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FORMATION OF SINGULARITIES OF SOLUTIONS FOR CAUCHY PROBLEM OF QUASILINEAR HYPERBOLIC SYSTEMS WITH DISSIPATIVE TERMS[1]

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Abstract In this paper, for a class of 2×2 quasilinear hyperbolic systems, we get existence theorems of the global smooth solutions of its Cauchy problem, under a certain hypotheses. In addition, for two concrete quasilinear hyperbolic systems, we study the formation of the singularities of the C^1-solution to its Cauchy problem.
Key Words Quasilinear hyperbolic systems; Cauchy problem; global smooth solution; singularity.
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1. Introduction

For the first order quasilinear hyperbolic systems

$$\begin{aligned} &u_t + \sigma(v)_x + 2\alpha u = 0, \quad \alpha > 0 \\ &v_t - u_x = 0 \end{aligned} \tag{1.1}$$

the existence and nonexistence of global smooth solutions of its Cauchy problem or initial-boundary problem had been studied by many scholars (see [1, 2, 3, 4, 5])

Suppose that there exists a constant $R > 0$, such that

$$\sigma'(v) < 0, \quad \forall |v| < R, \quad \sigma(v) \in C^2(|v| < R) \tag{1.2}$$

For (1.1), the initial datum are given by

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) \tag{1.3}$$

Let z, w be the Riemann invariants, i.e.,

$$z = u + \varphi(v), \qquad w = u - \varphi(v) \tag{1.4}$$

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where $\varphi(v) = \int_0^v \sqrt{-\sigma'(v)}dv$. Suppose that $z_0(x)$, $w_0(x)$, which are determined by the initial datum $u_0(x), v_0(x)$, satisfy

$$z_0(x), w_0(x) \in C^1(R)$$

Nishida had proved that Cauchy problem (1.1), (1.3) admits a global smooth solution, if the C^1-norm of $w_0(x)$, $z_0(x)$ is sufficiently small (see [1]).

Under the hypothesis of "smallness" to the C^1-norm of the initial datum, Li Tatsiens' have spread Nishida's result of existence into Cauchy problem of the general $n \times n$ systems with dissipative terms of the diagonal dominant ([6, 7, 8]):

$$U_t + F(U)_x + G(U) = 0 \tag{1.5}$$

However, for the nonlinear vector-function $G(U)$ and weakly diagonal dominant $A = B(0)\nabla G(0)B^{-1}(0)$, whether Cauchy problem to (1.5) admits a global smooth solution, Li Tatsiens' have not been studying this case, where $B(U)$ is the matrix of the eigenvector to (1.5), and det $B(U) \neq 0, B^{-1}(U)$ is a inverse matrix of $B(U)$. In this paper, for the case of that, we show that Cauchy problem of systems does not admit a global smooth solution, even if the smallness of C^1-norm or C^0-norm of the initial datum is ensured.

Under the hypothesis of the monotonic initial datum, Li Caizhongs' have shown that Cauchy problem of (1.1) admits a global smooth solution, if the oscillation of the initial datum is small, and Cauchy problem of (1.1) has a C^1-solution for only a finite time ([5]).

In view of the weakly diagonal dominant in (1.1), we compare Li Tatsiens' result of the strictly diagonal dominant, a query is easily arised: whether the singular result can be avoided in [5], if the dissipative terms are strengthened. In this paper, we show that the singular result is not yet avoided, even if the dissipative terms are strengthened.

In addition, we spread the existence result gotten by Li Caizhongs' to Cauchy problem of the general 2×2 systems:

$$\begin{aligned} &u_t + \lambda(u, v)u_x + f(u, v) = 0 \\ &v_t + \mu(u, v)v_x + g(u, v) = 0 \end{aligned} \tag{1.6}$$

