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THE CAUCHY PROBLEM FOR A SPECIAL SYSTEM OF QUASILINEAR EQUATIONS

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Abstrart We have obtained in this paper the existence of weak solutions to the Cauchy problem for a special system of quasilinear equations with physical interest of the form

$$\begin{cases} \dfrac{\partial}{\partial t}(u+qz)+\dfrac{\partial}{\partial x}f(u)=0 \\ \dfrac{\partial z}{\partial t}+k\varphi(u)z=0 \end{cases}$$

for the assumed smooth function $\varphi(u)$ by employing the viscosity method and the theory of compensated compactness.

Key Words Entropy pair; weak solution.
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1. Introduction

A physical model of combustion reads

$$\begin{cases} \dfrac{\partial}{\partial t}(u(x,t)+qz(x,t))+\dfrac{\partial}{\partial x}f(u(x,t))=0 \\ \dfrac{\partial}{\partial t}z(x,t)+k\varphi(u(x,t))z(x,t)=0, \quad (x,t)\in R_+^2 \end{cases} \tag{1.1}$$

where u denotes a lumped variable representing some features of density, velocity and temperature, z represents the density of unburn fraction in fluid while k is the rate of chemical reaction and q is specific binding energy, both of them are positive constants. $f(u)$ is a smooth function in R and $\varphi(u)=1$ for $u\geqslant 0$ and $\varphi(u)=0$ for $u<0$. This model was once mentioned by Majda [1]. Teng & Ying have widely investigated this problem; in particular, the existence and uniqueness of the solution on the Riemann problem for (1.1) has been obtained on condition that $f(u)$ is strongly convex for $u>0$ and $f''>0$ for $u\leqslant 0$ [2,3]. They also established the existence of generalized solutions when $k=+\infty$ under more restrictions on $f(u)$ by the difference scheme [4,5]. We use, here, the viscosity method and the theory of compensated compactness to achieve the existence of global weak solutions of the Cauchy problem for (1.1) for smooth $\varphi(u)$ when $f''(u)\neq 0$ a. e. in R. Note that (1. 1) reduces to

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f(u(x,t)) - kq\varphi(u(x,t))z(x,t) = 0 \\ \frac{\partial}{\partial t}z(x,t) + k\varphi(u(x,t))z(x,t) = 0, \quad (x,t)\in R_+^2 \end{cases} \tag{1. 2}$$

It is easy to see that the weak solutions of Cauchy problem for (1. 1) are equivalent to those for (1. 2). Thus we only pay our attentions to (1. 2) with the initial values

$$(u(x,0),z(x,0)) = (u_0(x),z_0(x)), \quad x\in R \tag{1. 3}$$

where $u_0(x), z_0(x)$ are bounded and measurable in R. The programme is as follows: firstly we shall establish the existence and a priori estimate of the global smooth solution $(u^\varepsilon(x,t),z^\varepsilon(x,t))$ for the following parabolic equations

$$\begin{cases} u_t^\varepsilon(x,t) + f(u^\varepsilon(x,t))_x - kq\varphi(u^\varepsilon(x,t))z^\varepsilon(x,t) = \varepsilon u_{xx}^\varepsilon(x,t) \\ z_t^\varepsilon(x,t) + k\varphi(u^\varepsilon(x,t))z^\varepsilon(x,t) = \varepsilon z_{xx}^\varepsilon(x,t), \quad \varepsilon > 0, (x,t)\in R_+^2 \end{cases} \tag{1. 4}$$

with the initial values

$$(u^\varepsilon(x,t),z^\varepsilon(x,t))|_{t=0} = (u^\varepsilon(x,0),z^\varepsilon(x,0)), \quad x\in R \tag{1. 5}$$

here $u^\varepsilon(x,0), z^\varepsilon(x,0)$ are step functions which are constants $u_n^\varepsilon, z_n^\varepsilon$ in the interval $n\varepsilon\leqslant x<(n+1)\varepsilon, n\in Z$ and converge to $u_0(x), z_0(x)$ almost everywhere in R, respectively; secondly we shall find out the subsequences of smooth functions $\{u^\varepsilon(x,t)\}, \{z^\varepsilon(x,t)\}$ such that the subsequence of $\{u^\varepsilon(x,t)\}$ converges in the sense of strong topology to a function $u(x,t)$ and the subsequence of $\{z^\varepsilon(x,t)\}$ converges in the sense of weak-star topology to a function $z(x,t)$. Finally we shall show the function pair $(u(x,t), z(x,t))$ is just the weak solution of (1. 2) and (1. 3).

