslant R$, by Lemma 3.3 and Lemma 3.5 we get (for $n \geqslant 5$),

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \frac{n-2}{2n}\int_\Omega a(x)|u|^{2n/(n-2)} dx - \int_\Omega G(x,u) dx \\ &\leqslant \frac{1}{2}t^2\left(\int_\Omega |\nabla v|^2 dx - \lambda \int_\Omega |v|^2 dx\right) - \frac{n-2}{2n}\int_\Omega a(x)|tv|^{2n/(n-2)} dx \\ &\quad + K_1 t^{2n/(n-2)} \varepsilon^{n(n-2)/(n+2)} d^{2n/(n-2)} \cdot \frac{n-2}{2n} \\ &\leqslant \frac{1}{n}\left[\frac{\int_\Omega |\nabla (I-P)\varphi|^2 dx - \lambda \int_\Omega |(I-P)\varphi|^2 dx}{\left(\int_\Omega a(x)|(I-P)\varphi|^{2n/(n-2)} dx - K_1 \varepsilon^{n(n-2)/(n+2)}\right)^{(n-2)/2}}\right]^{n/2} \\ &\leqslant \frac{1}{n}(\max_{\bar{\Omega}} a(x))^{(2-n)/2} S^{n/2} - C\varepsilon^2 \end{aligned}$$

Hence $C_0 < \frac{1}{n}(\max_{\bar{\Omega}} a(x))^{(2-n)/2} S^{n/2}$. Similarly, we have $C_0 < \frac{1}{n}(\max_{\bar{\Omega}} a(x))^{(2-n)/2} S^{n/2}$ (for $n = 4$). Then the functional $I(u)$ defined by (2.1) has a nontrivial critical point, the problem (P1) possesses at least one nontrivial solution. The proof of Theorem 3.1 is completed.

Theorem 3.8 *Assume that $a^{-1}(x) \in L^{(n-2)/2}(\Omega)$, $g(x,u)u \geqslant 0$ and $g(x,u)$ satisfies (G1), (G2), (G3) and (G4) (or (G5)). Then there is a constant λ_{k+1}^*, $(k = 0, 1, 2, \cdots)$ such that (P1) possesses at least one nontrivial solution for any $\lambda \in (\lambda_{k+1}^*, \lambda_{k+1})$, where*

$$\lambda_{k+1}^* = \max(\lambda_k, \lambda_{k+1} - (\max_{\bar{\Omega}} a(x))^{(2-n)/n} |a^{-1}(x)|_{(n-2)/2}^{(2-n)/n} S)$$

Proof From the proof of Theorem 3.1 we know that it is only needed to prove $I(u)$ defined by (2.1) satisfies the conditions in Theorem 2.1. We set

$$V = \bigoplus_{i \leqslant k} V_i \text{ (If } k = 0 \text{ we set } V = \varnothing), X = \bigoplus_{i \geqslant k+1} V_i, v = \varphi_{k+1}$$

where φ_{k+1} is the eigenfunction corresponding to λ_{k+1}.

It is obvious that there are two positive constants ρ and α such that $I(u)$ satisfies the condition (I_1) in Theorem 2. 1.

Now, we prove that there is a constant $R(>\rho)$ such that $I(u)$ satisfies the condition (I_2) in Theorem 2. 1.

By Hölder inequality, it is easy to prove that

$$\int_\Omega |u|^2 dx \leqslant \left(\int_\Omega |a^{-1}(x)|^{(n-2)/2} dx\right)^{2/n} \left(\int_\Omega a(x)|u|^{2n/(n-2)} dx\right)^{(n-2)/n}$$

for any $u \in H_0^1(\Omega)$. Hence

$$\int_\Omega a(x)|u|^{2n/(n-2)} dx \geqslant \left(\int_\Omega |a^{-1}(x)|^{(n-2)/2} dx\right)^{-2/(n-2)} \left(\int_\Omega |u|^2 dx\right)^{n/(n-2)}$$

for any $u \in H_0^1(\Omega)$. Since $\bigoplus_{i \leqslant k+1} V_i$ is a finite dimensional subspace of $H_0^1(\Omega)$, there is a positive constant K_2 such that

$$\int_\Omega |u|^2 dx \geqslant K_2 \int_\Omega |\nabla u|^2 dx, \quad \text{for all } u \in \bigoplus_{i \leqslant k+1} V_i$$

Then there is a positive constant R such that, for any $u \in \bigoplus_{i \leqslant k+1} V_i$ with $\| u \| \geqslant R$,

$$\begin{aligned} I(u) &\leqslant \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \frac{n-2}{2n}\int_\Omega a(x)|u|^{2n/(n-2)} dx \\ &\leqslant \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \cdot \frac{1}{2}\int_\Omega |\nabla u|^2 dx \\ &\quad - \frac{n-2}{2n}\left(\int_\Omega |a^{-1}(x)|^{-(n-2)/2} dx\right)^{-2/(n-2)} \left(\int_\Omega |u|^2 dx\right)^{n/(n-2)} \\ &\leqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \| u \|^2 - \frac{n-2}{2n} K_2^{n/(n-2)} |a^{-1}(x)|^{-1}_{(n-2)/2} \| u \|^{2n/(n-2)} \\ &\leqslant 0 \end{aligned} \tag{3. 25}$$