2. The Existence Theorems of the Global Smooth Solutions

Consider Cauchy problem of the quasilinear equations as

$$\begin{aligned} &u_t + \lambda(u, v)u_x + f(u, v) = 0 \\ &v_t + \mu(u, v)v_x + g(u, v) = 0 \end{aligned} \tag{2.1}$$

$$t = 0: \qquad u = u_0(x), v = v_0(x) \tag{2.2}$$

Suppose that

1° $\lambda(u,v), \mu(u,v) \in C^2, |\lambda(u,v) - \mu(u,v)| \geq \sigma > 0, \sigma = \text{const.}, \lambda_u \neq 0, \mu_v \neq 0$;

2° $f(u,v), g(u,v) \in C^2, f(0,0) = 0 = g(0,0), f_u(0,0) > 0, g_v(0,0) > 0$;

3° $u_0(x), v_0(x) \in C^1$, and $\|u_0(x)\|_{C^1} + \|v_0(x)\|_{C^1}$ is bounded,

$$\text{sign } (\lambda_u u_0'(x)) \geq 0, \qquad \text{sign } (\mu_v v_0'(x)) \geq 0, \qquad \forall x \in R, \qquad t \geq 0;$$

(H) As $\varepsilon_0 = \|u_0(x)\|_{C^0} + \|v_0(x)\|_{C^0}$ is sufficiently small, for any C^1-solution of Cauchy problem (2.1), (2.2), there exists a constant $D_0 > 0$, such that

$$\|u(x,t)\|_{C^0} + \|v(x,t)\|_{C^0} \leq D_0\varepsilon_0 \tag{2.3}$$

Theorem 2.1 *If ε_0 is sufficiently small, then* (2.1), (2.2) *possesses a unique global smooth solution on $t \geq 0$, under the hypotheses* 1°, 2°, 3° *and* (H).

Proof By the hypotheses in the Theorem 2.1, suppose that $\lambda_u(u,v) > 0, u_0'(x) \geq 0$, and there exist constants $\varepsilon_0^* \leq \varepsilon_0$, $\varepsilon^* \leq D_0\varepsilon_0$, such that

$$\|u_0(x)\|_{C^0} \leq \varepsilon_0^*, \quad \|v_0(x)\|_{C^0} \leq \varepsilon_0^*, \quad \|u(x,t)\|_{C^0} \leq \varepsilon^*, \quad \|v(x,t)\|_{C^0} \leq \varepsilon^* \tag{2.4}$$

Therefore, there exist constants $M, M_0 > 0$, such that $M_0 \to 0$, as $\varepsilon_0 \to 0$, and

$$|f(u,v)| \leq M_0, \qquad |g(u,v)| \leq M_0 \tag{2.5}$$

$$\|\lambda(u,v)\|_{C^2} + \|\mu(u,v)\|_{C^2} + \|f(u,v)\|_{C^2} + \|g(u,v)\|_{C^2} < M \tag{2.6}$$

(2.1) is differentiated with respect to x, one gets

$$(e^h u_x)' = -K(u,v)e^h u_x - f_v v_x e^h \tag{2.7}$$

where

$$' = \frac{\partial}{\partial t} + \lambda(u,v)\frac{\partial}{\partial x}$$

$$K(u,v) = f_u(u,v) + \lambda_u u_x + h_u(u,v)f(u,v) + \frac{\lambda_v g}{\lambda - \mu} \tag{2.8}$$

$$h(u,v) = \int_0^v \frac{\lambda_v(u,x)}{(\lambda - \mu)(u,x)}dx \tag{2.9}$$

By (H), and (2.5), let ε_0 be sufficiently small, such that

$$f_u(u,v) \geq \frac{3}{4}\beta \qquad (\beta = f_u(0,0)) \tag{2.10}$$

$$\left|h_u f + \frac{\lambda_v g}{\lambda - \mu}\right| \leq \frac{\beta}{4} \tag{2.11}$$

