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CLASSICAL SOLUTION TO THE ELECTROPAINTING PROBLEM

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Abstract The mathematical modelling of the electrodeposition phenomenon leads to a linear elliptic partial differential equation subject to nonlinear evolutionary mixed boundary conditions. In this paper, the existence, uniqueness and regularity of classical solution are proved for the electropainting problem when "dissolution current" is zero.
Key Words electropainting problem; classical solution.
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1. Introduction

We consider a time-dependent elliptic free boundary problem associated with an electropaint process. The problem is to find a pair $(v(x,t), h(x,t))$ such that in an annular region $\Omega \subset R^N (N \geqslant 2)$ with outer boundary S and inner boundary Γ there hold

$$\triangle v = 0 \quad \text{in } \Omega \tag{1.1}$$
$$v = 1 \quad \text{on } S \tag{1.2}$$
$$h v_n = v \quad \text{on } \Gamma \tag{1.3}$$
$$h_t = (v_n - \varepsilon)^+ \quad \text{on } \Gamma \tag{1.4}$$
$$h(x,0) = 0 \quad \text{on } \Gamma \tag{1.5}$$

where v_n is the inward normal derivative on Γ, $(z)^+ = \max(0, z)$, $\varepsilon > 0$ is a given constant.

Similar problems were considered by Hansen and McGeough ([1]), Aitchison, Lacey and Shillor ([2]), Caffarelli and Friedman ([3]), and Márquez and Shillor ([4]). The first two dealt with the modelling aspects of the electropaint process and numerical experiments. The latter two dealt with the time-discretized version of (1.1)—(1.5) and the electropainting problem with overpotentials respectively.

A problem of the type (1.1)—(1.5) can be considered as a model for the following process (see [1] or [2]):

A metal body with an outer surface Γ, to be painted, is immersed in a tank with an electrolytic solution. The solution occupies the region Ω such that $\partial\Omega = \Gamma \cup S$, where S is the inner surface of the tank. The metal part, which is usually called " the work piece", is connected to an electric potential source, the tank itself (S) serves as the other electrode and as a result of the flow of the electric current in the solution and into Γ, the process of paint deposition takes place on Γ. The unknown function v stands for the electric potential and in the boundary conditions on Γ, the unknown function h is the thickness of the paint coat. The existence of a " dissolution current" $\varepsilon>0$, that was postulated in [2], assures that there is paint deposition only at those points of Γ where the current v_n satisfies $v_n>\varepsilon$. When the dissolution current can be neglected, the model becomes

$$\triangle v=0 \quad \text{in } \Omega \tag{1.6}$$

$$v=1 \quad \text{on } S \tag{1.7}$$

$$hv_n=v \quad \text{on } \Gamma \tag{1.8}$$

$$h_t=v_n>0 \quad \text{on } \Gamma \tag{1.9}$$

$$h(x,0)=0 \quad \text{on } \Gamma \tag{1.10}$$

and we refer to (1.6)—(1.10) as problem (P). Hansen and McGeough first proposed this model in [1], and in addition, they also described two major features of the process, the " saturation effect" and the " levelling effect" ,via numerical experiment.

In all these papers mentioned above, however, it seems that no classical solution has been obtained for the electropainting problem (1.1)—(1.5) or (P).

In the present paper, we shall study the existence, uniqueness and regularity of classical solution to the problem (P)

Definition 1.1 *A classical solution to the problem (P) is a pair $(v(x,t),h(x,t))$ of functions such that*

(i) $v(x,t)\in C^0(\overline{\Omega}\times[0,\infty))$

$v(\cdot,t)\in C^1(\overline{\Omega})\cap C^2(\Omega) \quad \forall\ t\geq 0$

$h(x,t)\in C^0(\Gamma\times[0,\infty))$

$h_t(x,t)\in C^0(\Gamma\times(0,\infty))$

(ii) $\dfrac{\partial}{\partial n}\displaystyle\int_0^t v(x,\tau)d\tau = h(x,t) \quad$ *on* $\Gamma\times(0,\infty)$

(iii) (1.6)—(1.10) *are satisfied for any* $t>0$.