2. Global Smooth Solutions

To reach the existence of the global smooth solution to (1. 4) and (1. 5) we investigate the following integral equations (for simplicity we omit ε s of $u^\varepsilon(x,t)$ and $z^\varepsilon(x,t)$)

$$\begin{cases} u(x,t) = \int_{-\infty}^{\infty} u(\xi,0)G(x,t;\xi,0)d\xi + \int_0^t d\tau\int_{-\infty}^{\infty}[f(u(\xi,\tau))\frac{\partial G}{\partial\xi}(x,t;\xi,\tau) \\ \qquad + kq\varphi(u(\xi,\tau))z(\xi,\tau)G(x,t;\xi,\tau)]d\xi \\ z(x,t) = \int_{-\infty}^{\infty} z(\xi,0)G(x,t;\xi,0)d\xi - k\int_0^t d\tau\int_{-\infty}^{\infty}\varphi(u(\xi,\tau)) \\ \qquad \times z(\xi,\tau)G(x,t;\xi,\tau)d\xi \end{cases} \tag{2. 1}$$

where $G(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi\varepsilon(t-\tau)}}\exp\left\{-\frac{(x-\xi)^2}{4\varepsilon(t-\tau)}\right\}$.

It is not hard to see that (2.1) is equivalent to (1.4) and (1.5) if $u(x,t)$, $z(x,t)$ are smooth enough. Next we shall obtain the global solution of (2.1) by the consecutive method. Let

$$\begin{cases} u_0(x,t) = \int_{-\infty}^{\infty} u(\xi,0)G(x,t;\xi,0)d\xi \\ z_0(x,t) = \int_{-\infty}^{\infty} z(\xi,0)G(x,t;\xi,0)d\xi \\ u_n(x,t) = u_0(x,t) + \int_0^t d\tau \int_{-\infty}^{\infty} \Big[f(u_{n-1}(\xi,\tau)) \frac{\partial G}{\partial \xi} \\ \qquad + kq\varphi(u_{n-1}(\xi,\tau)) \times z_{n-1}(\xi,\tau)G(x,t;\xi,\tau) \Big] d\xi \\ z_n(x,t) = z_0(x,t) - k\int_0^t d\tau \int_{-\infty}^{\infty} \varphi(u_{n-1}(\xi,\tau))z_{n-1}(\xi,\tau))Gd\xi, n \geq 1 \end{cases} \tag{2.2}$$

Then we have the following lemmas.

Lemma 1 *Assume that* $f \in C^2(R), \varphi(u) \in C^1(R)$ *with* $0 \leqslant \varphi(u) \leqslant 1$. *Then* (2.1) *has a smooth solution* $(u(x,t), z(x,t))$ *in the region* $Q_1 = (0,t_1) \times (-\infty,\infty)$ *with the estimates* $\| u \|_{Q_1} \leqslant \sqrt{2}M$, $\| z \|_{Q_1} \leqslant \sqrt{2}M$ *for any bounded and measurable initials* $(u_0(x), z_0(x))$ *with* $\| u_0 \|_R \leqslant M$, $\| z_0 \|_R \leqslant M$, *where*

$$t_1 = \min\left[1, \frac{(3-2\sqrt{2})e\pi}{2(2L+kq\sqrt{e\pi})}, \frac{3-2\sqrt{2}}{2K^2}, \frac{e\pi}{2(2L+k(q+1)(1+\sqrt{2}Ms)\sqrt{e\pi})^2}\right]$$

$$L = \max_{|u| \leqslant \sqrt{2}M} |f'(u)|, \quad s = \max_{|u| \leqslant \sqrt{2}M} |\varphi'(u)|$$

Proof Without loss of generality we suppose that $f(0) = 0$. Noting that $\| u(x,0) \|_R \leqslant M$, $\| z(x,0) \|_R \leqslant M$ since $\| u_0(x) \| \leqslant M$, $\| z_0 \| \leqslant M$, we have from (2.2) that

$$|u_0(x,t)| \leqslant M\int_{-\infty}^{\infty} G(x,t;\xi,0)d\xi = M < \sqrt{2}M \quad \text{i.e. } \| u_0(x,t) \|_{Q_1} < \sqrt{2}M$$

$$|z_0(x,t)| \leqslant M\int_{-\infty}^{\infty} G(x,t;\xi,0)d\xi = M < \sqrt{2}M \quad \text{i.e. } \| z_0(x,t) \|_{Q_1} < \sqrt{2}M$$

Inductively we assume that $\| u_{n-1} \|_{Q_1} \leqslant \sqrt{2}M$ and $\| z_{n-1} \|_{Q_1} \leqslant \sqrt{2}M$. Then

$$|u_n(x,t)|_{Q_1} \leqslant M + \int_0^t d\tau \int_{-\infty}^{\infty} \left[L \| u_{n-1} \| \cdot \left|\frac{\partial G}{\partial \xi}\right| + kq \| z_{n-1} \| G \right] d\xi$$

$$\leqslant M + L \cdot \sqrt{2}M \cdot \frac{2\sqrt{t}}{\sqrt{e\pi}} + kq \cdot \sqrt{2}M \cdot t$$

$$\leqslant M+\sqrt{2}M\cdot\frac{2L+kq\sqrt{e\pi}}{\sqrt{e\pi}}\sqrt{t_1}\leqslant\sqrt{2}M$$

$$|z_n(x,t)|_{Q_1}\leqslant M+k\int_0^t d\tau\int_{-\infty}^{\infty}\|z_{n-1}\|Gd\xi\leqslant M+\sqrt{2}Mkt_1\leqslant\sqrt{2}M$$

