J. Partial Differential Equations
Vol. 4, *No.* 1(1991), 45—56

THE RELATION BETWEEN THE DIFFERENTIABILITY OF SOLUTION AND LOWER-ORDER TERMS OF THE CAUCHY PROBLEM FOR A CLASS OF WEAKLY HYPERBOLIC EQUATIONS WITH SINGULAR COEFFICIENTS

Jian Suwen

(Department of Mathematics , Yunnan Normal University)

(Received Dec. 29, 1988; revised Oct. 6, 1989)

Abstract This paper discusses a class of weakly hyperbolic equations with singular coefficients. We first set up the energy inequality, and then discuss the wellposedness of the Cauchy problem by means of the energy inequality, and the relation between the differentiability of solution and lower-order terms.

Key Words Differentiability of solution; Lower-order terms; Cauchy problem; Energy inequality.

Classification 35L80.

1. Introduction

Much study has been made to the relation between the differentiability of solution and lower-order terms of the Cauchy problem for weakly hyperbolic equations. V. Y. Ivrii and V. M. Petkof have shown in [1] that the Cauchy problem for a linear partial differential operator is L_2-wellposed with a loss of one derivative, with a necessary and sufficient condition that the operator is strictly hyperbolic. M. Zeman pointed out in [2] that if a linear hyperbolic operator has smooth double characteristics, its Cauchy problem is L_2-wellposed and the solution has a loss of at most two derivatives. T. Mandai proved in [3] that the differentiability of solution of the Cauchy problem for weakly hyperbolic operator with constant multiplicity characteristics is determined wholly by the multiplicity of its characteristics. These results show that for the operators mentioned-above, the loss of the differentiability of solution is determined wholly by the multiplicity of characteristics. But for general weakly hyperbolic operators, this conclusion will not remain true. The problem studied by Qi Minyou in [4] is exactly a powerful example in this case. T. Mandai studied in [3] the relation between the differentiability of solution and lower-order terms for general hyperbolic operators with energy inequality holding. This paper discusses a class of weakly hyperbolic equations with singular coefficients. We first set up the energy inequality, and then discuss the wellposedness of the Cauchy problem by means of the energy inequality, and the relation between the differentiability of solution and lower-order terms. Our results show that for the operator discussed here, the differentiability of solution is determined wholly by the norm of the lower-order operator with singular coefficients.

2. Notations and Definitions

We are concerned with the following equation

$$(\frac{\partial}{\partial t} - it^{\rho}a(x,t;D_x))(\frac{\partial}{\partial t} + it^{\rho}a(x,t;D_x))u + \frac{\beta(x,t;D_x)}{t}\frac{\partial u}{\partial t} = f(x,t) \qquad (1)$$

where $x \in R^n, D_x = (\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_n}), a(x,t;D_x)$ is a first order pseudo-differential operator with respect to x. t is regarded as a parameter with a real symbol $a_A = a(x,t;\xi)$, which is linearly homogeneous with respect to ξ and when $\xi \neq 0, a(x,t;\xi)$ is nonzero. $\beta(x,t;D_x)$ is an operator of order zero, $f(x,t)$ is a decently smooth function of x and t, and ρ is a positive constant.

We shall consider the homogeneous Cauchy problem for equation (1), i. e.,
$\left.\frac{\partial^j u}{\partial t^j}\right|_{t=0} = 0$, but the number of the initial conditions relates to β.

Definition 1 *We say that the Cauchy problem of a partial differential operator P of order m is well-posed in the sense of Hadamard, if it exists a unique solution which depends continuously on initial data and the function $f (Pu=f)$.*

Definition 2 *We say that the Cauchy problem of a partial differential operator P of order m is L_2-wellposed with the loss of r times derivatives, if it is well-posed in the sense of Hadamard and there is the following energy inequality*

$$\sum_{|\alpha| \leq m-r} \| D^{\alpha}u \|_s \leq C \| Pu \|_s \qquad (2)$$

where $\| v(\cdot, t) \|_s$ stands for the norm equipped to the Sobolev space H_s.

Denoting $Pu = f$, (2) shows that if $f \in H_s$, then $u \in H_{s+m-r}$.