Our main results are the existence theorem and the regularity theorem stated as follows:

Theorem 1.1 *Let $\partial\Omega\in C^4$. Then there exists a unique classical solution of the problem (P).*

Theorem 1.2 *Let $\partial\Omega\in C^{k+\alpha}(k\geq 4, 0<\alpha<1)$, and $v(x,t)$ be the classical solution of the problem (P). Then*

$$v(x,t)\in C^{k+\alpha}(\overline{\Omega}\times(0,\infty))$$

and moreover

$$D_t^m v(x,t)\in C^{k+\alpha}(\overline{\Omega}\times(0,\infty)),\ m=1,2,\cdots$$

Everywhere below, it will be assumed that $\partial\Omega \in C^1$ at least and $\frac{\partial}{\partial n}$ is the normal derivative into Ω.

2. Transformed Problem

Supposing (v,h) is a classical solution of the problem (P), we define Baiocchi transformation

$$u(x,t) = \int_0^t v(x,\tau)d\tau \quad (x,t) \in \bar{\Omega} \times [0,\infty)$$

and let one show how a problem for u can be obtained.

It is easy to verify that, for any fixed $t > 0$, $u(\cdot, t)$ is harmonic in Ω and $u=t$ on S. By (ii) of Definition 1.1, we have

$$u_n = h \quad \text{on } \Gamma$$

Then (1.8) and (1.9) yield

$$D_t(u_n^2) = 2v \quad \text{on } \Gamma$$

Integrating with respect to the time variable from 0 to t, we have

$$u_n^2 = 2u \quad \text{on } \Gamma$$

Thus the problem (P) is reduced to a transformed problem (P_0) which can be formulated as follows:

$$\triangle u = 0 \quad \text{in } \Omega \tag{2.1}$$

$$u = t \quad \text{on } S \tag{2.2}$$

$$u > 0 \quad \text{and } u_n = \sqrt{2u} \quad \text{on } \Gamma \tag{2.3}$$

In this section, we shall prove that

Theorem 2.1 *For any fixed $t \geq 0$, there exists a unique solution u to the problem (P_0) with $u \in C^{1+\alpha}(\bar{\Omega})$, any $\alpha \in (0,1)$.*

If we apply the maximum principle to a difference of any two solutions in $C^{1+\alpha}(\bar{\Omega})$, the uniqueness will easily be proved, and it is evident that the only solution for $t=0$ is $u=0$. To prove the solvability of the problem (P_0) for $t>0$, we introduce the following approximation

$$\triangle u = 0 \quad \text{in } \Omega \tag{2.4}$$

$$u = t \quad \text{on } S \tag{2.5}$$

$$u > 0 \text{ and } u_n = \sqrt{2u + \sigma^2} - \sigma \quad \text{on } \Gamma \tag{2.6}$$

where $t > 0$ is fixed and $\sigma \in (0,1)$.

For every $\sigma > 0$, by a suitable truncation of the boundary data on Γ, the problem (2.4)—(2.6), which will be called (P_σ) in the following, has a unique solution $u_\sigma \in C^3(\bar{\Omega})$ (*cf.* [5]). For convenience, in this section we will omit the subscript and denote by u the solution of the problem (P_σ).

By the maximum principle, it is seen that a priori

$$0 < u \leqslant t \quad \text{on } \bar{\Omega} \tag{2.7}$$

Furthermore, a uniform positive lower bound can be obtained for all the solutions of the problem (P_σ).

Lemma 2.2 *Let $t > 0$ be fixed. Then for any solution u of the problem (P_σ), there holds*

$$0 < \gamma \leqslant u \quad \text{in } \Omega \tag{2.8}$$

where $\gamma > 0$ is a constant independent of σ.