Thus we have

$$\|u_n\|_{Q_1}\leqslant\sqrt{2}M,\ \|z_n\|_{Q_1}\leqslant\sqrt{2}M\quad\text{for } n\geqslant 0 \tag{2.3}$$

Moreover we have in Q_1

$$\begin{aligned}
&|u_n(x,t)-u_{n-1}(x,t)|\\
&\leqslant\int_0^t d\tau\int_{-\infty}^{\infty}\Big[|f(u_{n-1})-f(u_{n-2})|\left|\frac{\partial G}{\partial\xi}\right|\\
&\quad+kq|\varphi(u_{n-1})z_{n-1}-\varphi(u_{n-2})z_{n-2}|G\Big]d\xi\\
&\leqslant\int_0^t d\tau\int_{-\infty}^{\infty}\Big[L\|u_{n-1}-u_{n-2}\|\left|\frac{\partial G}{\partial\xi}\right|\\
&\quad+kq(s\|u_{n-1}-u_{n-2}\|\times\sqrt{2}M+\|z_{n-1}-z_{n-2}\|)G\Big]d\xi\\
&\leqslant\left[\frac{2L}{\sqrt{e\pi}}+\sqrt{2}kqMs+kq\right]\sqrt{t_1}(\|u_{n-1}-u_{n-2}\|+\|z_{n-1}-z_{n-2}\|)\\
&|z_n(x,t)-z_{n-1}(x,t)|\\
&\leqslant k\int_0^t d\tau\int_{-\infty}^{\infty}|\varphi(u_{n-1})z_{n-1}-\varphi(u_{n-2})z_{n-2}|Gd\xi\\
&\leqslant(\sqrt{2}kMs+k)\sqrt{t_1}(\|u_{n-1}-u_{n-2}\|+\|z_{n-1}-z_{n-2}\|)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|u_n-u_{n-1}\|+\|z_n-z_{n-1}\|\\
&\leqslant\left[\frac{2L}{\sqrt{e\pi}}+\sqrt{2}kMs(q+1)+k(q+1)\right]\cdot\\
&\quad\cdot\sqrt{t_1}\times(\|u_{n-1}-u_{n-2}\|+\|z_{n-1}-z_{n-2}\|)\\
&\leqslant\left[\left[\frac{2L}{\sqrt{e\pi}}+\sqrt{2}kM(q+1)s+k(q+1)\right]\sqrt{t_1}\right]^{n-1}\cdot\\
&\quad\cdot(\|u_1-u_0\|+\|z_1-z_0\|)\\
&\leqslant 4\sqrt{2}M\cdot\left[\left[\frac{2L}{\sqrt{e\pi}}+\sqrt{2}kMs(q+1)+k(q+1)\right]\sqrt{t_1}\right]^{n-1}\\
&\overset{\triangle}{=}a_{n-1}
\end{aligned}$$

$\sum_{n\geq 1} a_{n-1}$ is a convergent series since $(2L/\sqrt{\varepsilon\pi}+k(q+1)(1+\sqrt{2}Ms))\sqrt{t_1}<1$. It follows that $\{u_n(x,t)\}$ and $\{Z_n(x,t)\}$ converge uniformly in Q_1, i. e. there exist functions $u(x,t)$ and $z(x,t)$ such that

$$u_n(x,t) \to u(x,t), z_n(x,t) \to z(x,t) \quad \text{uniformly in } Q_1 \text{ as } n\to\infty \tag{2.4}$$

There upon $u(x,t), z(x,t)$ satisfy (2. 1) with the estimates $\| u \|_{Q_1} \leqslant \sqrt{2}M$, $\| z \|_{Q_1} \leqslant \sqrt{2}M$.

We proceed to verify that $u(x,t), z(x,t)$ are smooth enough. Observe the fact that in Q_1 the function $w(x,t)=\int_0^t d\tau \int_{-\infty}^{\infty} f(u(\xi,\tau)) \frac{\partial G}{\partial \xi}(x,t;\xi,\tau)d\xi$ is uniformly Hölder continuous with respect to x with Hölder exponent $1/3$ since $f(u(\xi,\tau))$ is bounded in Q_1. In fact we have

$$|w(x,t)|_{Q_1} \leqslant \| f \|_{Q_1} \frac{2\sqrt{t}}{\sqrt{\varepsilon\pi}} \tag{2.5}$$