In the above definitions the loss of t-derivative and that of x-derivative are not distinguished. Since the variable t is in a more special position in our discussion, it is necessary to distinguish the loss of the x-differentiability of solution u from that of the t-differentiability. For this purpose we shall introduce the notation

$$\| u(\tau) \|_{q,s} = \left(\sum_{\substack{|\alpha| \leq q \\ j \leq s}} \int | D_x^{\alpha} D_t^j u(\tau) |^2 dx \right)^{1/2} \qquad (3)$$

Definition 3 *We say that the solution u of the Cauchy problem of a partial differential op-*

erator P of order m is L_2-wellposed with the loss of p times derivatives with respect to the x-differentiability and with the loss of q times derivatives to the t-differentiability, if it is well-posed in the sense of Hadamard and there is the following energy inequality

$$\| u(\tau) \|_{s+m-p,s_1+m-q} \leqslant C \| Pu \|_{s,s_1} \tag{4}$$

3. Energy Inequalities

Before setting up energy inequalities, we give three preparation theorems (See [5]).

Preparation theorem 1 *If*

$$ty'(t) \leqslant ay(t) + bt^K y(t) + M_1 t^M g(t) \tag{5}$$

where $g(t) \geqslant 0$ is integrable, a, b, M_1, K, M are positive constants, $M > a$, and $\lim\limits_{t\to 0} t^{-a} y(t) = 0$, then we have

$$y(t) \leqslant C_T t^a \int_0^t g(s)ds \qquad 0 \leqslant t \leqslant T \tag{6}$$

and

$$y'(t) \leqslant Ct^{a-1} \int_0^t g(s)ds \qquad 0 \leqslant t \leqslant T \tag{7}$$

Preparation theorem 2 *If $\tilde{v} \in L_2[0,T]$, and let*

$$\omega_1(t) = \int_t^\tau \tilde{v}(s)ds$$

$$\omega_2(t) = \int_t^\tau \omega_1(s)ds \omega_k(t) = \int_t^\tau \omega_{k-1}(s)ds$$

we have

$$\omega_k^2(t) \leqslant \tau^{2k-1} \int_0^\tau \tilde{v}^2(s)ds \qquad 0 \leqslant t \leqslant \tau \tag{8}$$

Preparation theorem 3 *Denote*

$$G_\tau = \{0 \leqslant t \leqslant \tau; x \in R^n\}, \quad \tau \leqslant T, \quad D_t = \frac{1}{i}\frac{\partial}{\partial t}$$

put $\tilde{v}(x,t) \in L_2(G)$, $D_t^k \widetilde{F} \in L_2(G_\tau)$, $D_t^k \tilde{v}\big|_{t=0} = 0$, $D_t^k \widetilde{F}\big|_{t=0} = 0$, $0 \leqslant k \leqslant N-1$, and let

$$[\tilde{F}]_N^2 = \int_{R^n} |D_t^N \tilde{F}(t)|^2 dx$$

then we have

$$\iint_{\Omega_\tau} \tilde{F}\tilde{v}\,dx\,dt \leqslant \delta \iint_{\Omega_\tau} \frac{\tilde{v}^2(t)}{t}\,dt\,dx + \frac{\tau^{2N+1}}{\delta}\int_0^\tau [\tilde{F}]_N^2 dt \tag{9}$$

where δ is a positive constant.

Again, let

$$v_1 = u$$

$$v_2 = \left(\frac{\partial}{\partial t} + it^\rho a(x,t;D_x)\right)v_1 \tag{10}$$

then equation (1) can be transformed into the system of equations,

$$\begin{cases} \dfrac{\partial v_1}{\partial t} = -\,it^\rho a(x,t;D_x)v_1 + v_2 \\ \dfrac{\partial v_2}{\partial t} = it^\rho a(x,t;D_x)v_2 + it^{\rho-1}\beta a(x,t;D_x)v_1 - \dfrac{\beta}{t}v_2 + f \end{cases} \tag{11}$$

$$v = {}^t(v_1, v_2), \quad A = \begin{pmatrix} 0 & -t \\ 0 & \beta \end{pmatrix}$$

$$B = \begin{pmatrix} -\,ita(x,t;D_x) & 0 \\ i\beta a(x,t;D_x) & ita(x,t;D_x) \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