Proof Denote by z_0 the solution of

$$\triangle z_0 = 0 \quad \text{in } \Omega; \quad z_0 = t \quad \text{on } S; \quad z_0 = t/2 \quad \text{on } \Gamma$$

Since z_0 attains its minimum everywhere on Γ, it implies that $(z_0)_n > 0$ on Γ and hence $\mu = \inf_\Gamma (z_0)_n > 0$. Now we choose $\gamma = \min\{t/2, \mu^2/2\}$ and denote by z the solution of

$$\triangle z = 0 \text{ in } \Omega; \quad z = t \quad \text{on } S; \quad z = \gamma \quad \text{on } \Gamma$$

Applying the strong maximum principle to the function $z - z_0$, which attains its minimum everywhere on Γ, we deduce that

$$z_n > (z_0)_n \geqslant \mu \geqslant \sqrt{2\gamma} = \sqrt{2z} \quad \text{on } \Gamma \tag{2.9}$$

For any solution u which solves the problem (P_σ), the function $u - z$ is harmonic in Ω, vanishes on S, and satisfies (from (2.6) and (2.9))

$$(u - z)_n < \sqrt{2u + \sigma^2} - \sigma - \sqrt{2z} < \sqrt{2u} - \sqrt{2z} \quad \text{on } \Gamma$$

Using the maximum principle again, we deduce that

$$u-z>0 \quad \text{in } \Omega$$

Noticing that

$$z>\gamma>0 \quad \text{in } \Omega$$

then (2.8) holds.

By standard Bernstein technique, we can obtain a uniform estimate on second derivatives of u, to which it is essential that u has a positive lower bound independent of σ.

Lemma 2.3 *Let $t>0$ be fixed. Then for any solution u of the problem (P_σ), there holds*

$$\sup_{\Omega}|D^2u|\leqslant c$$

where $c>0$ is a constant independent of σ.

3. Existence and Uniqueness

Based on the estimates in Section 2, we can assert that, for every fixed $t\geqslant 0$, the transformed problem (P_0) is uniquely solvable in $C^{1+\alpha}(\overline{\Omega})$, any $\alpha\in(0,1)$. The unique solution, which is the limit of a subsequence of solutions to the problems (P_σ) for $t>0$, is denoted by $u(x,t)$.

By the maximum principle, it is easy to get

Lemma 3.1 *$u(x,t)$ is an increasing function of t, for all $t\geqslant 0$.*

Lemma 3.2 $\sup_{\Omega}|u(x,t_1)-u(x,t_2)|\leqslant|t_1-t_2|, \quad \forall\ t_1,t_2\in[0,\infty)$.

It follows from Lemma 3.2 that

Theorem 3.3 $u(x,t)\in C^0(\overline{\Omega}\times[0,\infty))$.

Consider a linear boundary value problem

$$\triangle v=0 \quad \text{in } \Omega; \qquad v=1 \quad \text{on } S; \qquad v_n=v/\sqrt{2u} \quad \text{on } \Gamma$$

which is time-dependent due to the occurrence of u in the boundary condition on Γ. By $v(x,t)$ we denote the solution to this problem for $t>0$. The maximum principle implies that

$$0<v(x,t)\leqslant 1, \quad \forall\ (x,t)\in\overline{\Omega}\times(0,\infty) \tag{3.1}$$

It is desirable to know the relations between the functions u and v.

Theorem 3.4 $v(x,t)=D_tu(x,t), \quad (x,t)\in\overline{\Omega}\times(0,\infty)$.

Proof For a fixed $t>0$, as a function of x, the difference quotient $\triangle^\tau u=(u_\tau-u)/\tau$ solves the problem

$$\triangle z=0 \quad \text{in } \Omega; \qquad z=1 \quad \text{on } S; \qquad z_n=2z/(\sqrt{2u}+\sqrt{2u_\tau}) \quad \text{on } \Gamma$$

where $|\tau|<t/2, u_\tau=u(x,t+\tau)$.

Then the difference $\triangle^\tau u-v$ is harmonic in Ω, vanishes on S and satisfies

$$(\triangle^\tau u-v)_n=\frac{2(\triangle^\tau u-v)}{\sqrt{2u}+\sqrt{2u_\tau}}+\frac{2v(u-u_\tau)}{\sqrt{2u}(\sqrt{2u}+\sqrt{2u_\tau})^2}\quad \text{on } \Gamma$$

The monotonicity of u in t (see Lemma 3. 1) implies that

$$0<\inf_{\Omega} u(x,t/2)\leqslant u,\quad u_\tau\leqslant 3t/2 \tag{3. 2}$$

Applying the maximum principle to the function $\triangle^\tau u-v$, and noticing (3. 1), (3. 2) and Lemma 3. 2, we have

$$\sup_{\Omega}|\triangle^\tau u-v|\leqslant c\sup_{\Gamma}|u-u_\tau|\leqslant c|\tau| \tag{3. 3}$$

where $c>0$ is dependent only on t, but independent of τ.

