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ON THE NATURAL GROWTH QUASILINEAR ELLIPTIC EULER EQUATIONS①

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Abstract In this paper, we consider the eigenvalue problem and the Dirichlet problem of general Euler equations under the natural growth condition.

Key Words Natural growth; Elliptic Euler equation; Eigenvalue problem; Variational method.

Classifications 35J65; 49A40.

1. Preface and Assumptions

In this paper the eigenvalue of general Euler equations under the natural growth condition is first discussed:

$$\begin{cases} -\dfrac{d}{dx_i}F_i(x,u,Du) + F_u(x,u,Du) = \lambda |u|^{p-2}u, & x \in \Omega \\ u(x) - \omega(x) \in W_0^{1,m}(\Omega) \end{cases} \tag{1}$$

where

$$F_i = \frac{\partial}{\partial q_i}F(x,u,q), q = (q_1, \cdots, q_n), F_u = \frac{\partial}{\partial u}F(x,u,q)$$

Ω is a bounded domain in R^n, $n > m$, $m \leqslant p < \frac{nm}{n-m}$, $\omega(x) \in W^{1,m}(\Omega) \cap L_\infty(\partial\Omega)$. Both $W_0^{1,m}(\Omega)$ and $W^{1,m}(\Omega)$ are Sobolev spaces.

The special case of this problem, i.e. $F(x,u,q) = a_{ij}(x,u)q_iq_j + c(x)u^2$ and $\omega(x) = 0$, with the assumptions:

$$a|\xi|^2 \leqslant a_{ij}(x,u)\xi_i\xi_j \leqslant a_1|\xi|^2, \quad a > 0$$
$$-\frac{1}{2}u\partial_u a_{ij}(x,u)\xi_i\xi_j \leqslant \alpha a_{ij}(x,u)\xi_i\xi_j, \quad 0 < \alpha < 1, \partial_u = \frac{\partial}{\partial u}$$

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$$\lim_{u\to\infty} u\partial_u a_{ij}(x,u) = 0 \tag{2}$$

has been discussed in [1].

We consider multiple integrals of the form $I(u) = \int_\Omega F(x,u,Du)dx$, then we know that the variational problem:

$$I(u) = \inf_{v\in K} I(v)$$
$$K = \{u \mid u \in W^{1,m}(\Omega) \text{ and } u - \omega \in W_0^{1,m}(\Omega)\}$$

under some sufficient conditions relevant to $F(x,u,q)$, has its solution (see [2]). A similar method can be used to prove the existence of the solution to the variational problem:

$$I(u) = \inf_{v\in E} I(v), \quad E = \{u \mid u \in K, \| u \|_r = 1\} \tag{3}$$

here $\| u \|_r = \| u \|_{L_r}$.

However, when $F(x,u,q)$ grows naturally, $I(u)$ could be differentiable only when $u \in L_\infty(\Omega)$ (see [3]). Unfortunately, it is quite difficult to verify that the solution $u(x)$ to variational problem (3) is bounded, because $F(x,u,q)$ grows naturally, in addition, the problem is restricted on E.

In this paper this difficulty will be overcome. Shortly speaking, if $u(x)$ is the solution to (3), we firstly prove, for some special test functions φ, $\varphi(x) = \text{sign}u \cdot \max(|u|-k, 0)$, $I((u+t\varphi)/\| u+t\varphi \|_r)$ is differentiable about t, when $t=0$. Then, we derive:

$$\int_\Omega [F_i(x,u,Du)D_i\varphi + F_u(x,u,Du)\varphi]dx = \lambda \int_\Omega |u|^{r-2}u\varphi dx \tag{4}$$

Next this fact shows the boundedness of u, thus we learn (4) is satisfied for all $\varphi \in W_0^{1,m}(\Omega)$. The solution of the variational problem (3), consequently, is the weak solution of the problem (1). Here we can omit the condition (2), and we can also consider non-homogeneous Dirichlet problem, apart from these, $a_{ij}(x,u)$ need not to be uniformly bounded about u. We may apply our method to discuss the eigenvalue problem of general quasilinear Euler equations.

