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# EXISTENCE OF VISCOSITY SOLUTIONS OF SECOND ORDER FULLY NONLINEAR ELLIPTIC EQUATIONS①

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**Abstract** We consider the problem of existence for viscosity solutions of second order fully nonlinear elliptic partial differential equations $F(D^2u, Du, u, x)=0$. We prove existence results for viscosity solutions in $W^{1,\infty}$ under assumptions that function $F$ satisfies the natural structure conditions. We do not assume $F$ is convex.

**Key Words** Viscosity solutions; Second order fully nonlinear elliptic equations; Existence.

**Classification** 35J60.

## 1. Introduction

This paper deals with the problem of existence for solutions of second order fully nonlinear elliptic equations

$$F(D^2u, Du, u, x) = 0 \quad \text{in } \Omega \tag{1.1}$$

with Dirichlet boundary condition

$$u = g \quad \text{on } \partial\Omega \tag{1.2}$$

where $\Omega$ is a bounded domain in $R^n$ with $C^{1,1}$ boundary. Here $F$ is a real function on $\Gamma = S(n) * R^n * R * \Omega$, $S(n)$ denotes the $n * n$ real valued symmetric matrices, and $\Gamma'$ will denote set $S(n) * R^n * R$. We assume $g$ is a $C^2$ real function on $\overline{\Omega}$.

The existence results for such problems depend on both the properties of the function $F$ and the space in which solutions are taken. Using the method of continuity, we can establish existence result for classical solutions of (1.1) and (1.2) under some conditions on $F$ which include the convexity of $F$. Otherwise, some existence results of $W^{2,r}$ solutions of (1.1) and (1.2) can be obtained: for $F$ "linear at infinity" ([6]), for $F$ "close to linear" ([8]).

The definition of "viscosity solution" was introduced by [4] as a notion of weak solution for H-J equations in 1983. Under some assumptions, the uniqueness and exis-

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tence of viscosity solutions can be established. In [10] the definition of viscosity solution was extended to second order problems, and if $F$ is convex, the uniqueness of viscosity solutions was proved. In 1986, R. Jensen [9] proved uniqueness of viscosity solutions for (1. 1) and (1. 2). He does not assume $F$ is convex and not allow spatial dependence in $x$. We extended the result of [9] to the case that $F$ can be dependent on $x$ but we must assume $F$ is uniformly continuity in $x$ ([2]).

In this paper, we prove the following existence theorem.

**Theorem** *Let $F \in C^3(\Gamma)$ satisfy natural structure conditions and the following condition*

$$|F_{rx}|, |F_{rxx}|, |F_{px}|, |F_{pxx}|, |F_x|, |F_{xx}|, |F_{xxx}| \leqslant C(1 + |p|^2 + |r|)$$

*and suppose that $g \in C^2(\overline{\Omega})$. Then there exists a $W^{1,\infty}(\Omega)$ viscosity solution for problem* (1. 1) *and* (1. 2).

The method we use in the proof of the above theorem involves solving a sequence of approximate problems by the $m$-accretive operator technique, making $W^{1,\infty}$ estimates for $W^{2,p}$ $(p > 2n)$ solutions and passing to limits by means of a modification of G. Minty's Hilbert space method.

## 2. Preliminaries

We begin by some definitions.

**Definition 2. 1** *Let $u \in C(\overline{\Omega})$, the superdifferential $D^+ u(x)$ (subdifferential $D^- u(x)$) is defined as the set*

$$D^+ u(x) = \{(p, M) \in R^n * S(n) ; u(x + z) \leqslant u(x) + p * z + ((M/2) * z, z) + o(|z|^2)\}$$

$$(D^- u(x) = \{(p, M) \in R^n * S(n) ; u(x + z) \geqslant u(x) + p * z + ((M/2) * z, z) + o(|z|^2)\})$$

**Definition 2. 2** *$u \in C(\Omega)$ is a viscosity supersolution (subsolution) of* (1. 1) *if*

$$F(M, p, u(x), x) \leqslant 0 \quad \text{for all } (p, M) \in D^- u(x), x \in \Omega$$

$$(F(M, p, u(x), x) \geqslant 0 \quad \text{for all } (p, M) \in D^+ u(x), x \in \Omega)$$