The desired conclusion follows from (3. 3) upon taking $\tau\to 0$.

Replacing $\triangle^\tau u-v$ by $v(x,t_1)-v(x,t_2)$ and repeating the above process from which (3. 3) is deduced, we get

Lemma 3. 5 $\sup\limits_{\Omega}|v(x,t_1)-v(x,t_2)|\leqslant c\sup\limits_{\Omega}|u(x,t_1)-u(x,t_2)|$ *for any* $t_1,t_2\in[\delta,T]\subset(0,\infty)$, *where* $c>0$ *is a constant depending on* δ *and* T.

Let v_0 be the solution of the Dirichlet problem

$$\triangle v_0=0\quad \text{in } \Omega;\qquad v_0=1\quad \text{on } S;\qquad v_0=0\quad \text{on } \Gamma$$

Then, for any $t>0$, the function $v-v_0$ is harmonic in Ω, vanishes on S and satisfies

$$(v-v_0)_n=(v-v_0)/\sqrt{2u}-(v_0)_n\quad \text{on } \Gamma$$

By virtue of the maximum principle, it is seen that

$$\sup_{\Omega}|v-v_0|\leqslant\sup_{\Gamma}\sqrt{2u}\cdot\sup_{\Gamma}(v_0)_n\leqslant\sqrt{2t}\sup_{\Gamma}(v_0)_n \tag{3. 4}$$

which implies that v converges uniformly on $\bar{\Omega}$ to v_0 as $t\to 0^+$. Then, we can give a supplementary definition to $v(x,t)$ for $t=0$, that is

$$v(x,0)=v_0(x)\;,\quad x\in\bar{\Omega} \tag{3. 5}$$

Theorem 3. 6 $v(x,t) \in C^0(\overline{\Omega} \times [0,\infty))$.

Proof It is enough to verify the continuity of v at an arbitrarily given point $(x', t') \in \overline{\Omega} \times [0,\infty)$.

If $t' > 0$, by means of Lemma 3. 2 and 3. 5, we get

$$|v(x,t) - v(x',t')| \leqslant c|t - t'| + |v(x,t') - v(x',t')| \tag{3. 6}$$

In the case of $t' = 0$, (3. 5) and (3. 4) can be used to get

$$|v(x,t) - v(x',0)| \leqslant \sqrt{2t}\sup_{\Gamma}(v_0)_n + |v_0(x) - v_0(x')| \tag{3. 7}$$

Due to the continuity of $v(x,t')$ and $v_0(x)$ with respect to x, the desired conclusion follows from (3. 6) and (3. 7).

Now we are able to prove the existence theorem.

Proof of Theorem 1. 1 By virtue of Theorem 3. 3, 3. 4 and 3. 6, it is readily checked that the function $v(x,t)$ defined above together with the function

$$h(x,t) = \sqrt{2u(x,t)}, \quad (x,t) \in \Gamma \times [0,\infty)$$

makes up a classical solution to the problem (P).

Assuming that $\{v_1, h_1\}$ and $\{v_2, h_2\}$ are two classical solutions of (P), we reduce our consideration to the functions

$$u_i(x,t) = \int_0^t v_i(x,\tau)d\tau, \quad (x,t) \in \overline{\Omega} \times [0,\infty), \quad i = 1,2 \tag{3. 8}$$

Clearly

$$u_i(x,0) = 0, \quad x \in \overline{\Omega}, \quad i = 1,2$$

For any fixed $t > 0$, in view of Definition 1. 1, we infer that

$$(u_i)_n = h_i \quad \text{on } \Gamma, \quad i = 1,2 \tag{3. 9}$$

and both u_1 and u_2 solve the problem (P_0). Then the maximum principle implies that

$$u_1(x,t) = u_2(x,t), \quad x \in \overline{\Omega}$$

Hence $u_1(x,t)$ coincides with $u_2(x,t)$ for all $(x,t) \in \overline{\Omega} \times [0,\infty)$. By differentiation, it follows from (3. 8) and (3. 9) that

$$v_1 \equiv v_2 \text{ and } h_1 \equiv h_2$$

Thus the uniqueness is proved.