Therefore, by (2.9), (2.10), (2.11), (2.6), one gets

$$K(u,v) \geq \frac{\beta}{2} + \lambda_u u_x \tag{2.12}$$

Integrate (2.7) along the first characteristics $x = x_1(t,\alpha), x_1(0,\alpha) = \alpha$, and notice $v' = (\lambda - \mu)v_x - g(u,v)$, one has

$$u_x(x,t)e^{h(x,t)} = u_0'(\alpha)e^{h(\alpha,0)-\int_0^t K(s)ds} - \int_0^t \frac{e^h(f_v V' + g)}{\lambda - \mu}\Big|_{x=x_1(t,\alpha)} e^{-\int_\tau^t K(s)ds}d\tau \tag{2.13}$$

where $K(s) = K(u(s, x_1(s,\alpha)), v(s, x_1(s,\alpha)))$, $h(x,t) = h(u(x,t), v(x,t))$. Let

$$q(u,v) = \int_0^v \frac{f_v(u,y)e^{h(u,v)}}{(\lambda - \mu)(u,y)}dy \tag{2.14}$$

For the time being, suppose that

$$K(s) \geq \beta/4 \tag{2.15}$$

Therefore, by (2.13), (2.14) and 3°, there exists a constant $M_1 > 0$, such that

$$|u_x(x,t)| \leq M_1 + B \tag{2.16}$$

where B is a positive constant depending on ε_0, and $B \to 0$, as $\varepsilon_0 \to 0$.

As $t = 0$, by $u_0'(x) \geq 0$, one may get

$$\lambda_u(u_0(x), v_0(x)) \geq 0 \tag{2.17}$$

thus, $K(0) > \beta/4$. Hence, (2.15) is valid at $t = 0$. By (2.13), (2.15), one gets $u_x(x,t) \geq -B$. By $|\lambda_u(u,v)| \leq M$, $\lambda_u(u,v) \neq 0$, let B be sufficiently small, such that $B < \dfrac{\beta}{4M}$, thus, one can get $K(s) > \beta/4$. Hence, (2.15) is valid on $t > 0$.

Similarly, as $u_0'(x) \leq 0$, $\lambda_u(u,v) < 0$, (2.16) can be got.

So far, the priori estimate of $|u_x|$ is got, for that of $|v_x|$, it is done similarly. Therefore, Cauchy problem (2.1), (2.2) admits a global smooth solution on $t \geq 0$.

By 2°, (2.1) can be rewritten as

$$\begin{aligned} u_t + \lambda(u,v)u_x &= -au - bv - A_1(u,v)u^2 - B_1(u,v)uv - C_1(u,v)v^2 \\ v_t + \mu(u,v)v &= -cu - dv - A_2(u,v)u^2 - B_2(u,v)uv - C_2(u,v)v^2 \end{aligned} \tag{2.18}$$

where $A_i, B_i, C_i \in C^1$, $i = 1,2$, $a = a(0,0) > 0$, $b = b(0,0)$, $c = c(0,0)$, $d = d(0,0) > 0$.

Lemma 2.2 *Suppose that* $A = \begin{pmatrix} a & -|b| \\ -|c| & d \end{pmatrix}$, *and the real parts of all eigenvalues of A are greater than a constant* $\alpha (\alpha > 0)$, *Cauchy problem* (2.18), (2.2) *has a priori estimate for the* C^1*-solution*

$$\|U(x,t)\|_{C^0} \leq D\|U_0(x)\|_{C^0}e^{-\alpha t} \tag{2.19}$$

for $t \geq 0$, *as long as the* C^1*-solution exists, where* $U(x,t) = (u(x,t),\ v(x,t))^T$, $D=$ *constant* >0.

Corollary 2.3 *Under the hypotheses* 1°, 2°, 3°, *if the conditions of Lemma* 2.2 *is hold and* C^0 *norm of* $u_0(x)$, $v_0(x)$ *is sufficiently small, then Cauchy problem* (2.2), (2.18) *possesses a unique global smooth solution on* $t \geq 0$.