*$u \in C(\overline{\Omega})$ is a viscosity solution of* (1. 1) *if it is both a viscosity supersolution and subsolution.*

For superdifferential and subdifferential, we have (see [6])

**Lemma 2. 3** *Suppose $u \in W_g^{1,p}(\Omega)$ $(p > n)$ and let $x_0 \in \Omega$. Then for any pair $(p, M)$ $\in D^- u(x_0)$ (or $D^+ u(x_0)$), there exists a sequence $\{\varphi_k\} \subset C_g^\infty(\Omega)$ such that*

(i) $\varphi_k(x_0) \to u(x_0)$, $D\varphi_k(x_0) \to p$, $D^2 \varphi_k(x_0) \to M$

(ii) $\varphi_k(x_0) - u(x_0) = \| \varphi_k - u \|_{C(\overline{\Omega})} > \varphi_k(x) - u(x)$

(*or* $u(x_0) - \varphi_k(x_0) = \| u - \varphi_k \|_{C(\overline{\Omega})} > u(x) - \varphi_k(x)$)

$$\text{for } x\in\Omega\ x\neq x_0$$

*where* $W^{1,p}_g(\Omega)=\{u\in W^{1,p}(\Omega): u|_{\partial\Omega}=g\}$, $C^\infty_g(\Omega)=\{u\in C^\infty(\Omega): u|_{\partial\Omega}=g\}$.

Let us now assume that $F$ satisfies the uniform ellipticity

(F1) $\qquad \lambda I\leqslant (F_{r_{ij}})\leqslant \Lambda I \qquad 0<\lambda\leqslant\Lambda<\infty$

**Proposition 2. 4** *Let* $\varepsilon>0$, $F_\varepsilon$ *be continuous functions on* $\Gamma$ *satisfying* (F1) *and* $u_\varepsilon$ *be viscosity solutions of*

$$F_\varepsilon(D^2u_\varepsilon, Du_\varepsilon, u_\varepsilon, x)=0 \quad \text{in } \Omega$$

*We assume that* $F_\varepsilon$ *converges on compact subsets of* $\Gamma$ *to some function* $F$ *and that* $u_\varepsilon$ *converges on compact subsets of* $\Omega$ *to some functions* $u$, *then* $u$ *is a viscosity solution of* (1. 1).

**Proof** This is Proposition I. 3 in P. L. Lions [10].

Constructing approximate solutions depends on a "quasilinearization" representation of fully nonlinear function $F$. This representation arose in L. C. Evans [6] for simpler functions.

**Lemma 2. 5** *Suppose that* (F1) *holds. Then for* $(r,p,u,x)\in\Gamma$

$$F(r,p,u,x)=\max_{(s,q,v)\in\Gamma}\ \min_{(y,k,w)\in\Gamma}\{a_{ij}(l,x)(r_{ij}-s_{ij}) + b_i(l,x)(p_i-q_i)+c(l,x)(u-v)+F(s,q,v,x)\}$$

*where*

$$a_{ij}(l,x)=\int_0^1 F_{r_{ij}}(A,B,C,x)dt,\qquad b_i(l,x)=\int_0^1 F_{p_i}(A,B,C,x)dt$$

$$c(l,x)=\int_0^1 F_z(A,B,C,x)dt,\qquad l=((s,q,v),(y,k,w))\in\Gamma*\Gamma \tag{2.1}$$

$$A=(1-t)s+ty,\qquad B=(1-t)q+tk,\qquad C=(1-t)v+tw$$

## 3. $W^{1,\infty}$-estimates for $W^{2,p}$-solutions

Letting $\Gamma_k=S(n)*R^n*[-k,k]*\Omega$ for $k\in R^+$, we adopt the following structural conditions in this section

(F2) $\qquad F_z(r,p,z,x)\leqslant-\mu_1,\qquad |F(0,0,0,x)|\leqslant\mu_2$

(F3) $\qquad |F(0,p,z,x)|\leqslant\mu_3(1+|p|^2)\qquad \forall\ (p,z,x)\in\Gamma_k$

(F4) $\qquad |p|\,|F_p(r,p,z,x)|,\ \delta F(r,p,z,x)\leqslant\mu_4(|p|^2+|r|)$

for all $(r,p,z,x)\in\Gamma_k$ with $|p|\geqslant M$

where $\mu_1,\mu_2,\mu_3=\mu_3(k),\mu_4=\mu_4(k)$ and $M$ are positive constants and

$$\delta F=F_z+\sup_{|\xi|=1}\{(|p.\xi|+1)^{-1}|F_\xi|\}$$

Consider now the Dirichlet problem of the elliptic equation

$$F(D^2u,Du,u,x)=0 \quad \text{a. e. in } \Omega \tag{3.1}$$

$$u=g \quad \text{on } \partial\Omega \tag{3.2}$$

At first, a priori estimate for solutions of (3.1) and (3.2) follows from the maximum principle of Bony ([3]) and the local maximum principle for strong solutions ([12]).