For $\delta\geqslant 1$ we have

$$|w(x+\delta,t)-w(x,t)| \leqslant 2\| f \|_{Q_1} \frac{2\sqrt{t_1}}{\sqrt{\varepsilon\pi}} \leqslant 4\| f \|_{Q_1} \frac{\delta^{1/3}}{\sqrt{\varepsilon\pi}}$$

For $0<\delta<1$ letting η be a positive parameter and recalling (2. 5) we have

$$\begin{aligned}
&|w(x+\delta,t)-w(x,t)| \\
&\leqslant \int_{t-\eta}^{t} d\tau \int_{-\infty}^{\infty} \left| f(u(\xi,\tau)) \frac{\partial G}{\partial \xi}(x+\delta,t;\xi,\tau) \right| d\xi \\
&\quad + \int_{t-\eta}^{t} d\tau \int_{-\infty}^{\infty} \left| f(u(\xi,\tau)) \frac{\partial G}{\partial \xi}(x,t;\xi,\tau) \right| d\xi \\
&\quad + \int_{0}^{t-\eta} d\tau \int_{-\infty}^{\infty} |f(u(\xi,\tau))| \left| \frac{\partial G}{\partial \xi}(x+\delta,t;\xi,\tau) - \frac{\partial G}{\partial \xi}(x,t;\xi,\tau) \right| d\xi \\
&\leqslant 4\| f \|_{Q_1} \frac{\eta^{1/2}}{\sqrt{\varepsilon\pi}} + \int_0^{t-\eta} d\tau \int_{-\infty}^{\infty} \| f \|_{Q_1} d\xi \int_x^{x+\delta} \left| \frac{\partial^2 G}{\partial \xi \partial y}(y,t;\xi,\tau) \right| dy \\
&\leqslant 4\| f \|_{Q_1} \frac{\eta^{1/2}}{\sqrt{\varepsilon\pi}} + \| f \|_{Q_1} \cdot \frac{\delta}{\varepsilon} \ln \frac{t}{\eta} \\
&\leqslant 4\| f \|_{Q_1} \frac{\eta^{1/2}}{\sqrt{\varepsilon\pi}} + \frac{1}{\varepsilon} \| f \|_{Q_1} \cdot \frac{\delta}{\eta}
\end{aligned} \tag{2.6}$$

Choosing $\eta=\delta^{2/3}$, we have from (2. 6) that

$$|w(x+\delta,t)-w(x,t)|\leqslant 4\|f\|_{Q_1}\frac{1}{\sqrt{\varepsilon\pi}}\delta^{1/3}+\frac{1}{\varepsilon}\|f\|_{Q_1}\delta^{1/3}$$

provided that $\delta^{2/3}<t$. On the other hand, for $\delta^{2/3}\geqslant t$ (2.5) reduces to $|w(x+\delta,t)-w(x,t)|\leqslant 4\|f\|_{Q_1}\frac{1}{\sqrt{\varepsilon\pi}}\delta^{1/3}$. Hence

$$|w(x+\delta,t)-w(x,t)|\leqslant\sqrt{2}\,LM\left[\frac{4}{\sqrt{\varepsilon\pi}}+\frac{1}{\varepsilon}\right]\delta^{1/3}\text{ for any }\delta>0 \quad (2.7)$$

since $\|f\|_{Q_1}\leqslant\|f(u)-f(0)\|_{Q_1}\leqslant\|u\|_{Q_1}\leqslant\sqrt{2}\,LM$. Furthermore

$$\begin{aligned}\left|\int_{-\infty}^{\infty}u(\xi,0)\frac{\partial G}{\partial x}(x,t;\xi,0)d\xi\right| &=\left|\lim_{N\to\infty}\sum_{n=-N}^{N}\int_{n\varepsilon}^{(n+1)\varepsilon}u_n\frac{\partial G}{\partial\xi}(x,t;\xi,0)d\xi\right|\\ &=\left|\lim_{N\to\infty}\sum_{n=-N+1}^{N-1}(u_{n-1}-u_n)G(x,t;n\varepsilon,0)\right|\\ &\leqslant 2M\lim_{N\to\infty}\sum_{\eta=-N+1}^{N-1}G(x,t;n\varepsilon,0)\\ &=\frac{2M}{\varepsilon}\int_{-\infty}^{\infty}G(x,t;\xi,0)d\xi=\frac{2M}{\varepsilon}\end{aligned} \quad (2.8)$$

Thus (2.1), (2.8)and (2.7) reduce to

$$|u(x+\delta,t)-u(x,t)|\leqslant\text{const }(\delta+\delta^{1/3}),(x,t)\in Q_1 \quad (2.9)$$

where the constant is independent of x,t and δ.