then (11) can be rewritten as

$$\frac{dv}{dt} = -\frac{1}{t}Av + t^{\rho-1}Bv + F \tag{12}$$

or

$$t\frac{dv}{dt} + Av - t^\rho Bv = tF \tag{13}$$

which is just the singular hyperbolic systems of type (a,ρ) introduced by H. Tahara in [6]. Here $a=1$, and $\rho - a + 1 = \rho > 0$. Let us make the following assumption for (12):

$$\left.\frac{\partial^j v}{\partial t^j}\right|_{t=0} = 0 \qquad 0 \leqslant j \leqslant N$$
$$\left.\frac{\partial^j F}{\partial t^j}\right|_{t=0} = 0 \qquad 0 \leqslant j \leqslant N-1 \tag{14}$$

where $N = [\sup_t \| A \|]$, $[x]$ stands for the integer part of x. If we denote

$$L = \frac{d}{dt} + \frac{A}{t} - t^{\rho-1}B$$

then equation (12) can be rewritten as $Lv = F$.

Theorem 1 *If $B+B^*$ is a L_2-bounded operator under the assumption of* (14), *then for the operator L there is an energy inequality*

$$\int |v(\tau)|^2 dx \leqslant C_T \int_0^T \| Lv \|_{0,N}^2 dt \tag{15}$$

where $N = [\sup_t \| A \|]$.

Proof Multiplying both sides of (12) by $\bar{v}$, and integrating on G_τ yields

$$\iint_{G_\tau} \text{Re } v_t \cdot \bar{v} dx dt = -\int_0^\tau \frac{\text{Re}(Av, v)}{t} dt + \int_0^\tau t^{\rho-1} \text{Re}(Bv, v) dt + \int_0^\tau \text{Re}(F, v) dx$$

Applying preparation theorem 3 by (14) to the last term of the right side of the above expression, and noting the L_2-boundedness of A and $B+B^*$, we have

$$\int_0^\tau \frac{d}{dt} \| v \|^2 dt = \| v(\tau) \|^2$$

$$\leqslant \sup_t \| A \| \int_0^\tau \frac{\| v(t) \|^2}{t} dt + b\tau^\rho \int_0^\tau \frac{\| v(t) \|^2}{t} dt$$
$$+ \delta \int_0^\tau \frac{\| v(t) \|^2}{t} dt + M_1 \tau^{2N+1} \int_0^\tau [F]_N^2 dt$$

$$\| v(\tau) \|^2 \leqslant a \int_0^\tau \frac{\| v(t) \|^2}{t} dt + b\tau^\rho \int_0^\tau \frac{\| v(t) \|^2}{t} dt$$
$$+ M_1 \tau^{2N+1} \int_0^\tau [F]_N^2 dt$$

since $\| v(0) \|^2 = 0$, the integration of the right side of the above expression makes sense. If let

$$y(\tau)=\int_0^\tau \frac{\| v(t) \|^2}{t}dt$$

then

$$\tau y'(\tau) = \| v(\tau) \|^2$$

and we have

$$\tau y'(\tau) \leqslant ay(\tau) + b\tau^e y(\tau) + M_1\tau^{2N+1}g(\tau) \tag{16}$$

where $a=\sup_t \| A \| +\delta$, and δ can be taken as a sufficiently small positive number such that $[\sup_t \| A \| +\delta]=[\sup_t \| A \|]$. Since $N=[\sup_t \| A \|]$, it is obvious that

1) $2N+1>a$;
2) $\lim_{t\to 0} t^{-a}y(t)=0$.

Thus (16) satisfies the condition of preparation theorem 1 and we have

$$\| v(\tau) \|^2 \leqslant C_T\tau^{2N}\int_0^\tau g(s)ds$$

Since

$$g(s)=\int_0^s [F]_N^2 dt, \quad [F]_N^2=\int_{R^n} |D_t^N F|^2 dx,$$

this expression can be rewritten as

$$\| v(\tau) \|^2 \leqslant C_T\int_0^\tau \| F \|_{0,N}^2 dt = C_T\int_0^\tau \| Lv \|_{0,N}^2 dt$$

Because the conjugate equation of the system (12) has essentially the similar form to that of (12), the inequality with the similar form to (15) can be also proved for the conjugate equation. Denoting by L^* the conjugate operator of L, we have

$$\| v^*(\tau) \|^2 \leqslant C_T^*\int_0^\tau \| L^* v^* \|_{0,N}^2 dt \tag{15*}$$

It follows from the well-known theorem (cf. [7]) that under the assumption of Theorem 1, the system (12) has a weak solution under the additional condition (14). And by (15) the uniqueness and continuous dependency of the solution are obvious.