Because (2) is omitted, the proof of (4) about some special test functions φ becomes rather complex and different from the other paper. Nevertheless, we easily apply the properties of Giorgi functions in [2] to prove $u \in L_\infty(\Omega)$ finally.

Surely, we can also use this method to discuss Dirichlet problem of general quasilinear Euler equations.

Some assumptions about $F(x,u,q)$:

(i) Suppose $F(x,u,q)$ is measurable about x, and is continuously differentiable for u and q. Besides, $F(x,u,q)\geqslant 0$.

(ii) $F(x,u,q)$ is convex about q, i.e.

$$F(x,u,q)-F(x,u,\bar{q})-F_i(x,u,\bar{q})(q_i-\bar{q}_i)>0,\quad \forall\, q\not\equiv\bar{q} \tag{5}$$

(iii) There is $u_0\in E$, such that

$$I(u_0)<+\infty \tag{6}$$

(iv) Weak elliptic condition:

$$\sigma(|u|)|q|^m\leqslant F_i(x,u,q)q_i\leqslant c_1(\sigma(|u|)|q|^m+|q|^{m_1}) \tag{7}$$

here $1\leqslant m_1<m$, $\sigma(t)\geqslant c_1>0$, moreover, when $1\leqslant c_2<2$, there exists c_3, such that

$$\sigma(c_2t)\leqslant c_3\sigma(t),\quad \forall\, t>0 \tag{7'}$$

(v) A condition about F_u:

$$-c_4F_i(x,u,q)q_i\leqslant u\,F_u(x,u,q)\leqslant c_5(\sigma(|u|)|q|^m+|q|^{m_1}+|u|^s) \tag{8}$$

here $0\leqslant c_4<1$, $c_5\geqslant 0$; $m\leqslant s<\frac{nm}{n-m}$.

2. Eigenvalue Problem

Theorem 1 *If $F(x,u,q)$ satisfies (i)—(v), then the problem (1) has a weak solution u, and $u\in L_\infty(\Omega)$.*

Proof Our proof is divided into four steps.

a) According to the above conditions, a method similar to Theorem 2. 1 of Chapter 5 in [2] may show the existence of the solution $u(x)$ to the variational problem (3), furthermore

$$I(u)=\inf_{v\in E}I(v)=d\leqslant I(u_0)<+\infty$$

However, by the condition (7), we have

$$\begin{aligned}F(x,u,q)&=\int_0^1\frac{dF(x,u,tq)}{dt}dt+F(x,u,0)\\&=\int_0^1F_i(x,u,tq)q_idt+F(x,u,0)\\&\geqslant\int_0^1\sigma(|u|)\frac{t^m|q|^m}{t}dt+F(x,u,0)\end{aligned}$$

$$\geqslant \frac{1}{m}\sigma(|u|)|q|^m \tag{9}$$

Thus, the solution of the variational problem (3) satisfies:

$$\int_\Omega \sigma(|u|)|Du|^m dx \leqslant md \leqslant mI(u_0) < +\infty \tag{10}$$

b) Next, we will prove (4), with $\varphi(x) = \text{sign}\, u \cdot \max(|u|-k, 0)$. Because

$$\frac{u + t\varphi}{\|u + u\varphi\|_r} \in E$$

if $\frac{d}{dt} I\left(\frac{u+t\varphi}{\|u+t\varphi\|_r}\right)$ exists when $t=0$, then