(i) If $u \in \bar{B}_R \cap V$, then

$$I(u) \leqslant \frac{1}{2}(\lambda_k - \lambda)\int_\Omega |u|^2 dx - \frac{n-2}{2n}\int_\Omega a(x)|u|^{2n/(n-2)} dx - \int_\Omega G(x,u) dx \leqslant 0$$

(ii) If $u \in \partial(\bar{B}_R \cap V) \oplus \{tv \mid 0 \leqslant t \leqslant R\}$, or $u \in (\bar{B}_R \cap V) \oplus \{Rv\}$, then $\| u \| \geqslant$

R, $u\in \bigoplus\limits_{i\leqslant k+1} V_i$. By (3.25) we get $I(u)\leqslant 0$.

(i) and (ii) imply that $I|_{\partial Q}\leqslant 0$.

Last, we prove $C_0<\frac{1}{n}(\max\limits_{\bar{\Omega}} a(x))^{(2-n)/2}S^{n/2}$ for $\lambda\in(\lambda_{k+1}^*,\lambda_{k+1})$. In fact we have

$$
\begin{aligned}
I(u)&\leqslant \frac{1}{2}(\lambda_{k+1}-\lambda)\int_\Omega |u|^2dx-\frac{n-2}{2n}\int_\Omega a(x)|u|^{2n/(n-2)}dx-\int_\Omega G(x,u)dx\\
&\leqslant \frac{1}{2}(\lambda_{k+1}-\lambda)\left(\int_\Omega |a^{-1}(x)|^{(n-2)/2}dx\right)^{2/n}\cdot\left(\int_\Omega a(x)|u|^{2n/(n-2)}dx\right)^{(n-2)/n}\\
&\quad -\frac{n-2}{2n}\int_\Omega a(x)|u|^{2n/(n-2)}dx\\
&\leqslant \frac{1}{n}(\lambda_{k+1}-\lambda)^{n/2}\left(\int_\Omega |a^{-1}(x)|^{(n-2)/2}dx\right)\\
&<\frac{1}{n}(\max_{\bar{\Omega}} a(x))^{(2-n)/2}S^{n/2},\quad \text{for } u\in Q
\end{aligned}
$$

Hence $C_0\leqslant\max\limits_{u\in Q} I(u)<\frac{1}{n}(\max\limits_{\bar{\Omega}} a(x))^{(2-n)/2}S^{n/2}$. The proof of Theorem 3.8 is completed.

Remark 2 If $0<\lambda<\lambda_1$, $\lim\limits_{u\to\infty}\frac{g(x,u)}{a(x)|u|^{(n+2)/(n-2)}}=0$ (uniformly in $x\in\Omega$), then $g(x,u)$ satisfies (G2) and (G4).

Remark 3 If there are $\theta\in(0,\frac{1}{2})$ and $M>0$ such that

$$0<G(x,u)\leqslant\theta g(x,u)u\quad\text{for}\quad |u|\geqslant M, x\in\Omega$$

then $g(x,u)$ satisfies (G4).

Corollary *Suppose $a(x)$ is a smooth function which is nonnegative and positive somewhere, If $a(x)$ satisfies (A1) (or (A2)), and $\lambda\notin\sigma(-\triangle)$, then the problem*

$$
(P2)\qquad\begin{cases}-\triangle u=a(x)|u|^{4/(n-2)}u+\lambda u, & x\in\Omega\\ u=0, & x\in\partial\Omega\end{cases}
$$

possesses at least one nontrivial solution.

If $a^{-1}(x)\in L^{(n-2)/2}(\Omega)$, then there exists a constant $\lambda_{k+1}^*(k=0,1,2,\cdots)$, such that (P2) possesses at least one nontrivial solution for any $\lambda\in(\lambda_{k+1}^*,\lambda_{k+1})$, where

$$\lambda_{k+1}^*=\max(\lambda_k,\lambda_{k+1}-(\max_{\bar{\Omega}} a(x))^{(2-n)/n}|a^{-1}(x)|_{(n-2)/2}^{(2-n)/n}S)$$

4. Appendix

Proof of Theorem 2.1 It is obvious that $C\geqslant a>0$ since $S=\partial B_\rho\cap X, Q=(B_R\cap V)\cup\{u+tv\mid u\in\partial B_R\cap V, 0\leqslant t\leqslant R\}\cup\{u+Rv\mid u\in B_R\cap V\}(R>\rho)$ link (see [6]).

Now we prove that there exists a sequence $\{u_j\}\subset E$ such that $I'(u_j)\to 0$ and $I(u_j)\to C_0$. If there exists no sequence $\{u_j\}\subset E$ such that $I'(u_j)\to 0$ and $I(u_j)\to C_0$, then there are constants b and $\bar{\varepsilon}$ satisfying

$$\| I'(u)\| \geqslant b \quad \text{for} \quad u\in (A_{C_0+\bar{\varepsilon}}-A_{C_0-\bar{\varepsilon}}) \tag{4.1}$$

where $A_C=\{u\in E\mid I(u)\leqslant C\}, b>0$ and $\bar{\varepsilon}>0$.