From (2.9) we have

$$\begin{aligned}&\left|\int_0^t d\tau\int_{-\infty}^{\infty}f(u(\xi,\tau))\frac{\partial^2 G}{\partial\xi\partial x}d\xi\right|\\ &=\left|\int_0^t d\tau\int_{-\infty}^{\infty}(f(u(\xi,\tau))-f(u(x,\tau)))\frac{\partial^2 G}{\partial\xi\partial x}d\xi\right|\\ &\leqslant\text{const}\int_0^t d\tau\int_{-\infty}^{\infty}(|x-\xi|+|x-\xi|^{1/3})\left(\frac{1}{2\varepsilon(t-\tau)}+\frac{(x-\xi)^2}{4\varepsilon^2(t-\tau)^2}\right)Gd\xi\\ &\leqslant\text{const}\end{aligned}$$

Thus $\frac{\partial u}{\partial x}$ exists in Q_1, and

$$\begin{aligned}\frac{\partial u}{\partial x}(x,t)=&\int_{-\infty}^{\infty}u(\xi,0)\frac{\partial G}{\partial x}(x,t;\xi,0)d\xi\\ &+\int_0^t d\tau\int_{-\infty}^{\infty}\left[\frac{\partial f}{\partial u}(u(\xi,\tau))\cdot\frac{\partial u}{\partial\xi}(\xi,\tau)\frac{\partial G}{\partial\xi}(x,t;\xi,\tau)\right.\end{aligned}$$

$$+ kq\varphi(u(\xi,\tau))z(\xi,\tau)\frac{\partial G}{\partial x}(x,t;\xi,\tau)\Big]d\xi \quad (2.10)$$

Furthermore

$$\| \frac{\partial u}{\partial x} \|_{Q_1} \leqslant \text{const} \quad (2.11)$$

where the constant is independent of x,t.

Similarly (2.10),(2.11) imply that

$$\left| \frac{\partial u}{\partial x}(x+\delta,t) - \frac{\partial u}{\partial x}(x,t) \right|_{Q_1} \leqslant \text{const}(t^{-1/2}\delta + \delta^{1/3}) \quad (2.12)$$

the constant is independent of x,t,δ.

As a result the integrals

$$\int_0^t d\tau \int_{-\infty}^{\infty} \frac{\partial f}{\partial u}(u(\xi,\tau)) \frac{\partial u}{\partial \xi}(\xi,\tau) \frac{\partial^2 G}{\partial x \partial \xi}(x,t;\xi,\tau)d\xi$$

$$\int_0^t d\tau \int_{-\infty}^{\infty} kq\varphi(u(\xi,\tau))z(\xi,\tau) \frac{\partial^2 G}{\partial x^2}(x,t;\xi,\tau)d\xi$$

converge in Q_1. Thus $\frac{\partial^2 u}{\partial x^2}$ exists in Q_1. Noting that

$$\frac{\partial u}{\partial t} = \varepsilon u_{xx} - \frac{\partial f}{\partial u}(u(x,t))\frac{\partial u}{\partial x}(x,t) + kq\varphi(u(x,t))z(x,t)$$

we get that $\frac{\partial u}{\partial t}$ exists in Q_1. In the anologous way we can obtain that $\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial z}{\partial t}$ exist in Q_1.

Lemma 2 *Suppose that $u_0(x) \to 0, z_0(x) \to 0$ as $|x| \to \infty$. Then the smooth solution $(u(x,t),z(x,t))$ obtained in Lemma 1 satisfies $u(x,t) \to 0, z(x,t) \to 0$ as $|x| \to \infty$ uniformly in Q_1.*

Proof Since $u(\xi,0) \to 0, z(\xi,0) \to 0$ as $|\xi| \to \infty$, we have $u_0(x,t) \to 0, z_0(x,t) \to 0$ uniformly in Q_1. Inductively we can get $u_n(x,t) \to 0, z_n(x,t) \to 0$ as $|x| \to 0$ uniformly in Q_1 for any $n \geqslant 1$. Thus $u(x,t) \to 0$ and $z(x,t) \to 0$ as $|x| \to \infty$ uniformly in Q_1.

Lemma 2 implies that there exists N which is independent of x and t such that

$$|u(x,t)| \leqslant 1, |z(x,t)| \leqslant 1 \quad as\ |x| \geqslant N\ for\ t \in (0,t_1) \quad (2.13)$$

Lemma 3 *Assume that the conditions of Lemma 1 and Lemma 2 hold, then for any $T > 0$, $\| u \|_{Q_1} \leqslant M+2+kq(M+1)Te^{kT}$, $\| z \|_{Q_1} \leqslant (M+1)e^{kT}$ so long as $t_1 \leqslant T$.*

Proof Let $\eta(z) = \text{ch}(mz)$ for any large integer m. Multiplying the second equation of (1.4) by $\eta'(z)$, integrating in the region $(-N,N) \times (0,t)$ $(0<t<t_1)$, we obtain

$$\int_{-N}^{N}\eta(z(x,t))dx-\int_{-N}^{N}\eta(z(x,0))dx$$
$$=\int_0^t \varepsilon\eta'(z(x,\tau))z_x(x,\tau)\,|_{-N}^{N}d\tau-\varepsilon\int_0^t\int_{-N}^{N}\eta''(z(x,\tau))z_x^2(x,\tau)dxd\tau$$
$$-\int_0^t\int_{-N}^{N}k\varphi(u)z\eta'(z)dxd\tau \tag{2.14}$$

where N is the same as in (2.13).