Consider Cauchy problem of the equations as

$$\begin{aligned} &u_t + \lambda(u,v)u_x = -au - bv \qquad (a>0)\\ &v_t + \mu(u,v)v_x = -cu - dv \qquad (d>0)\\ &t = 0 : u = u_0(x), \qquad v = v_0(x)\end{aligned} \tag{2.20}$$

Lemma 2.4 *If matrix* $A = \begin{pmatrix} a & -|b| \\ -|c| & d \end{pmatrix}$ *has eigenvalues with non-negative real parts, then Cauchy problem* (2.20) *has a priori estimate*

$$\|u(x,t)\|_{C^0} \leq \text{ const. } \|u_0(x)\|_{C^0}, \quad \|v(x,t)\|_{C^0} \leq \text{ const. } \|v_0(x)\|_{C^0} \tag{2.21}$$

for $t \geq 0$, *as long as the* C^1*-solution exists.*

Corollary 2.5 *Under the hypotheses* 1°, 2°, 3°, *if the conditions of Lemma* 2.4 *hold and* C^0 *norm of* $u_0(x)$, $v_0(x)$ *is sufficiently small, then* (2.20) *possesses a unique global smooth solution on the upper-half plane* $t \geq 0$.

At last, we take an example of quasilinear hyperbolic systems to show that the solutions of its Cauchy problem will blow up in a finite time, no matter how small C^0 or C^1 norm of initial data is.

Consider Cauchy problem of the equations as

$$\begin{aligned} &z_t + \lambda(z,w)z_x = -z - w - w^2\\ &w_t + \mu(z,w)w_x = w^2\\ &t = 0 : z = z_0(x) = \varepsilon, \qquad w = w_0(x) = 1/M\end{aligned} \tag{2.22}$$

where $\lambda, \mu \in C^1$, $\lambda \neq \mu$, $\lambda_z \neq 0$, $\mu_w \neq 0$, M =const. >0.

Theorem 2.6 *The problem* (2.22) *has a* C^1*-solution for a finite time.*

Proof For any point $(t,x)(0 \leq t < M)$, we take two characteristics L_1, L_2 of (2.22), with intersecting the x-axis at θ, η respectively. Integrate (2.22) along L_1, L_2 respectively, one can get

$$\begin{aligned} &z(x,t)e^t = z_0(\theta) - \int_0^t (w + w^2)|_{L_1} e^s ds\\ &w(x,t) = 1/(M-t), \quad (0 \leq t < M)\end{aligned} \tag{2.23}$$

$w = 1/(M-t)$ is valid for any point $(x,t)(0 \le t < M)$. Naturally it is valid on L_1. Thus, one can get

$$z(x,t) = \varepsilon e^{-t} - 1/(M-t) + \frac{e^{-t}}{M} \quad (0 \le t < M)$$

Therefore, as $t \to M^-$, $z(x,t) \to -\infty$, $w(x,t) \to +\infty$, no matter how small ε, $1/M$ are.

3. Formation of Singularities of the Solutions

Consider the following Cauchy problem of equations

$$\begin{aligned} &u_t + p(v)_x + (2\alpha + \varepsilon)u = 0 \\ &v_t - u_x + \frac{2\varepsilon v}{1-\gamma} = 0 \end{aligned} \tag{3.1}$$

$$t = 0 : u = u_0(x), \qquad v = v_0(x) \tag{3.2}$$

where $v > 0$, $p(v) = k^2 v^{-\gamma}$, ε, α, k, γ=const. >0, $\gamma = 1 + 2\sigma$, $\sqrt{\frac{\alpha}{2\alpha+\varepsilon}} < \sigma < 1$

The Riemann invariants are taken as

$$s = u + \varphi(v), \qquad r = u - \varphi(v) \tag{3.3}$$

where $\varphi(v) = \int_v^\infty \sqrt{-p'(v)}dv$. Therefore, (3.1) can be rewritten as

$$\begin{aligned} &r_t + \lambda(r,s) r_x = -\alpha(r+s) - \varepsilon r \\ &s_t + \mu(r,s) s_x = -\alpha(r+s) - \varepsilon s \end{aligned} \tag{3.4}$$

where λ, μ are characteristic roots of (3.1) and

$$\lambda = -\sqrt{-p'(v)} = \lambda(s-r) < 0, \quad \mu = \sqrt{-p'(v)} = \mu(s-r) > 0 \tag{3.5}$$

Suppose that,

$$t = 0 : s = s_0(x) = u_0(x) + \varphi(v_0(x)), r = r_0(x) = u_0(x) - \varphi(v_0(x)) \tag{3.6}$$