Let $\varepsilon\in(0,\bar{\varepsilon})$ and $\varepsilon<\min(b^2/2, 1/8, C_0/2), A=\{u\in E\mid I(u)\geqslant C_0+\bar{\varepsilon}$ or $I(u)\leqslant C_0-\bar{\varepsilon}\}$ and $B=\{u\in E\mid C_0-\varepsilon\leqslant I(u)\leqslant C_0+\varepsilon\}$. Then $A\cap B=\varnothing$. Define $g(u)=\| u-A\|\cdot(\| u-A\|+\| u-B\|)^{-1}, h(s)=1$ if $s\in[0,1], h(s)=s^{-1}$ if $s\geqslant 1$. Consider the ordinary differential equation

$$\frac{d\eta_t}{dt}=-g(\eta_t)h(\| v(\eta_t)\|)v(\eta_t), \quad \eta_0(u)=u \quad \text{for} \quad u\in E \tag{4.2}$$

where v is a pseudo-gradient vector field for $I(u)$. It is obvious that the solution $\eta_t(u)$ of (4.2) is a homeomorphism of E onto E, $\eta_t(u)=u$ for $u\notin I^{-1}[C_0-\varepsilon, C_0+\varepsilon]$, and $\eta_1(A_{C_0+\varepsilon})\subset A_{C_0-\varepsilon}$ (see [9]). In fact, the semigroup property for solutions of ordinary differential equation (4.2) implies that $\eta_t(u)$ is a homeomorphism of E onto E and $\eta_t(u)=u$ for $u\notin I^{-1}[C_0-\varepsilon, C_0+\varepsilon]$. We are going to prove that $\eta_1(A_{C_0+\varepsilon})\subset A_{C_0-\varepsilon}$.

Since

$$\frac{d}{dt}I(\eta_t(u))=-g(\eta_t(u))h(\| v(\eta_t(u))\|)\cdot\langle I'(\eta_t(u)), v(\eta_t(u))\rangle\leqslant 0$$

$$I(\eta_t(u))\leqslant C_0-\varepsilon \quad \text{for } u\in A_{C-\varepsilon}$$

we need only to prove that $I(\eta_1(u))\leqslant C_0-\varepsilon$ for $u\in Y=A_{C_0+\varepsilon}-A_{C_0-\varepsilon}$.

Suppose $\eta_t(u)\in Y$ for all $t\in[0,1], u\in Y$. Then

$$\frac{dI(\eta_t(u))}{dt}\leqslant -h(\| v(\eta_t(u))\|)\cdot\| I'(\eta_t(u))\|^2 \tag{4.3}$$

If for some t we have $\| v(\eta_t(u))\|\leqslant 1$ and $h(\| v(\eta_t(u))\|)=1$, then by (4.1), (4,2),

$$\frac{dI(\eta_t(u))}{du}\leqslant -\| I'(\eta_t(u))\|^2<-b^2 \tag{4.4}$$

If we have $\| v(\eta_t(u))\|>1$ and $h(\| v(\eta_t(u))\|)=\| v(\eta_t(u))\|^{-1}$, then by (4.1) and (4.3),

$$\frac{dI(\eta_t(u))}{dt}\leqslant -\frac{\| I'(\eta_t(u))\|^2}{\| v(\eta_t(u))\|}\leqslant -\frac{\| v(\eta_t(u))\|}{4}\leqslant -\frac{1}{4} \tag{4.5}$$

Combining (4.5) with (4.4) we deduce

$$I(\eta_1(u)) - I(u) \leqslant -\min(b^2, \frac{1}{4})$$

that is

$$C_0 - \varepsilon \leqslant I(\eta_1(u)) \leqslant I(u) - \min(b^2, \frac{1}{4}) < C_0 + \varepsilon - \min(b^2, \frac{1}{4})$$

which violates $\varepsilon < \min(\frac{b^2}{2}, \frac{1}{8}, \frac{C_0}{2})$. So we have proved that $\eta_1(A_{C_0+\varepsilon}) \subset A_{C_0-\varepsilon}$.

For $0 < \varepsilon < \min(\bar{\varepsilon}, 1/8, \frac{b^2}{4}, \frac{C_0}{2})$, (I_3) implies that there exists $h_0 \in \Gamma$ such that $\max_{u \in Q} I(h_0(u)) \leqslant C_0 + \varepsilon$. Hence $h_0 \circ \eta_1 \in \Gamma$ and $\max_{u \in Q} I(\eta_1(h_0(u))) < C_0 - \varepsilon$, that is

$$C_0 = \inf_{h \in \Gamma} \max_{u \in Q} I(h(u)) \leqslant \max_{u \in Q} I(\eta_1(h_0(u))) < C_0 - \varepsilon$$

which is a contradiction. Theorem 2.1 is completely proved.

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