$$\frac{d}{dt} I((u + t\varphi)/\|u + t\varphi\|_r)|_{t=0} = 0 \tag{11}$$

By virtue of the Mean-value theorem, we have

$$\lim_{t\to 0^+} (I((u + t\varphi)/\|u + t\varphi\|_r) - I(u))/t$$

$$= \lim_{t\to 0^+} \int_\Omega \frac{1}{t}\left(F\left(x, \frac{u + t\varphi}{\|u + t\varphi\|_r}, \frac{Du + tD\varphi}{\|u + t\varphi\|_r}\right) - F(x, u, Du)\right) dx$$

$$= \lim_{t\to 0^+} \int_\Omega F_i\left(x, \frac{u + t\xi\varphi}{\|u + t\xi\varphi\|_r}, \frac{Du + t\xi D\varphi}{\|u + t\xi\varphi\|_r}\right) \cdot$$

$$\cdot \|u + t\xi\varphi\|_r^{-2} \cdot [\|u + t\xi\varphi\|_r D_i\varphi - (D_i u + t\xi D_i\varphi) \cdot$$

$$\cdot \|u + t\xi\varphi\|_r^{1-r} \int_\Omega |u + t\xi\varphi|^{r-2}(u + t\xi\varphi)\varphi dx] dx$$

$$+ \lim_{t\to 0^+} \int_\Omega F_u\left(x, \frac{u + t\xi\varphi}{\|u + t\xi\varphi\|_r}, \frac{Du + t\xi D\varphi}{\|u + t\xi\varphi\|_r}\right) \cdot \|u + t\xi\varphi\|_r^{-2} \cdot$$

$$\cdot [\|u + t\xi\varphi\|_r \varphi - (u + t\xi\varphi) \cdot \|u + t\xi\varphi\|_r^{1-r} \int_\Omega |u + t\xi\varphi|^{r-2}\varphi(u + t\xi\varphi) dx] dx$$

$$= \lim_{t\to 0^+} I_1(t) + \lim_{t\to 0^+} I_2(t)$$

Now, let's prove $\varphi = \text{sign}u \cdot \max(|u|-k, 0)$ belong to $W_0^{1,m}$. We have

$$\varphi(x) = \max(u - k, 0) - \max(-u - k, 0) = \varphi_1(x) - \varphi_2(x)$$

By [2] and $\|u\|_{L_\infty(\partial\Omega)} < k$, we know that φ_1 and $\varphi_2 \in W_0^{1,m}(\Omega)$, hence $\varphi \in W_0^{1,m}$. Moreover, we learn:

$$D_i\varphi = D_iu, \text{ when } x \in A_k = \{x | x \in \Omega, |u| > k\}$$
$$D_i\varphi = 0, \text{ when } x \in \Omega \setminus A_k$$

Now set $\psi(x) = \varphi(x)/u(x)$, we have $0 \leqslant \psi(x) \leqslant 1$. When t is sufficiently small, we have $1/2 \leqslant \|u + t\xi\varphi\|_p \leqslant 2$.

By (iv) we have the following estimate:

$$\left| F_i\left(x, \frac{u + t\xi\varphi}{\|u + t\xi\varphi\|_p}, \frac{Du + t\xi D\varphi}{\|u + t\xi\varphi\|_p}\right) \frac{D_i\varphi}{\| u + t\varphi\|_p} \right|$$
$$= \left| F_i\left(x, \frac{u + t\xi\varphi}{\| u + t\xi\varphi\|_p}, \frac{Du + t\xi D\varphi}{\|u + t\xi\varphi\|_p}\right) \frac{D_iu + t\xi D_i\varphi}{\| u + t\xi\varphi\|_p} \cdot \frac{D_i\varphi}{D_iu + t\xi D_i\varphi} \cdot \frac{\|u + t\xi\varphi\|_p}{\|u + t\varphi\|_p} \right|$$
$$\leqslant C\left[\sigma(|u + t\xi\varphi|) \frac{|Du + t\xi D\varphi|^m}{\|u + t\xi\varphi\|_p^m} + \frac{|Du + t\xi D\varphi|^{m_1}}{\|u + t\xi\varphi\|_p^{m_1}}\right] \left| \frac{1}{1 + t\xi} \right|$$
$$\leqslant C[\sigma(|u|)|Du|^m + |Du|^{m_1}]$$