It follows from $u=v_1$ in transformation (10) that $\| u(\tau) \|^2 \leqslant \| v(\tau) \|^2$. Since $Lv=F, F={}^t(0,f)$, $\| F \|^2=\| f \|^2$, if denote

$$Pu = (\frac{\partial}{\partial t} - it^{\beta}a(x,t;D_x))(\frac{\partial}{\partial t} + it^{\beta}a(x,t;D_x))u + \frac{\beta}{t}\frac{\partial u}{\partial t}$$

then by (15), we have

$$\| u(\tau) \|^2 \leqslant C_T \int_0^\tau \| Pu \|_{0,N}^2 dt$$

Integrating both sides of this expression with respect to τ yields

$$\int_0^T \| u(\tau) \|^2 d\tau \leqslant C_T \int_0^T \int_0^\tau \| Pu(t) \|_{0,N}^2 dt d\tau$$

$$= C_T \int_0^T (\tau - t) \| Pu(t) \|_{0,N}^2 dt \leqslant C_T \int_0^T \| Pu(\tau) \|_{0,N}^2 d\tau$$

i. e.,

$$\| u \| \leqslant C \| Pu \|_{0,N}$$

Note the relation between u and v and condition (14). If we add the following Cauchy condition to equation (1)

$$\left.\frac{\partial^j u}{\partial t^j}\right|_{t=0} = 0 \qquad 0 \leqslant j \leqslant N+1 \tag{14_1}$$

and demand it to satisfy the condition

$$\left.\frac{\partial^j f}{\partial t^j}\right|_{t=0} = 0 \qquad 0 \leqslant j \leqslant N-1 \tag{14_2}$$

then by definition 3 we get the following

Theorem 2 *If $B+B^*$ is a L_2-bounded operator and $N=[\sup_t \| A \|]$, then under the additional conditions (14_1) and (14_2) the Cauchy problem for equation (1) is L_2-wellposed with the loss of two derivatives with respect to variable x and with the loss of $N+2$ times derivatives with respect to variable t.*

Theorems 1 and 2 are derived under the assumption the $B+B^*$ is a L_2-bounded operator. Generally speaking, this condition is rather stronge, which will be relaxed in what follows. For this purpose we shall transform equation (12) by means of the method offered by *A.* Menikoff in [8]. Let

$$K = \begin{pmatrix} t & 0 \\ -\beta/2 & 1 \end{pmatrix}$$

then

$$K^{-1}=\begin{pmatrix} 1/t & 0 \\ \beta/(2t) & 1 \end{pmatrix}$$

and

$$K^{-1}BK=\begin{pmatrix} -ita(x,t;\xi) & 0 \\ 0 & ita(x,t;\xi) \end{pmatrix}=D$$

Let G be the pseudo-differential operator with symbol $g=(KK^*)^{1/2}$, then the symbol of the operator $G^{-1}BG$ is $(KK^*)^{-1/2}BKK^*(KK^*)^{-1/2}$. Since $BKK^*=KDK^*$ is symmetric, it is easy to find out that $(KK^*)^{-1/2}BKK^*(KK^*)^{-1/2}$ is also symmetric, then the operator $G^{-1}BG$ is a symmetric operator. This fact is very important to the following discussion. Again,

$$K=\begin{pmatrix} \frac{t}{2T} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2T & 0 \\ -\frac{\beta}{2} & 1 \end{pmatrix}$$