From (2.13) there exists a constant A which is independent of x and t such that $|\varepsilon\eta'(z(x,\tau))z_x(x,\tau)\,|_{-N}^{N}|\leqslant Ame^m$. Letting $w(t)=\sup\limits_{x\in(-N,N)}|z(x,t)|$, recalling that $|\eta(z(x,0))|\leqslant e^{mM}$, we get from (2.14) that

$$\int_{-N}^{N}\eta(z(x,t))dx\leqslant 2Ne^{mM}+ATme^m+km\int_0^t w(\tau)d\tau\int_{-N}^{N}\eta(z)dz \tag{2.15}$$

Using Gronwall's inequality we have

$$\int_{-N}^{N}\eta(z(x,t))dx\leqslant(2Ne^{mM}+ATme^m)\exp\left\{km\int_0^t w(\tau)d\tau\right\} \tag{2.16}$$

Raising on both sides of (2.16) to the power $1/m$ and letting $m\uparrow\infty$ we obtain

$$w(t)\leqslant M+1+k\int_0^t w(\tau)d\tau \tag{2.17}$$

Using Gronwall's inequality again we get from (2.17) that

$$w(t)\leqslant e^{kt}(M+1)\leqslant(M+1)e^{kT}\quad(t<t_1\leqslant T) \tag{2.18}$$

Thus (2.13) and (2.18) lead to

$$\|z\|_{Q_1}\leqslant(M+1)e^{kT} \tag{2.19}$$

Now we turn our attentions to the estimate of $u(x,t)$. Let

$$\begin{cases}\eta(u)=\mathrm{ch}(mu)\\ \psi(u)=m\,\mathrm{sh}(mu)f(u)-m^2\int_0^u\eta(s)f(s)ds\end{cases} \tag{2.20}$$

where m is any large integer.

Multiplying the first equation of (1.4) by $\eta'(u)$, integrating on $(-N,N)\times(0,t)$ $(0<t<t_1)$ we have

$$\int_{-N}^{N}\eta(u(x,t))dx-\int_{-N}^{N}\eta(u(x,0))dx+\int_0^t\psi(u(x,\tau))\,|_{-N}^{N}d\tau$$
$$=\int_0^t\varepsilon\eta'(u)u_x\,|_{-N}^{N}d\tau-\int_0^t\int_{-N}^{N}\varepsilon\eta''(u)u_x^2dxd\tau+\int_0^t\int_{-N}^{N}kq\varphi(u)z\eta'(u)dxd\tau \quad (2.21)$$

Similarly we can reach the following estimate from (2.21), (2.20), (2.13), (2.19)

$$\|u\|_{Q_t}\leqslant M+2+kq(M+1)Te^{kT} \quad (2.22)$$

Lemma 3 plays an important role in extending local smooth solutions globally. To reach the estimates in Lemma 3 we choose the special entropy pairs $(\eta,0)$, (η,ψ) instead of employing the extremum principles [8]. From Lemma 3 we can assert that the iteration may step upwards. Actually, for any $T>0$ the length of each iteration interval keeps fixed

$$\tau_T=\min\left[1,\frac{(3-2\sqrt{2})\varepsilon\pi}{2(2L_T+kq\sqrt{\varepsilon\pi})^2},\frac{3-2\sqrt{2}}{2k^2},\frac{\varepsilon\pi}{2(2L_T+k(q+1)(1+\sqrt{2}MS_T)\sqrt{\varepsilon\pi})^2}\right]$$

where $L_T=\max\limits_{|u|\leqslant M_T}|f'(u)|$, $S_T=\max\limits_{|u|\leqslant M_T}|\varphi'(u)|$, $M_T=M+2+2kq(M+1)Te^{kT}$, since the L^∞ norms of z,u are less than $(M+1)e^{2kT}$ and M_T respectively after each iteration. Thus the smooth solution exists in the region $Q_T=(0,T)\times(-\infty,\infty)$, i.e., we have

Theorem 4 *Suppose that $u_0(x)$, $z_0(x)$ are bounded and measurable with*

$$\|u_0\|_{L^\infty(R)}\leqslant M,\ \|z_0\|_{L^\infty(R)}\leqslant M,\ \text{and } u_0(x)\to 0,\ z_0(x)\to 0 \quad \text{as } |x|\to\infty$$