**Lemma 3.1** *Let $u\in W^{2,p}(p>n)$ satisfy (3.1) and (3.2) and suppose that (F1), (F2) and (F3) hold. Then*

$$\|u\|_{C(\overline{\Omega})}+\|u\|_{C^{\alpha}(\overline{\Omega})}\leqslant C \tag{3.3}$$

*where $\alpha>0$ depends on $n,\lambda,\Lambda,\mu_3(M_0)$ and $M_0=\|u\|_{C(\overline{\Omega})}$ and $C$ depends, in addition, on $\mu_1$, $\mu_2$ and* diam$\Omega$.

By means of well known barrier techniques the boundary gradient estimate of classical solutions may be extended to that of the $W^{2,p}(p>n)$ solutions ([13]).

**Lemma 3.2** *Let $u\in W^{2,p}(p>n)$ satisfy (3.1) and (3.2) and suppose that (F1) and (F3) hold. Then there is a constant $C$ depending only on $n,\lambda,\Lambda,\mu_3(M_0)$, $\|g\|_{C^2}$ and $\Omega$ such that*

$$|u(x)-g(y)|\leqslant C|x-y|, \quad \text{for all } x\in\overline{\Omega}, y\in\partial\Omega \tag{3.4}$$

Finally we have the following interior gradient estimates.

**Lemma 3.3** *Let $u\in W^{2,p}(\Omega)$ $(p>2n)$ satisfy (3.1) and (3.2), and suppose that (F1) and (F4) hold. Then there exist constants $C$ and $\theta$ depending only on $n,\lambda,\Lambda$, and $\mu_4(M_0)$ such that if $B=B_{2R}(y)$ is any ball in $\Omega$ and $\alpha=\operatorname{osc}_{B_R}u\leqslant\theta$, then*

$$|Du(y)|\leqslant C\alpha^{1/2}/R+24M \tag{3.5}$$

*Further the estimate (3.5) remains valid if $B$ is replaced by $B\cap\Omega$ for any ball $B=B_{2R}(y)$ with $y\in\Omega$ and $|Du|\leqslant M$ on $\partial\Omega\cap B$.*

**Proof** Without loss of generality we can take $y=0$. For $x\in B_R(0)$, $\xi\in B_1(0)$, and $h\in(0,R)$ set

$$\begin{aligned}&\eta_1(x)=(1-|x|^2/R^2)^2, \eta_2(\xi)=(1-|\xi|^2)^2, \eta=\eta_1\eta_2\\ &v=v_h(x,\xi)=[u(x+h\xi)-u(x)]/h, \overline{w}=\eta_1v^2\end{aligned} \tag{3.6}$$

Taking the difference quotients for (3.1) we obtain

$$\begin{aligned}a_{ij}D_{ij}\overline{w}+B_iD_i\overline{w}=&v^2[a_{ij}D_{ij}\eta_1+b_iD_i\eta_1-\frac{2}{\eta_1}a_{ij}D_i\eta_1D_j\eta_1]\\ &+2\eta_1(a_{ij}D_ivD_jv-cv^2-fv)\end{aligned} \tag{3.7}$$

where

$$a_{ij}=\int_0^l F_{r_{ij}}(D^2u_\theta, Du_\theta, u_\theta, x+\theta h\xi)d\theta$$

$$b_i=\int_0^l F_{p_i}d\theta,\quad c=\int_0^l F_z d\theta,\quad B_i=b_i-\frac{2}{\eta_1}a_{ij}D_j\eta_1$$