(A) $\quad r_0(x), s_0(x) \in C^1, \|r_0(x)\|_{C^1} + \|s_0(x)\|_{C^1}$ is bounded

$$\delta = \inf s_0(x) - \sup r_0(x) > 0$$

Theorem 3.1 *Under the hypothesis* (A), *if* $s_0'(x) \ge 0$, $r_0'(x) \ge 0$, $\forall x \in R$, *and* C^0 *norm of* $r_0(x)$, $s_0(x)$ *is sufficiently small, then problem* (3.1), (3.2) *possesses a unique global smooth solution on the upper-half plane* $t \ge 0$.

Theorem 3.2 *Under the hypothesis* (A), *if*

i) $s_0'(x) > 0, r_0'(x) = 0, \qquad \forall x \in [\bar{\eta}, \bar{\beta}];$ (3.7)

ii) $\bar{\beta} - \bar{\eta}$ *is proper large;*

iii) $\eta^* \in [\bar{\eta}, \beta], \bar{\beta} - \eta^*$ *is sufficiently small;* (3.8)

iv) $M = s_0(\bar{\beta}) - s_0(\eta^*)$ *is sufficiently large;*

v) $\delta^* = s_0(\eta^*) - s_0(\bar{\eta})$ *is sufficiently small;*

then, the first derivative of $r(x,t)$ *which is the solution of Cauchy problem* (3.4), (3.6) *must blow up in curve-characteristic triangle of with the base* $\bar{\eta}\bar{\beta}$.

(3.4), (3.6) can be rewritten as

$$\begin{aligned} w_t + \bar{\lambda} w_x &= -\alpha(z+w) \\ z_t + \bar{\mu} z_x &= -\alpha(z+w) \end{aligned} \tag{3.9}$$

$$t = 0 : z = z_0(x) = s_0(x), \quad w = w_0(x) = r_0(x) \tag{3.10}$$

where

$$w = re^{\varepsilon t}, \quad z = se^{\varepsilon t} \tag{3.11}$$

$$\bar{\lambda} = \lambda(e^{-\varepsilon t}(z-w)), \quad \bar{\mu} = \mu(e^{-\varepsilon t}(z-w)) \tag{3.12}$$

According to concrete case considered, by analysis, one can get

$$\bar{\mu} = e^{-\theta t}\mu(z-w) \qquad \left(\theta = \frac{\gamma+1}{\gamma-1}\varepsilon\right) \tag{3.13}$$

Lemma 3.3 *Suppose that* $z_0'(x) \geq 0$, $w_0'(x) \geq 0$, $\forall x \in [\eta, \beta]$, $z_0(x) - w_0(x)$ *is a monotonic increasing function to* x *in* $[\eta, \beta]$, $w_0(x)$, $z_0(x) \in C^1$, $\|z_0(x)\|_{C^1}$, $\|w_0(x)\|_{C^1}$ *are bounded, then for any* C^1*-solution of* (3.9), (3.10) *satisfies*

$$z_0(\eta) - w_0(\beta) \leq (z-w)(x,t) \leq z_0(\beta) - w_0(\beta) - e^{-2\alpha t}(z_0(\beta) - w_0(\eta)) \tag{3.14}$$

for $t \geq 0$, *where* β, η *are respectively* x*-coordinates, as two characteristics passing through* (t, x) *intersect the* X*-axis.*

Proof of Theorem 3.2

At first, differentiating (3.9) with respect to x and using (3.13), one can get

$$w_x'' = -\mu' e^{-\theta t} w_x^2 + \mu' e^{-\theta t} w_x z_x - \alpha(z_x + w_x) \tag{3.15}$$

$$z_x`` = -\mu' e^{-\theta t} z_x^2 + \mu' e^{-\theta t} w_x z_x - \alpha(z_x + w_x) \tag{3.16}$$

where $'' = \frac{\partial}{\partial t} + \bar{\lambda}\frac{\partial}{\partial x}$, $` = \frac{\partial}{\partial t} + \bar{\mu}\frac{\partial}{\partial x}$. By (3.7), (3.10), ii), iv), one has

$$z_0'(x) > 0, \quad w_0'(x) = 0, \quad \forall x \in [\bar{\eta}, \bar{\beta}] \tag{3.17}$$

$$\begin{aligned} M = z_0(\bar{\beta}) - z_0(\bar{\eta}) &\text{ is sufficiently large} \\ \delta^* = z_0(\eta^*) - z_0(\bar{\eta}) &\text{ is sufficiently small} \end{aligned} \tag{3.18}$$