We remark that the existence result remains valid if (1. 7), the boundary condition on S, is replaced by

$$v = g \quad \text{on } S \tag{1. 7'}$$

where $g(x) > 0$ is a sufficiently smooth function on S, which stands for the electric potential on the anode (S) as in the electropaint process. Moreover, it can be proved that the classical solution $\{v, h\}$ varies continuously depending on the variation of g.

Theorem 3. 7 *Suppose that*

$$0 < m \leqslant g_i \leqslant M \tag{3. 10}$$

and $\{v_i, h_i\}$ *is the relevant classical solution where* $i = 1, 2$. m *and* M *are constants. Then we have*

$$\sup_{\bar{\Omega} \times [\delta, T]} |v_1(x,t) - v_2(x,t)| \leqslant c \sup_S |g_1 - g_2| \tag{3. 11}$$

$$\sup_{\Gamma \times [\delta, T]} |h_1(x,t) - h_2(x,t)| \leqslant c \sup_S |g_1 - g_2| \tag{3. 12}$$

where $c > 0$ *is a constant depending on* m, M, δ *and* T.

Proof Corresponding to (1. 7′), in the transformed problem (P_0) the boundary condition on S, (2. 2), is replaced by

$$u = t \cdot g(x) \text{ on } S \tag{2. 2'}$$

By the maximum principle, it is easily seen that

$$\sup_{\Omega} |u_1 - u_2| \leqslant t \sup_S |g_1 - g_2|, \quad t > 0 \tag{3. 13}$$

where u_i is the solution of the transformed problem relevant to g_i, $i = 1, 2$.

The hypothesis (3. 10) implies that (cf. Lemma 2. 2) there exists a constant $\gamma > 0$ depending on m and δ, such that

$$u_i(x,t) \geqslant \gamma > 0, \quad \forall\ (x,t) \in \bar{\Omega} \times [\delta, T], \quad i = 1. 2$$

Recalling that

$$h_i(x,t) = \sqrt{2u_i(x,t)}\ , \quad (x,t) \in \Gamma \times (0, \infty), i = 1, 2$$

we have

$$|h_1(x,t) - h_2(x,t)| \leqslant |u_1(x,t) - u_2(x,t)| / \sqrt{2\gamma} \qquad (3.14)$$

for any $(x,t) \in \Gamma \times [\delta,T]$. (3.14) and (3.13) yield (3.12).

To prove (3.11), we apply the maximum principle to the difference $v_1 - v_2$ which satisfies

$$\triangle(v_1 - v_2) = 0 \qquad \text{in } \Omega$$

$$v_1 - v_2 = g_1 - g_2 \qquad \text{on } S$$

$$(v_1 - v_2)_n = \frac{v_1 - v_2}{h_1} + \frac{v_2(h_2 - h_1)}{h_1 h_2} \quad \text{on } \Gamma$$

Then we obtain

$$\sup_{\bar{\Omega}\times[\delta,T]} |v_1(x,t) - v_2(x,t)| \leqslant \sup_S |g_1 - g_2| + \sup_{\Gamma\times[\delta,T]} h_1 \cdot \sup_{\Gamma\times[\delta,T]} \left| \frac{v_2(h_2 - h_1)}{h_1 h_2} \right|$$

Because of that

$$\sqrt{2\gamma} \leqslant h_i(x,t) \leqslant \sqrt{2TM}, \quad \forall\, (x,t) \in \Gamma \times [\delta,T]; 0 < v_i \leqslant M, \quad i = 1,2$$

thus (3.11) follows.

4. Regularity

In this section, we shall discuss the regularity of the classical solution.