$$u_\theta=\theta u(x+h\xi)+(1-\theta)u(x)$$

By the Sobolev imbedding theorem we know $u\in C^{1,\beta}(\overline{\Omega})$ with $\beta=1-n/p$. By using structure condition (F1) and (F4), it follows that, on the set $S_0=\{(x,\xi)\in B_R * B_1: |Du|\geqslant 2M\}$ for $h<(M/\|u\|_{C^{1,\beta}(\overline{\Omega})})^{1/\beta}$

$$a_{ij}D_{ij}\overline{w}+B_iD_i\overline{w}\geqslant 2\lambda\eta_1|Dv|^2-16\mu_4\eta_1(1+v^2)(|Du|^2+|D^2u|)$$
$$-v^2[36n\Lambda/R^2+4\mu_4\eta_1^{1/2}/R(|Du|+|D^2u|/|Du|)-\varepsilon_3 \tag{3.8}$$

where

$$\varepsilon_3=\varepsilon_3(h)=4\mu_4(v^2+2+\varepsilon_2)(\varepsilon_1+\varepsilon_2)$$
$$+4\mu_4v^2/R[\varepsilon_1/M+\varepsilon_2(1+|D^2u|/M)]$$
$$\varepsilon_1=\varepsilon_1(h)=|D^2u(x+h\xi)-D^2u(x)|$$
$$\varepsilon_2=|Du(x+h\xi)-Du(x)|^2+||Du(x+h\xi)|^2-|Du(x)|^2|$$
$$+|Du(x+h\xi)-Du(x)|+||v(x)|^2-|D_\xi u(x)|^2|+|v(x)-D_\xi u(x)|$$

Now we set

$$M_R=\sup_{B_R}u,\qquad M_h=\sup_{B_R\times B_1}\eta v^2,\qquad \alpha=\operatorname*{osc}_{B_R}u$$
$$u^*=\exp((u-M_R)/\alpha),\qquad w=w_h(x,\xi)=\eta v^2+\alpha^{1/2}M_hu^*$$
$$S=\{(x,\xi)\in S_0: w\geqslant\frac{3}{4}\sup_{B_R\times B_1}w\}$$

By the inequality $u^*\leqslant 1$, we infer that $3M_h/4\leqslant w\leqslant\eta v^2+\alpha^{1/2}M_h$ on the set $S$, that is $\eta v^2\geqslant(3/4-\alpha^{1/2})M_h$. Take $\theta_1=(1/16)^2$, then for $\alpha\leqslant\theta_1$ we have $\eta v^2\geqslant M_h/2$ on set $S$. Furthermore using $u^*\geqslant e^{-1}\geqslant 1/3$, we know that if $\alpha\leqslant\theta_1$ and $M_h\geqslant 2$ then

$$a_{ij}D_{ij}w+B_iD_iw\geqslant 2\lambda\eta|Dv|^2-C_1\frac{M_h}{\alpha^{1/2}}|D^2u|+\frac{\lambda}{3\alpha^{3/2}}M_h|Du|^2$$
$$-v^2[32\mu_4\eta|Du|^2+36n\Lambda R^{-2}+4\mu\eta^{1/2}R^{-1}|Du|]$$
$$-\frac{M_h}{\alpha^{1/2}}[\mu_3|Du|^2+\frac{8n\Lambda}{\eta_1^{1/2}R}|Du|]-\varepsilon_4 \tag{3.9}$$

where $C_1$ is a constant depending on $n,\lambda,\Lambda$ and $\mu_3$ and

$$\varepsilon_4 = \varepsilon_3 - M_k u^* \alpha^{-1/2}[(|Du| + |D^2u|/M)\varepsilon_2 + 2\varepsilon_1] + 2\mu_3 M_k^{1/2} |D^2u| (RM)^{-1}\varepsilon_2$$

Writting $D_{i+n}$ for $\partial/\partial\xi_i, i=1,2,\cdots,n,$ and let $\sigma,\tau$ be indices running from 1 to 2n, setting

$$A_{\sigma\tau} = \begin{cases} a_{ij} & \sigma = i, \tau = j \\ C_1(2\alpha^{1/2})^{-1} v\mathrm{sgn}\{D_{ij}u(x+h\xi)\} & \sigma = n+i, \tau = j \text{ or } \sigma = i, \tau = n+j \\ C_1^2 v^2 (\lambda\alpha)^{-1}\delta_{ij} & \sigma = n+i, \tau = n+j \end{cases}$$