Thus, by (3.15), (3.17), one can get

$$w_x''(0, s) = -\alpha z_0'(s) < 0, \forall s \in [\bar{\eta}, \bar{\beta}] \tag{3.19}$$

On the other hand, $w_x(0,s) = w_0'(s) = 0$ ($\forall s \in [\bar{\eta}, \bar{\beta}]$). So, for any given point $\beta_0 \in [\bar{\eta}, \bar{\beta}]$, there exists a neighborhood D_1 of $(0, \beta_0)$ in the upper-half plane $t > 0$, such that $w_x < 0$ in D_1 (without segment in the x-axis). Since $z_0'(\beta_0) > 0$, thus, there exists a neighborhood D_2 of $(0, \beta_0)$ in the upper-half plane $t > 0$, such that $z_x > 0$ in D_2. Therefore, one may pick out a characteristic triangle K_0 (see Fig. 3.1). In K_0 (without segment $\eta\beta$), one can get

$$z_x > 0, \qquad w_x < 0 \tag{3.20}$$

and show that (3.20) is always valid in the cured characteristic triangle K with the base $\bar{\eta}\bar{\beta}$ (without segment $\bar{\eta}\bar{\beta}$). The reasoning of (3.20) is seen in [5]. We get

$$(z - w)'' = -2\mu e^{-\theta t} z_x \tag{3.21}$$

Let

$$\begin{aligned} h &= \frac{1}{2} \ln \mu(z - w), \\ g &= \alpha \int_{\delta}^{z-w} \frac{e^{h(x)}}{2\mu(x)} dx \end{aligned} \tag{3.22}$$

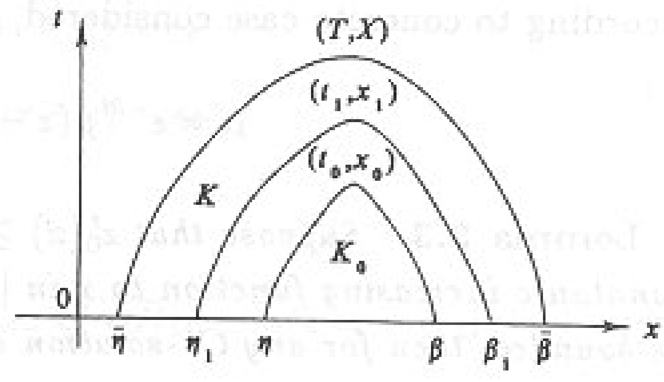

Fig. 3.1

The method in [5] is borrowed, one can get

$$(e^h w_x)'' = -\mu' e^{-\theta t} w_x^2 e^h - \alpha e^h w_x + e^{\theta t} g'' \tag{3.23}$$

$$(e^h w_x)'' = -\mu' e^{-\theta t} w_x^2 e^h - \alpha e^h (z_x + w_x) \tag{3.24}$$

Secondly, we discuss the growth of w_x along $x = x_1(t, \bar{\beta})$ $(t > 0)$. Integrate (3.23) along $x = x_1(t, \bar{\beta})$ and notice $w_0'(\bar{\beta}) = 0$, one can get

$$(e^h w_x)(t, x_1(t, \bar{\beta})) = e^{-\alpha t}\Big(-\int_0^t \mu' e^{h - \theta s} w_x^2 e^{\alpha s} ds + \int_0^t g'' e^{(\alpha+\theta)s} ds\Big) \tag{3.25}$$

Set

$$U(s,x_1(s,\bar\beta)) = U(s) = -(e^h w_x)e^{\alpha s}(s,x_1(s,\bar\beta)) \tag{3.26}$$