For $\omega\in R^2$ we have

$$(KK^*\omega,\omega)=(K^*\omega,K^*\omega)=|K^*\omega|^2\geqslant Ct^2|\omega|^2$$

for $0\leqslant t\leqslant T$, because the matrix

$$\begin{pmatrix} 2T & 0 \\ -\frac{\beta}{2} & 1 \end{pmatrix}^{-1}=\begin{pmatrix} \frac{1}{2T} & 0 \\ -\frac{\beta}{4T} & 1 \end{pmatrix}$$

is bounded independent of x, t and ξ. $\begin{pmatrix} \frac{t}{2T} & 0 \\ 0 & 1 \end{pmatrix}^{-1}$ is bounded by $\frac{1}{2t}$ if $t\leqslant T$. Thus $(g\omega,\omega)\geqslant Ct|\omega|^2$. By the Lax-Nirenberg theorem [9], we can choose a lower-order term of G so that

$$\mathrm{Re}(Gu,u)\geqslant Ct|u|^2,\quad u\in C_0^\infty(R^n)^2$$

(The lower-order term depends on x- and ξ-derivatives of $g(x,t,\xi)$ and therefore is bounded by at worst $1/t$, which as will be seen does not hurt our argument.) In addition G will be bounded above. Thus $\|G_t\|\leqslant C$ and $\|G^{-1}\|\leqslant C/t$. Make the change of variables $GU=v$. Thus equation (12)

$$\frac{dv}{dt}=t^{\rho-1}Bv-\frac{1}{t}Av+F$$

becomes

$$\frac{dU}{dt}=t^{\rho-1}G^{-1}BGU+G^{-1}G_tU-\frac{1}{t}G^{-1}AGU+G^{-1}F \tag{17}$$

By the symmetry of $G^{-1}BG$, among which each element with first order operator is pure imaginary, and the properties of G, G^{-1} and G_t, equation (17) can be rewritten into

$$\frac{dU}{dt}=t^{\rho-1}HU-\frac{H_0}{t}U+\widetilde{F} \tag{17'}$$

or

$$t\frac{dU}{dt}+H_0U-t^{\rho}HU=F_1 \tag{17''}$$

where $F_1=t\widetilde{F}$, $H+H^*$ and H_0 are L_2-bounded operators and equation (17′) has the same form as (12). The results corresponding to theorem 1 and theorem 2 can be obtained from the analogous discussion with a modification that here N is $[\sup_t \| H_0 \|]$, H_0 depends on G and A, G and A depend on β, therefore H_0 depends on β. In a word, for equation (17′) if we assume

$$\left.\frac{\partial^j U}{\partial t^j}\right|_{t=0}=0 \qquad 0\leqslant j\leqslant N$$
$$\left.\frac{\partial^j F}{\partial t^j}\right|_{t=0}=0 \qquad 0\leqslant j\leqslant N-1 \tag{18}$$

where $N=[\sup_t \| H_0 \|]$. Denote

$$\widetilde{L}=\frac{d}{dt}+\frac{H_0}{t}-t^{\rho-1}H,$$

then we have the following

Theorem 3 *If condition* (18) *holds then for the operator* $\widetilde{L}$ *there is an energy inequality*

$$\int |U(\tau)|^2 dx\leqslant C_T\tau^{2N+2}\int_0^{\tau}\| \widetilde{L}U \|^2_{0,N}d\tau \tag{19}$$

where $N=[\sup_t \| H_0 \|]$.

Since $v=GU, U=G^{-1}v$, and G^{-1} is bounded and each differentiation by t adds a factor of $1/t$ to the bound (see [8]). We obtain that

$$\| v(t) \|^2 \leqslant C_T \sum_{k=1}^{N+1} \int_0^t \| D_t^k F \|^2 dt$$

of course it works under condition (18). Since

$$U = G^{-1}v, \widetilde{F} = G^{-1}F, G^{-1}$$

will add a factor of $1/t$ at each differentiation by t. In order to ensure condition (18) it needs only that

$$\left.\frac{\partial^j v}{\partial t^j}\right|_{t=0} = 0 \qquad 0 \leqslant j \leqslant N+1$$
$$\left.\frac{\partial^j F}{\partial t^j}\right|_{t=0} = 0 \qquad 0 \leqslant j \leqslant N \tag{18'}$$

holds. We obtain the following

Theorem 4 *Assume that condition* (18′) *holds, then the Cauchy problem for equation* (12) *is well-posed, and there is an energy inequality*

$$\| v(t) \|^2 \leqslant C_T \| F(t) \|^2_{0,N+1} \tag{19'}$$

From the relation between v and u, it is immediate to transform this result to the Cauchy problem for equation (1).