and $B_\sigma = 0, \sigma = n+i, i=1,2,\cdots,n,$ then we obtain by calculation and estimate, if $\alpha \leqslant \theta_2$

$$A_{\sigma\tau}D_{\sigma\tau}w + B_\sigma D_\sigma w \geqslant |Du|^2[\lambda(6\alpha^{3/2})^{-1}M_k - C_2(\alpha^{1/2}R)^{-1}M_k^{1/2} - 36n\Lambda/R^2] - \varepsilon(h)$$
$$\text{for a. e. } (x,\xi) \in S_k \tag{3.10}$$

where

$$\begin{aligned} C_2 &= 4\mu_4 + 16n\Lambda + 24C_1 \\ S_k &= \{(x,\xi) \in S : |\xi|^2 \leqslant k/(k+1)\}, k = 1,2,\cdots \\ \theta_2 &= \lambda^2[6(33\mu_4 + 4C_1\lambda^{-1} + 16C_1^2(k+1)^2\lambda^{-1} \\ &\quad + (4n+16)C_1^2(k+1)\lambda^{-1})]^{-2} \\ \varepsilon(h) &= A\varepsilon_1(h) + B\varepsilon_2(h) + Ch|\triangle u(x+h\xi)| \end{aligned}$$

$A,B,C$ are all independent of $h$. Setting

$$C_3 = 36(36n\Lambda + C_2)^2/\lambda^2, \qquad \theta_k = \min\{\theta_1,\theta_2\}$$
$$C = \max\{C_3, 1\}, \qquad h_0 = \min\{R/2, (M/\|u\|_{C^{1,\beta}(\bar{\Omega})})^{1/\beta}\}$$

Then we have from (3.10) for $\alpha \leqslant \theta_k, h \leqslant h_0$ and $M_k \geqslant C\alpha/R^2 + 2$

$$A_{\sigma\tau}D_{\sigma\tau}w + B_\sigma D_\sigma w \geqslant -\varepsilon(h) \quad \text{for a. e. } (x,\xi) \in S_k \tag{3.11}$$

Using Alexsandrov maximum principle we obtain from (3.11)

$$\sup_{S_k} w \leqslant \sup_{\partial S_k} w + B(h)\left[\int_{S_k} |\varepsilon(h)/D^*|^{2n} dx d\xi\right]^{1/(2n)} \tag{3.12}$$

where $D^* = \det(A_{\sigma\tau})$ and $B(h)$ depending only on $n$, $\mathrm{diam}(S_k)$ and $\|B_i/D^*\|_{L^{2n}}$. By estimation we obtain from the definition of $\varepsilon(h), B_i$ and $A_{\sigma\tau}$

$$\delta_k = B(h)\left[\iint_{S_k} |\varepsilon(h)/D^* |^{2n} dx d\xi\right]^{1/(2n)} \to 0 \quad \text{as } h \to 0 \qquad (3.13)$$

provided $a \leqslant \theta_k, M_h \geqslant C^a/R^2+2$, uniformly with respect to $k$. Furthermore we have by the condition $u \in C^{1,\beta}(\overline{\Omega})$ $(\beta = 1-n/p > 1/2), v_k(x,\xi) \to D_\xi u(x)$ $(h \to 0)$ uniformly with respect to $(x,\xi)$, and $M_k \to \sup\limits_{B_R \times B_1} \eta |(D_\xi u(x)|^2 = M_1$.

Setting $A_k = \{(x,\xi) \in B_R * B_1 : |Du(x)| \geqslant 2M, |\xi|^2 \leqslant k/(k+1)\}$ we have

$$\sup_{A_k} w \leqslant \max\{\sup_{A_k \setminus S_k} w, \sup_{S_k} w\}$$

Case 1 $\sup\limits_{A_k} w \leqslant \sup\limits_{A_k \setminus S_k} w$. This implies $\sup w \leqslant \frac{3}{4} \sup\limits_{B_R \times B_1} w$. Setting

$$B_k = \{(x,\xi) \in B_R * B_1 : |\xi|^2 \leqslant k/(k+1)\}$$

and letting $h \to 0$, we thus obtain

$$\sup_{B_1} \eta |D_\xi u(x)|^2 \leqslant 4M^2 + \frac{3}{4}(1+a^{1/2})(1+\frac{1}{k})^3 \sup_{B_1} \eta |D_\xi u|^2$$

Using inequality $a \leqslant \theta_1$ and taking $k = k_0 = [(20/17)^{1/3} - 1]^{-1}$ we have by calculation

$$\sup_{B_k} \eta |D_\xi u(x)|^2 \leqslant 64M^2$$

Consequently $|Du(0)| \leqslant 24M$.