If $f\in C^2(R)$ and $\varphi(u)\in C^1(R)$ with $0\leqslant\varphi(u)\leqslant 1$, then (1.4) and (1.5) has a global smooth solution $(u^\varepsilon(x,t),z^\varepsilon(x,t))$ which satisfies

$$\|z\|_{L^\infty(Q_T)}\leqslant(M+1)e^{2kT}$$
$$\|u\|_{L^\infty(Q_T)}\leqslant(M+2)+2kq(M+1)Te^{2kT}\quad \text{for any } T>0 \quad (2.23)$$

3. Global Weak Solutions

From (2.23) in Section 2 we easily gain $u^\varepsilon\in L^\infty(\Omega)$, $z^\varepsilon\in L^\infty(\Omega)$, and $\varepsilon u_x^\varepsilon\in L^\infty(\Omega)$ for any bounded open set. Multiplying the first equation of (1.4) by $u^\varepsilon(x,t)$, then integrating over Ω and using (2.23), we have

$$\sqrt{\varepsilon}\, u_x^\varepsilon \in L^2(\Omega) \tag{3.1}$$

In view of the background of functional analysis there exist subsequences of $\{u^\varepsilon(x,t)\}$, $\{z^\varepsilon(x,t)\}$ and functions $u(x,t)$, $z(x,t)$ such that

$$u^\varepsilon(x,t) \xrightarrow{*} u(x,t),\ z^\varepsilon(x,t) \xrightarrow{*} z(x,t) \quad \text{in } L^\infty(\Omega) \text{ as } \varepsilon \to 0 \tag{3.2}$$

(Without loss of generality we always regard the subsequence as the original one). Next we shall prove that there exists a subsequence of $\{u^\varepsilon(x,t)\}$ such that $u^\varepsilon(x,t) \to u(x,t)$ a. e. in Ω if $f''(u) \neq 0$ a. e. in R by using the theory of compensated compactness. To do this we introduce

Lemma 5 *Let*

$$(\eta_1(u), q_1(u)) = (u^\varepsilon - r, f(u^\varepsilon) - f(r))$$

$$(\eta_2(u^\varepsilon), q_2(u^\varepsilon)) = (f(u^\varepsilon) - f(r), \int_r^u f_s^2(s)ds)$$

where r is any real number. Then $(\frac{\partial \eta_i}{\partial t} + \frac{\partial q_i}{\partial x})(u^\varepsilon(x,t))$ *lies in a compact set of* $H_{loc}^{-1}(\Omega)$, $i = 1,2$.

Proof For any $\eta \in W_0^{1,2}(\Omega)$ we have

$$I^\varepsilon(\eta) = \iint_\Omega \left(\frac{\partial \eta_2}{\partial t} + \frac{\partial q_2}{\partial x}\right) u^\varepsilon(x,t)\eta(x,t)dxdt$$

$$= \iint_\Omega f'(u^\varepsilon)\eta(x,t)(\varepsilon u_{xx}^\varepsilon + kq\varphi(u^\varepsilon)z^\varepsilon)dxdt = I_1^\varepsilon(\eta) + I_2^\varepsilon(\eta)$$

$$I_1^\varepsilon(\eta) = -\iint_\Omega \varepsilon u_x^\varepsilon f'(u^\varepsilon)\eta dxdt$$

$$I_2^\varepsilon(\eta) = \iint_\Omega (-\varepsilon f''(u^\varepsilon)(u_x^\varepsilon)^2 + kq\varphi(u^\varepsilon)z^\varepsilon f'(u^\varepsilon))\eta(x,t)dxdt$$

(3.1) implies that

$$|I_1^\varepsilon(\eta)| \leqslant \text{const} \cdot \sqrt{\varepsilon} \, \| \sqrt{\varepsilon}\, u_x^\varepsilon \|_{L^2(\Omega)} \cdot \|\eta\|_{H_0^1(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0$$

i. e. $I_1^\varepsilon \in H_{loc}^{-1}(\Omega)$.

(3.1) and (2.23) yield that $|I_2^\varepsilon(\eta)| \leqslant \text{const} \cdot \|\eta\|_{C^0}$, i. e. $\|I_2^\varepsilon\|_{(C^0)^*} \leqslant$ const. So that I_2^ε lies in a compact set of $W^{-1,q_0}(\Omega)$, $1 < q_0 < 2$. Thus I^ε lies in a compact set of $W^{-1,q_0}(\Omega)$, $1 < q_0 < 2$. We easily get from (3.1) and (2.23) that I^ε lies in a boundary set of $W^{-1,r}(\Omega)$, $r > 1$. Therefor I^ε lies in a compact set of $H_{loc}^{-1}(\Omega)$ ([6]). Similarly we can get $\left(\frac{\partial \eta_1}{\partial t} + \frac{\partial q_1}{\partial x}\right)(u^\varepsilon(x,t))$ lies in a compact set of $H_{loc}^{-1}(\Omega)$.