4. The Relation Between the Loss of the Diffferentiability of Solution and Lower-order Terms

For equation (1)

$$\left(\frac{\partial}{\partial t} - it^\rho a(x,t;D_x)\right)\left(\frac{\partial}{\partial t} + it^\rho a(x,t;D_x)\right)u + \frac{\beta}{t}\frac{\partial u}{\partial t} = f \tag{1}$$

what does Theorem 2 (Theorem 4) show? From

$$\| u(t) \| \leqslant C \| f \|_{0,N} \tag{20}$$

it follows that

1) If $f \in H_N$, then $u \in H_0$. Since $m=2$, and it can be also regarded as $u \in H_{m+N-(2+N)}$, therefore the Cauchy problem for equation (1) is L_2-wellposed with a loss of $N+2$ times derivatives. But $N=[\sup_t \|A\|](\sup_t \|H_0\|)$, H_0 depends on G and A, G and A depend on B, therefore the greater the norm of the operator β is, the greater the loss of the differentiability of u is.

2) If only the loss of the x—differentiability of solution u is concerned, then it has a loss of at most two derivatives, which coincides with the results by M. Zeman in [2] (depending only on the multiplicity 2 of characteristics). If the loss of the t-differentiability of solution u is concerned, then it has a loss of $N+2$ times derivatives, which depends on β. The greater the norm of β is, the greater the loss of the t-differentiability of u is.

3) If $\beta \equiv 0$, there is no longer singular coefficients in equation (1). This fact was discussed by M. Zeman in [2]. If t is taken as sufficiently small in Theorem 2 so that $\sup_t \|A\|=0$, then our results coincide with that by M. Zeman in [2].

The following example is a special one of equation (1):

$$P(t,x;D_t,D_x) = D_t^2 - t^2 D_x^2 - iD_x \qquad \text{for } x \in R$$

Here $\beta \equiv 0, k=1$. Since

$$P(t,x;\tau,\xi) = \tau^2 - t^2\xi^2 + \frac{1}{t}(-it\xi)$$

$$= Q_2(t,x;\tau,t\xi) + \frac{1}{t}Q_1(t,x;\tau,t\xi)$$

where

$$Q_2(t,x;\tau,t\xi) = \tau^2 - t^2\xi^2, \quad Q_1(t,x;\tau,t\xi) = -it\xi$$

Here

$$A = \begin{pmatrix} 0 & -t \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} itD_x & 0 \\ 0 & -i\tau D_x \end{pmatrix}$$

hence $B+B^*$ is L_2-bounded, so is A of course. Therefore by Theorem 2 there is an energy inequality

$$\|u\|_0 \leqslant \|Pu\|_N$$

By calculation,

$$\text{Im}\,\frac{Q_1(0,0,\pm 1)+\frac{1}{2i}(\pm 1)\partial_\tau^2 Q_2(0,0;\pm 1,1)}{\partial_\tau Q_1(0,0;\pm 1,1)}=0$$

and $q+m-1-p=1+N$, therefore the result of Theorem (3. 3) of [3] holds. (see [3])

References

[1] Ivrii V. Y. and Petkof V. M. ,*Usp: Math. Nauk.* ,27(5) (1974), 3—70.

[2] Zeman M. ,*Comm. In Partial diff. equa.* ,3(11) (1978), 1041—1052.

[3] Mandai T. ,*Publ. RIMS. Kyoto Univ.* ,18(1982), 187—240.

[4] Qi Minyou. ,*Acta Mathematica.* , 8(1958), 521—529.

[5] Jian Suwen. , *Acta Mathematica Scientia.* , 9(1) (1989), 67—75.

[6] Tahara H. , *J. Fac. Sci. Univ. Tokyo* sect ,1A 26(1979), 312—238.

[7] Nirenberg L. ,Proc. Symp. Pure Math. 16. Amer. Math. Soc. Providence 1970, 149—167.

[8] Menikoff A. , *American Journal of Math.* , 93(2)(1975), 548—558.

[9] Lax P. D. and Nirenberg L. , *Comm. Pure Appl. Math.* , 19(1966), 473—492.