Case 2 $\sup\limits_{A_k} w \leqslant \sup\limits_{S_k} w$. W. l. o. g. we can assume $M_h \geqslant C^a/R^2+2$. By the inequality (3.12) we have

$$\sup_{A_k} w \leqslant \sup_{\partial S_k} w + \delta_h$$

According to the structure of set $\partial S_k$ and relation (3.13) we obtain by the same argument with the case 1

$$|Du(0)| \leqslant 24M$$

By combining the case 1 wtih case 2, it follows that, for $a \leqslant \theta = \theta_{k_0}$ and $C = \max\{C_3(k_0), 1\}$

$$|Du(0)| \leqslant Ca^{1/2}/R + 24M$$

Now we obtain

**Theorem 3. 4** *Let $u \in W^{2,p}(\Omega)$ $(p > 2n)$ satisfy (1. 1) and (1. 2) and suppose that structure conditions (F1)—(F4) hold. Then there is positive constant C depending only on $n$, $\lambda$, $\Lambda$, $\mu_1$, $\mu_2$, $\mu_3$, $\mu_4$ and* diam$\Omega$ *such that*

$$\| u \|_{W^{1,\infty}(\Omega)} \leqslant C$$

## 4. Existence Results

Let us assume that $F \in C^3(\Gamma)$ and augment conditions (F1)—(F4) by

(F5) $|F_{rz}|, |F_{rxx}|, |F_{px}|, |F_{pxx}|, |F_z|, |F_{zz}|, |F_{zzz}|$
$\leqslant \mu_5(1 + |p|^2 + |r|)$ for all $(r,p,z,x) \in \Gamma_{M_0}$

where $\mu_5$ is a positive constant.

The purpose of this section and the fundamental result of this paper is the following existence theorem.

**Theorem 4. 1** *Suppose that F satisfies structure conditions (F1)—(F5). Then there exists a $u \in W^{1,\infty}(\Omega)$ solving (1. 1) and (1. 2).*

**Proof** (See the appendix of [6] for accretive and $m$-accretive operator theory).

Let us define for $(r,p,z,x) \in \Gamma$

$$H(r,p,z,x) = F(r,p,z,x) - \frac{\lambda}{2}\,\text{trace}(r) \tag{4. 1}$$

It is clear that $H$ satisfies (F1)—(F5) (elliptic constant is changed) also. Define again

$$\overline{H}(r,p,z,x) = -H(-r, -p, -z, x) \tag{4. 2}$$

then $\overline{H}$ satisfies (F1)—(F5) and (3. 1) can be rewritten into

$$-\frac{\lambda}{2}\triangle u + \overline{H}(-D^2u, -Du, -u, x) = 0 \qquad \text{a. e. in } \Omega \tag{4. 3}$$

Define $a_{ij}, b_i, c$ by (2. 1) with $F = \overline{H}$ and

$$f(l,x) = \overline{H}(s,q,v,x) - a_{ij}(l,x)S_{ij} - b_i(l,x)q_i - c(l,x)v$$

By Lemma 2. 5 (applied to $\overline{H}$ instead of $F$)

$$\overline{H}(-D^2u, -Du, -u, x) =$$

$$= \max_{(s,q,v)\in\Gamma'} \min_{(y,k,w)\in\Gamma'} \{-a_{ij}(l,x)D_{ij}u - b_i(l,x)D_iu - c(l,x)u + f(l,x)\} \quad (4.4)$$

Hypothesis (F1) and (4. 2) imply

$$\frac{\lambda}{2}I \leqslant (a_{ij}(l,x)) \leqslant (\Lambda - \frac{\lambda}{2})I$$

Let

$$L_lu = -a_{ij}(l,x)D_{ij}u - b_i(l,x)D_iu - c(l,x)u$$

for $u\in D(L_l)=\{u\in W^{2,p}(\Omega)\cap W_g^{1,p}(\Omega)\ (n\leqslant p<\infty), L_lu\in C(\overline{\Omega})\}$, Then $L_l$ is the uniformly elliptic linear operator for each $l\in\Gamma\times\Gamma$ and

$$\overline{H}(-D^2u, -Du, u, x) = \max_{(s,q,v)\in\Gamma'} \min_{(y,k,w)\in\Gamma'} \{L_lu + f(l,x)\} \quad (4.5)$$

is the max-min of affine elliptic operator.