For concrete case considered, one can get

$$\mu(y) = cy^{\frac{1+\sigma}{\sigma}} \tag{3.27}$$

where $c = k\sqrt{1+2\sigma}\Big(\frac{\sqrt{2}\sigma}{2k\sqrt{1+2\sigma}}\Big)^{\frac{1+\sigma}{\sigma}}$. By (3.22), it is easily shown that $g'' < 0$, therefore, by (3.27), one can get

$$e^{-\alpha t}\int_0^t g'' e^{(\alpha+\theta)s}ds = \frac{\alpha\sigma}{\sqrt{c}(1-\sigma)}e^{-\alpha t+(\alpha+\theta)\bar t}[(z_0(\bar\beta)-w_0(\bar\beta))^{\frac{\sigma-1}{2\sigma}} - (z-w)^{\frac{\sigma-1}{2\sigma}}(x,t)] \tag{3.28}$$

where $0<\bar t<t$. By (3.7), (3.10) and Lemma 3.3, one can get

$$(z-w)(x,t) \le (z_0(\bar\beta)-w_0(\bar\beta)) - e^{-2\alpha t}(z_0(\bar\beta)-z_0(\eta)) \tag{3.29}$$

where η is x-coordinate of intersection point which is produced by the second characteristics passing through $(t,x)=(t,x_1(t,\bar\beta))$ and the x-axis. Take $\eta^* = \eta(t_0,\bar\beta)$ from (3.28), (3.29), one can get

$$(e^h w_x)|_{t_0} \le -\frac{\alpha\sigma}{(1-\sigma)\sqrt{c}}e^{-\alpha t_0+(\alpha+\theta)\bar t}\Big\{[(z_0(\bar\beta)-w_0(\bar\beta)) - e^{-2\alpha t_0}(z_0(\bar\beta)-z_0(\eta^*))]^{\frac{\sigma-1}{2\sigma}} - (z_0(\bar\beta)-w_0(\bar\beta))^{\frac{\sigma-1}{2\sigma}}\Big\} = -Ae^{-\alpha t_0} \tag{3.30}$$

Since $w_x<0$ on $x=x_1(\tau,\bar\beta)$, one can get

$$U(s) = -(e^{h+\alpha s}w_x)(s,x_1(s,\bar\beta)) > 0 \tag{3.31}$$

Let $\bar\beta$ be replaced by $x_0(x_0=x_1(t,\bar\beta))$, integrating (3.23) from t_0, using (3.26), (3.30), and then multiplying the two sides by $e^{\alpha t}$, one can get

$$e^{\alpha t}(e^h w_x)(x,t) = -e^{\alpha t_0}(e^h w_x)|_{t_0} + \int_{t_0}^t \mu' e^{-h-\theta s-\alpha s}U^2(s)ds - \int_{t_0}^t g'' e^{\alpha s+\theta s}ds \tag{3.32}$$

Accordingly, one can get

$$U(t) \ge A + \int_{t_0}^t (\mu' e^{-h-\alpha s-\theta s})U^2(s)ds \tag{3.33}$$

By Lemma 4.3, in [5], one can get

$$U(t) \ge A\Big/\Big(1 - A\int_{t_0}^t \mu' e^{-h-\alpha s-\theta s}ds\Big) \tag{3.34}$$

Since

$$\int_{t_0}^{t} \mu' e^{-h-\alpha s-\theta s} ds \geq \frac{\sqrt{c}(1+\sigma)}{(\alpha+\theta)\sigma}(z_0(\bar{\eta}) - w_0(\bar{\beta}))^{\frac{1-\sigma}{2\sigma}} e^{-(\alpha+\theta)t_0}(1 - e^{-(\alpha+\theta)(t-t_0)}) \quad (3.35)$$