For general evolutionary elliptic free boundary problems, it is a rather knotty problem to study the regularity with respect to the time parameter t. Owing to the introduction of Baiocchi transformation, our discussion in this section, as well as in Section 3, is essentially based on the continuous dependence of solutions of linear elliptic problems on the boundary data.

We begin with a statement of regularity of $u(x,t)$, the solution of the transformed problem (P_0).

Let $\partial\Omega \in C^{k+\alpha}(k \geqslant 4, 0 < \alpha < 1)$, then it is evident that

$$u(\cdot,t) \in C^{k+\alpha}(\bar{\Omega}), \quad \forall\, t \geqslant 0 \qquad (4.1)$$

Using the C^2 estimate in Section 2 and performing the standard $C^{k+\alpha}$ estimates on $u(\cdot,t)$ and $u(\cdot,t_1) - u(\cdot,t_2)$ respectively, we obtain (cf. [6])

$$\| u(\cdot,t) \|_{C^{k+\alpha}(\bar{\Omega})} \leqslant c, \quad \forall\, t \in [\delta,T] \qquad (4.2)$$

$$\| u(\cdot,t_1) - u(\cdot,t_2) \|_{C^{k+\alpha}(\bar{\Omega})} \leqslant c|t_1 - t_2|, \quad \forall\, t_1,t_2 \in [\delta,T] \qquad (4.3)$$

which yield

$$|D_x^p u(x,t) - D_x^p u(x',t')| \leqslant c\{|x-x'|^\alpha + |t-t'|\}, \quad |p| \leqslant k \quad (4.4)$$

for any (x,t), $(x',t') \in \bar{\Omega} \times [\delta, T]$, where $c>0$ depends on δ and T, $p=(p_1,\cdots,p_N)$ is a multi-index, $|p|=p_1+\cdots+p_N$ and

$$D_x^p = \frac{\partial^{|p|}}{\partial x_1^{p_1} \cdots \partial x_N^{p_N}}$$

For convenience, we rewrite $u^{(1)}=v$ where v is the classical solution of the problem (P) and also the derivative of u with respect to t as shown in Theorem 3. 4. For any positive integer $m \geqslant 2$, by $u^{(m)}(x,t)$, $(x,t) \in \bar{\Omega} \times (0,\infty)$, we denote the solution to the following linear boundary value problem

$$\triangle u^{(m)} = 0 \quad \text{in } \Omega; \qquad u^{(m)} = 0 \quad \text{on } S$$

$$u_n^{(m)} = u^{(m)}/\sqrt{2u} + G_m[u, u^{(1)}, \cdots, u^{(m-1)}] \quad \text{on } \Gamma$$

where

$$G_m[u,u^{(1)},\cdots,u^{(m-1)}] = \sum_{\substack{2 \leqslant i \leqslant m \\ i_1+\cdots+i_r=i \\ 1 \cdot i_1+\cdots+r \cdot i_r=m}} \frac{m!(-1)^{i-1}(2i-3)!!}{\prod_{s=1}^{r} (i_s)!(s!)^{i_s}} (\sqrt{2u})^{1-2i} \prod_{s=1}^{r} (u^{(s)})^{i_s}$$

By induction, $u^{(m)}$ is well-defined and it can be claimed that the assertions (4. 1) —(4. 4) are valid also for $u^{(m)}$. Namely we have

Lemma 4. 1 *Let $\partial\Omega \in C^{k+\alpha}$ $(k \geqslant 4, 0<\alpha<1)$. Then for any nonnegative integer m, there hold*

$$u^{(m)}(\cdot,t) \in C^{k+\alpha}(\bar{\Omega}), \quad \forall t \geqslant 0 \quad (4.5)$$

$$\| u^{(m)}(\cdot,t) \|_{C^{k+\alpha}(\bar{\Omega})} \leqslant c, \quad \forall t \in [\delta, T] \quad (4.6)$$

$$\| u^{(m)}(\cdot,t_1) - u^{(m)}(\cdot,t_2) \|_{C^{k+\alpha}(\bar{\Omega})} \leqslant c|t_1-t_2|, \quad \forall t_1,t_2 \in [\delta,T] \quad (4.7)$$

$$|D_x^p u^{(m)}(x,t) - D_x^p u^{(m)}(x',t')| \leqslant c\{|x-x'|^\alpha + |t-t'|\}, \quad |p| \leqslant k \quad (4.8)$$

for any (x,t), $(x',t') \in \bar{\Omega} \times [\delta,T]$, where $c>0$ depends on δ and T, p is a multi-index as before.