According to standard elliptic theory the operator $L_l: D(L_l)\subset C(\overline{\Omega})\to C(\overline{\Omega})$ is $m$-accretive in $C(\overline{\Omega})$ ([5]). Fix $\theta>0$, let $J_\theta(l)$ denote the $\theta$-th resolvent of $L_l$ and $A_\theta(l)$ denote $\theta$-th Yosida approximation of $L_l$. We know that each $A_\theta(l)$ is a defined everywhere, Lipschitz, accretive operator on $C(\overline{\Omega})$ ([6]).

Next let $\varepsilon>0$ and select some smooth function $\beta=\beta_\varepsilon(x)$ such that

$$\begin{aligned} &\beta(x) = x, \quad |x| \leqslant 1/\varepsilon - 1 \\ &\beta(x) = 1/\varepsilon, \quad |x| \geqslant 1/\varepsilon \\ &0 \leqslant \beta \leqslant 1 \end{aligned} \quad (4.6)$$

Now we define the nonlinear operator

$$B_\theta(u) = \beta\{\max_{(s,q,v)\in\Gamma'} \min_{(y,k,w)\in\Gamma'} (A_\theta(l)u + f(l,x))\} \quad (4.7)$$

Since $f(l,x)$ is uniformly continuous with respect to $x$ for $|s|, |q|, |v|, |y|, |k|, |w| \leqslant (1/\theta)^{1/5}$, $B_\theta$ is defined on all of $C(\overline{\Omega})$. Furthermore $B_\theta$ is Lipschitz (since each $A_\theta(l)$ is Lipschitz with the same constant $2/\theta$) and $B_\theta$ is accretive on $C(\overline{\Omega})$ (see [6]).

Hence the Perturbation Lemma (applied to $A=-\frac{\lambda}{2}\triangle$ and $B=B_\theta$) from Evans [6] implies the existence of unique $u_\theta=u_\theta(\varepsilon)\in W^{2,p}(\Omega)\cap W_g^{1,p}(\Omega)$ solving

$$\theta u_\theta - \frac{\lambda}{2}\triangle u_\theta + B_\theta(u_\theta) = 0 \qquad \text{a. e. in } \Omega \quad (4.8)$$

Since $|B_\theta|\leqslant \sup|\beta_\varepsilon|\leqslant 1/\varepsilon$, we have

$$\| u_\theta \|_{W^{2,p}} \leqslant C(p,\varepsilon) \quad \text{for each } p \geqslant n \tag{4.9}$$

the constant depends only on $p$ and $\varepsilon$.

Owing to (4.9) there exists a sequence $\theta_k \to 0$ and a function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &u_{\theta_k} \to u \quad \text{weakly in } W^{2,p}(\Omega)\ (p \geqslant n) \\ &Du_{\theta_k} \to Du \quad \text{uniformly on } \overline{\Omega} \\ &u_{\theta_k} \to u \quad \text{uniformly on } \overline{\Omega} \end{aligned} \tag{4.10}$$

By calculation we have for some given $\varphi \in C_0^\infty(\Omega)$ with $\| \varphi \|_{C^2(\overline{\Omega})} \leqslant (1/\theta)^{1/5}$

$$\| A_\theta(l)\varphi - L_l\varphi \|_{C(\overline{\Omega})} \leqslant \theta \| (L_l)^2\varphi \|_{C(\overline{\Omega})}$$

$$\max_{|s|,|q|,|v| \leqslant \theta^{-1/5}} \ \min_{|y|,|k|,|w| \leqslant \theta^{-1/5}} \{L_l\varphi + f(l,x)\} = \overline{H}(-D^2\varphi, -D\varphi, -\varphi, x)$$