Consequently, by (10) and the Dominated convergence theorem we have

$$\lim_{t \to 0^+} \dot{I}_1(t) = \int_\Omega F_i(x, u, Du) D_i\varphi dx$$
$$- \int_\Omega F_i(x, u, Du) D_iudx \cdot \int_\Omega |u|^{p-2} u\varphi dx \qquad (12)$$

By (v) we have the estimate:

$$\left| F_u\left(x, \frac{u + t\xi\varphi}{\|u + t\xi\varphi\|_p}, \frac{Du + t\xi D\varphi}{\|u + t\xi\varphi\|_p}\right) \frac{\varphi}{\|u + t\varphi\|_p} \right|$$
$$= \left| F_u\left(x, \frac{u + t\xi\varphi}{\|u + t\xi\varphi\|_p}, \frac{Du + t\xi D\varphi}{\|u + t\xi\varphi\|_p}\right) \cdot \frac{u + t\xi\varphi}{\|u + t\xi\varphi\|_p} \cdot \frac{\varphi}{u + t\xi\varphi} \cdot \frac{\|u + t\xi\varphi\|_p}{\|u + t\varphi\|_p} \right|$$
$$\leqslant C\left[\sigma(|u|) \frac{|Du + t\xi D\varphi|^m}{\|u + t\xi\varphi\|_p^m} + \frac{|Du + t\xi D\varphi|^{m_1}}{\|u + t\xi\varphi\|_p^{m_1}} + \frac{|u + t\xi\varphi|^s}{\|u + t\xi\varphi\|_p^s}\right]$$
$$\leqslant C[\sigma(|u|)|Du|^m + |Du|^{m_1} + |u|^s]$$

Similarly, by (10) and the Dominated convergence theorem we have

$$\lim_{t \to 0^+} I_2(t) = \int_\Omega F_u(x, u, Du)\varphi dx - \int_\Omega F_u(x, u, Du) udx \cdot \int_\Omega |u|^{p-2} u\varphi dx \qquad (13)$$

Applying (13), (12) to (11), we derive:

$$\int_\Omega [F_i(x, u, Du) D_i\varphi + F_u(x, u, Du)\varphi] dx = \lambda \int_\Omega |u|^{p-2} u\varphi dx \qquad (14)$$

where

$$\varphi = \text{sign } u \cdot \max(|u| - k, 0)$$

and

$$\lambda = \int_{\Omega} [F_i(x,u,Du)D_iu + F_u(x,u,Du)u]dx$$

By (7), (8) and (10), we know that $|\lambda| < +\infty$.

c) Now we will prove the boundedness of u.

Putting $\varphi(x) = \text{sign } u \cdot \max(|u| - k, 0)$ into (14), we have

$$\int_{A_k} [F_i(x,u,Du)D_iu + \text{sign } u \cdot (|u| - k)F_u(x,u,Du)]dx$$
$$= \lambda \int_{A_k} |u|^{p-1}(|u| - k)dx$$

Hence,

$$\int_{A_k} \left[\frac{|u| - k + k}{|u|} F_i(x,u,Du)D_iu + \frac{|u| - k}{|u|} F_u(x,u,Du)u\right]dx$$
$$= \lambda \int_{A_k} |u|^{p-1}(|u| - k)dx$$

By means of (v),

$$(1 - c_4)\int_{A_k} \frac{|u| - k}{|u|} F_i(x,u,Du)D_iudx + \int_{A_k} \frac{k}{|u|} F_i(x,u,Du)D_iudx$$
$$\leqslant \lambda \int_{A_k} |u|^{p-1}(|u| - k)dx$$