Lemma 6 *If $f''(u)\neq 0$ a. e. in R and the conditions in Theorem 4 hold, then there exists a subsequence of $\{u^\varepsilon(x,t)\}$ such that the subsequence converges almost everywhere in Ω as $\varepsilon\to 0$.*

Proof From the theory of compensated compactness ([6]), we have

$$\overline{(u^\varepsilon - r)\int_r^{u^\varepsilon}(f'(s))^2ds - (f(u^\varepsilon)-f(r))^2}$$

$$=\overline{u^\varepsilon - r}\,\overline{\int_r^{u^\varepsilon}(f'(s))^2ds} - \overline{(f(u^\varepsilon)-f(r))}^2 \quad \text{a. e. in } \Omega \tag{3. 3}$$

where $\overline{g(u^\varepsilon)}$ denotes the weak limit of $g(u^\varepsilon)$ in $L^\infty(\Omega)$ as $\varepsilon\to 0$.

Referring to [7], we have from (3. 3) that

$$\overline{(u^\varepsilon - u)\int_u^{u^\varepsilon}(f'(s))^2ds - (f(u^\varepsilon)-f(u))^2 +}$$

$$+\overline{(f(u^\varepsilon)-f(u))}^2 = 0 \quad \text{a. e. in } \Omega \tag{3. 4}$$

Noting that

$$(u^\varepsilon - u)\int_u^{u^\varepsilon}(f'(s))^2ds - (f(u^\varepsilon)-f(u))^2$$

$$=\int_u^{u^\varepsilon}ds\int_u^{w}(f'(s)-f'(w))^2dw \geqslant 0$$

we have from (3. 4) that

$$\lim_{\varepsilon\to 0}\int_\Omega\left[(u^\varepsilon - u)\int_u^{u^\varepsilon}(f'(s))^2ds - (f(u^\varepsilon)-f(u))^2\right]dxdt = 0 \tag{3. 5}$$

Since $f''(u)\neq 0$, a. e. in R we see that

$$\int_u^{u^\varepsilon}ds\int_u^{w}(f'(s)-f'(w))^2dw = \int_u^{u^\varepsilon}ds\int_u^{w}dw\left[\int_w^s f''(p)dp\right]^2 \geqslant c(a) \quad \text{if } |u^\varepsilon - u| > a$$

where a is any positive number, and (3. 5) implies that

$$\lim_{\varepsilon\to 0}\text{mes}(\{(x,t)\,|\,(x,t)\in\Omega, |u^\varepsilon - u| > a\}) = 0$$

Therefore there exists a subsequence of $\{u^\varepsilon(x,t)\}$ such that

$$u^\varepsilon(x,t)\to u(x,t) \quad \text{a. e. in } \Omega \text{ as } \varepsilon\to 0 \tag{3. 6}$$

Finally we reach the main result in this paper

Theorem 7 *Suppose that $f\in C^2(R)$ with $f''(u)\neq 0$ a. e. in R, $\varphi(u)\in C^1(R)$ with 0*

$\leqslant \varphi(u) \leqslant 1$. *Then* (1. 1) *and* (1. 3) *has a weak solution for any bounded and measurable initial data* $u_0(x), z_0(x)$ *which satisfy* $u_0(x) \to 0, z_0(x) \to 0$ *as* $|x| \to \infty$.

Proof For any $\eta \in C_0^1(R)$ we have from (1. 4) that

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty (u^\varepsilon \eta_t + f(u^\varepsilon)\eta_x + kq\varphi(u^\varepsilon)z^\varepsilon \eta)dxdt \\ \quad + \int_{-\infty}^\infty u^\varepsilon(x,0)\eta(x,0)dx = \int^0 \infty \int_{-\infty}^\infty \varepsilon u_x^\varepsilon \eta_x dxdt \\ \int_0^\infty \int_{-\infty}^\infty (z^\varepsilon \eta_t - k\varphi(u^\varepsilon)z^\varepsilon \eta)dxdt + \int_{-\infty}^\infty z^\varepsilon(x,0)\eta(x,0)dx \\ = \int_0^\infty \int_{-\infty}^\infty \varepsilon z_x^\varepsilon \eta_x dxdt \end{cases} \tag{3. 7}$$

Letting $\varepsilon \to 0_+$ in (3. 7) we get

$$\begin{cases} \int_0^\infty \int_{-\infty}^\infty (u\eta_t + f(u)\eta_x + kq\varphi(u)z\eta)dxdt + \int_{-\infty}^\infty u_0(x)\eta(x,0)dx = 0 \\ \int_0^\infty \int_{-\infty}^\infty (z\eta_t - k\varphi(u)z\eta)dxdt + \int_{-\infty}^\infty z_0(x)\eta(x,0)dx = 0 \end{cases}$$

by virtue of (3. 6), (3. 2), (3. 1) and (2. 23), i. e. $(u(x,t), z(x,t))$ is the weak solution of (1. 1) and (1. 3).

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