Therefore by (4.7) and structure condition (F5) we obtain for $\theta$ small enough

$$\begin{aligned} &\| B_\theta(\varphi) - \beta(\overline{H}(-D^2\varphi, -D\varphi, -\varphi, x)) \|_{C(\overline{\Omega})} \\ &\leqslant \theta C \max_{\substack{|s|,|q|,|v| \leqslant \theta^{-1/5} \\ |y|,|k|,|w| \leqslant \theta^{-1/5}}} \{(1 + |s|^2 + |y|^2 + |k|^4 + |q|^4) \| \varphi \|_{C^4(\overline{\Omega})}\} \leqslant \overline{C}\theta^{1/5} \end{aligned}$$

Consequently

$$B_\theta(\varphi) \to \beta(\overline{H}(-D^2\varphi, -D\varphi, -\varphi, x)) \tag{4.11}$$

uniformly on $\overline{\Omega}$ as $\theta \to 0$.

Now according to the accretiveness of $-\frac{\lambda}{2}\triangle + B_\theta$ we have

$$[u_\theta - \varphi, -\frac{\lambda}{2}\triangle u_\theta + B_\theta(u_\theta) - (-\frac{\lambda}{2}\triangle\varphi + B_\theta(\varphi))]_+ \geqslant 0$$

for any $\varphi \in C_0^\infty(\Omega)$; (4.8) then implies

$$[u_\theta - \varphi, -\theta u_\theta + \frac{\lambda}{2}\triangle\varphi - B_\theta(\varphi)]_+ \geqslant 0$$

Let $\theta = \theta_k \to 0$, by relations (4.10) and (4.11) and the upper semicontinuity of $[\cdot, \cdot]_+$ with respect to uniform convergence

$$[u-\phi, \frac{\lambda}{2}\triangle\varphi-\beta(\overline{H}(-D^2\varphi, -D\varphi, -\varphi, x))]_+ \geqslant 0, \text{for any } \varphi\in C_?^\infty(\Omega) \tag{4.12}$$

Fix a point $x_0\in\Omega$ and a pair $(p, M)\in D^- u(x_0)$. By the characterization of $[\cdot, \cdot]_+$ ([6]) and Lemma 2.3 there exists a sequence $\varphi_k\in C_?^\infty(\Omega)$ such that

$$\frac{\lambda}{2}\triangle\varphi_k(x_0)-\beta(\overline{H}(-D^2\varphi_k(x_0), -D\varphi_k(x_0), -\varphi_k(x_0), x_0))\leqslant 0 \tag{4.13}$$

Let $k\to\infty$ and use Lemma 2.3 to find

$$\frac{\lambda}{2}M-\beta(\overline{H}(-M, -p, -u(x_0), x_0))\leqslant 0 \tag{4.14}$$

This inequality implies that $u$ is a viscosity supersolution of the following equation

$$\frac{\lambda}{2}\triangle u-\beta(\overline{H}(-D^2u, -Du, -u, x)=0 \tag{4.15}$$

In the same way we can prove $u$ is a viscosity subsolution of (4.15), so that $u$ is a viscosity solution of (4.15).

We can rewrite (4.15) into

$$G_\varepsilon[u]=\frac{\lambda}{2}\triangle u-\beta_\varepsilon(-F(D^2u, Du, u, x)+\frac{\lambda}{2}\triangle u)=0 \tag{4.16}$$

Next we remove the $\beta_\varepsilon$ from equation (4.16) and denote by $u_\varepsilon$ the viscosity solution of (4.16) constructed above.

According to Theorem 3.4 (applied to $G_\varepsilon$ instead of $F$) we have

$$\| u_\varepsilon \|_{W^{1,\infty}(\Omega)}\leqslant C \tag{4.17}$$

where $C$ is independent of $\varepsilon$.

Estimate (4.17) implies the existence of a subsequence (also denoted by $u_\varepsilon$) and a function $u\in W^{1,\infty}(\Omega)\cap W_?^{1,p}(\Omega)$ such that

$$u_\varepsilon\rightharpoonup u \quad \text{weakly in } W^{1,\infty}$$
$$u_\varepsilon\to u \quad \text{uniformly on } \overline{\Omega}$$

Furthermore by (4.6) we see that $G_\varepsilon$ converges on compact subsets of $\Gamma$ to $F$. So according to Lemma 2.4 we know that $u$ is the viscosity solution of (1.1) and (1.2).

This completes the proof of Theorem 4.1.

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