According to (iv) and $\int_{A_k} |u|^p dx \leqslant 1$, we have

$$\int_{A_k} |Du|^m dx \leqslant C\int_{A_k} |u|^p dx \leqslant C\left(\int_{A_k} |u|^p dx\right)^{m/p}$$
$$\leqslant C\left\{\left[\int_{A_k} |u - k|^p dx\right]^{m/p} + k^m(\text{mes}A_k)^{m/p}\right\}$$

Then using Theorem 5.1 in Chapter 2 of [2], we know that $u(x)$ is bounded, since

$$\frac{m}{p} = 1 - \frac{m}{n} + \varepsilon, \quad \varepsilon = \frac{mn + mp - np}{np} > 0$$

d) By $u \in L_\infty(\Omega)$, and [3], we know that (14) holds for all $\varphi \in W_0^{1,m}(\Omega)$.

Example Put

$$F(x,u,q) = \frac{1}{m}(a_{ij}(x,u)q_iq_j)^{m/2} + c(x)|u|^m$$

where $m\geqslant 2, c(x)\geqslant 0$, moreover, $a_{ij}(x,u)$ satisfy: $a_{ij}(x,u)\in C^1(R)$ and $C(\overline{\Omega})$ for u and x.

$$C'|\xi|^2\leqslant\sigma_0(|u|)|\xi|^2\leqslant a_{ij}(x,u)\xi_i\xi_j\leqslant C''\sigma_0(|u|)|\xi|^2,\quad C'>0, C''\geqslant 1$$

$$-\frac{1}{2}u\partial_u a_{ij}(x,u)\xi_i\xi_j\leqslant \alpha a_{ij}(x,u)\xi_i\xi_j,\quad 0\leqslant\alpha<1$$

$$|u\partial_u a_{ij}(x,u)|\leqslant\sigma_0(|u|),\quad \text{for } \sigma_0, (7') \text{ is hold.}$$

Then $F(x,u,q)$ satisfies (i)$-$(v).

3. Dirichlet Problem

We consider:

$$\begin{cases}\dfrac{d}{dx_i}F_i(x,u,Du)-F_u(x,u,Du)=0\\ u-\omega\in W_0^{1,m}(\Omega)\end{cases}\tag{15}$$

Then we have

Theorem 2 *If $F(x,u,q)$ satisfies conditions* (i)$-$(v), *then the weak solutions to problem* (15) *exist. Furthermore u satisfies the maximum principle, i. e.*

$$\|u\|_{L_\infty(\Omega)}\leqslant\|\omega\|_{L_\infty(\partial\Omega)}$$

Proof All proof is similar to Theorem 1, except the second step, i. e. b), seems more simple here, since we only consider the differentiability of $I(u+t\varphi)$ at $t=0$. Apart from it, step c) is also different. In fact we have

$$(1-c_4)\int_{A_k}F_i(x,u,Du)D_iu\,dx\leqslant 0$$

Thus we have

$$(1-c_4)\int_{A_k}F_i(x,u,D\varphi)D_i\varphi dx\leqslant 0\ .$$

here $\varphi(x)=\text{sign}\,u\cdot\max(|u|-k)$. By (iv), we have

$$\int_{\Omega}|D\varphi|^{m}dx \leqslant 0.$$

For

$$k > \|u\|_{L_{\infty}(\partial\Omega)}, \quad \varphi \in W_0^{1,m}(\Omega)$$

we use Poincaré inequality

$$\int_{\Omega}|\varphi|^{m}dx \leqslant c\int_{\Omega}|D\varphi|^{m}dx.$$

Hence $\varphi \equiv 0$. Then we have

$$\|u\|_{L_{\infty}(\Omega)} \leqslant k, \quad \forall\, k > \|u\|_{L_{\infty}(\partial\Omega)} = \|\omega\|_{L_{\infty}(\partial\Omega)}$$

thus we derive $\|u\|_{L_{\infty}(\Omega)} \leqslant \|\omega\|_{L_{\infty}(\partial\Omega)}$.

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