Lemma 4. 1 provides the regularity of $u^{(m)}$, by which it may be concluded that (with $u^{(0)}=u$)

Corollary 4. 2 $\quad u^{(m)} \in C^{k+\alpha}(\bar{\Omega} \times (0,\infty))$, $\quad m=0,1,2,\cdots$

As a matter of fact, $u^{(m)}$ is exactly the m-th order derivative of u with respect to t.

Theorem 4. 3

$$u^{(m)}(x,t)=D_t^m u(x,t)\ ,\quad (x,t)\in\bar{\Omega}\times(0,\infty),\quad m=1,2,\cdots$$

Proof It suffices to prove that

$$\sup_{\Omega}|\triangle^\tau u^{(m-1)}-u^{(m)}|\leqslant C_m|\tau|\ ,\quad m=1,2,\cdots \tag{4.9}$$

where $t>0$ is fixed, $|\tau|<t/2$, $C_m>0$ is dependent on t, but independent of τ.

We proceed by induction.

On account of (3.3), (4.9) is valid for $m=1$.

Assuming that (4.9) is verified for all $1\leqslant j\leqslant m$, we now show the truth of (4.9) for $m+1$. As in the proof of Theorem 3.4, it is seen that the difference $\triangle^\tau u^{(m)}-u^{(m+1)}$ is harmonic in Ω, vanishes on S and satisfies

$$(\triangle^\tau u^{(m)}-u^{(m+1)})_n=(\triangle^\tau u^{(m)}-u^{(m+1)})/\sqrt{2u}+F_m\quad \text{on } \Gamma$$

where

$$F_m=[\triangle^\tau(1/\sqrt{2u})+u^{(1)}/(\sqrt{2u})^3]u_\tau^{(m)}+(u^{(m)}-u_\tau^{(m)})u^{(1)}/(\sqrt{2u})^3$$
$$+\triangle^\tau G_m-\{G_{m+1}[u,u^{(1)},\cdots,u^{(m)}]+u^{(1)}u^{(m)}/(\sqrt{2u})^3\}$$
$$u_\tau^{(m)}=u^{(m)}(x,t+\tau)$$

By calculation, we have

$$D_t(1/\sqrt{2u})=-u^{(1)}/(\sqrt{2u})^3$$

and

$$D_tG_m=G_{m+1}[u,u^{(1)},\cdots,u^{(m)}]+u^{(1)}u^{(m)}/(\sqrt{2u})^3$$

Then with our inductive assumption, we can deduce that

$$\sup_{\Omega}|\triangle^\tau(1/\sqrt{2u})+u^{(1)}/(\sqrt{2u})^3|\leqslant c|\tau| \tag{4.10}$$

$$\sup_{\Omega}|\triangle^\tau G_m-(G_{m+1}[u,u^{(1)},\cdots,u^{(m)}]+u^{(1)}u^{(m)}/(\sqrt{2u})^3)|\leqslant c|\tau| \tag{4.11}$$

where $c>0$ is dependent on $C_1,C_2,\cdots,C_{m-1}$ and t, but independent of τ. Moreover, (4.7) gives

$$\sup_{\Omega}|u^{(m)}-u_\tau^{(m)}|\leqslant c|\tau| \tag{4.12}$$

From (4.10), (4.11) and (4.12), we have

$$|F_m| \leqslant C_m |\tau|$$

Then applying the maximum principle to $\triangle^\tau u^{(m)} - u^{(m+1)}$ we obtain

$$\sup_\Omega |\triangle^\tau u^{(m)} - u^{(m+1)}| \leqslant \sup_\Gamma \sqrt{2u} \cdot \sup_\Gamma |F_m| \leqslant C_m |\tau|$$

Thus we have come to the desired conclusion.

Finally, the regularity result (Theorem 1. 2) follows from Corollary 4. 2 and Theorem 4. 3.

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