Putting (3.35) into (3.34), one can get

$$U(t) = -e^{-h} w_x e^{\alpha t} \geq A \Big/ (1 - Q(1 - e^{-(\alpha+\theta)(t-t_0)})) \quad (3.36)$$

where

$$Q = \frac{\alpha(1+\sigma)}{(\alpha+\theta)(1-\sigma)} e^{-(\alpha+\theta)(t_0-\bar{t})} \Big\{ [(z_0(\bar{\beta}) - w_0(\bar{\beta})) - e^{-2\alpha t_0}(z_0(\bar{\beta}) - z_0(\eta^*))]^{\frac{\sigma-1}{2\sigma}}$$

$$-[z_0(\bar{\beta}) - w_0(\bar{\beta})]^{\frac{\sigma-1}{2\sigma}} \Big\} (z_0(\bar{\eta}) - w_0(\bar{\beta}))^{\frac{1-\sigma}{2\sigma}} \quad (3.37)$$

Therefore, from (3.11), one can get

$$|r_x(x,t)| \geq \frac{1}{\sqrt{\mu_*}} \frac{Ae^{-(\alpha+\varepsilon)t}}{1 - Q(1 - e^{-(\alpha+\theta)(t-t_0)})} \quad (3.38)$$

where $\mu_* = \mu(z_0(\bar{\beta}) - w_0(\bar{\beta}))$.

In order to explain the question, suppose that

$$w_0(\bar{\beta}) = 0, \qquad z_0(\bar{\eta}) = \bar{\delta} > \delta > 0 \quad (3.39)$$

Therefore, by (3.19), (3.18), one may get

$$Q = e^{-(\alpha+\theta)(t_0-\bar{t})} B((\bar{\delta} + \delta^* + M - Me^{-2\alpha t_0})^{\frac{\sigma-1}{2\sigma}} - (\bar{\delta} + \delta^* + M)^{\frac{\sigma-1}{2\sigma}} \bar{\delta}^{\frac{1-\sigma}{2\sigma}}) \quad (3.40)$$

where $B = \dfrac{\alpha(1+\sigma)}{(\alpha+\theta)(1-\sigma)} > 1$ $\left(\text{notice } \sqrt{\dfrac{\alpha}{2\alpha+\varepsilon}} < \sigma < 1\right)$. By iii), $\bar{\beta} - \eta^*$ is sufficiently small, then, $t_0 \to 0$. By $0 < \bar{t} < t_0$, one can get

$$e^{-(\alpha+\theta)(t_0-\bar{t})} \to 1 \text{ as } \bar{\beta} - \eta^* \to 0$$

Hence,

$$Q \to B\Big(1 + \frac{\delta^*}{\bar{\delta}}\Big)^{\frac{\sigma-1}{2\sigma}} \Big(1 - \Big(\frac{\bar{\delta} + \delta^* + M}{\bar{\delta} + \delta^*}\Big)^{\frac{\sigma-1}{2\sigma}}\Big) = \bar{Q} \quad (\bar{\beta} - \eta^* \to 0) \quad (3.41)$$

Take positive integral numbers n, m, such that

$$B(1 - 1/n)(1 - 1/m) > 1 \quad (3.42)$$

Fix n and m, let M be sufficiently large, and δ^* be small, such that

$$\Big(\frac{1}{2} + \frac{M}{2\bar{\delta}}\Big)^{\frac{\sigma-1}{2\sigma}} < 1/m \quad (3.43)$$

$$\delta^* < \bar{\delta}, \quad \left(1+\frac{\delta^*}{\bar{\delta}}\right)^{\frac{\sigma-1}{2\sigma}} > 1-\frac{1}{n} \tag{3.44}$$

Thus

$$\left(\frac{\delta^*+\bar{\delta}+M}{\bar{\delta}+\delta^*}\right)^{\frac{\sigma-1}{2\sigma}}=\left(1+\frac{M}{(\bar{\delta}+\delta^*)}\right)^{\frac{\sigma-1}{2\sigma}}<(1/2+M/(2\bar{\delta}))^{\frac{\sigma-1}{2\sigma}}\le 1/m \tag{3.45}$$

Accordingly, one can get $\bar{Q} > 1$. Thus, as $\bar{\beta}-\eta^*$ is sufficiently small, then $Q > 1$. So, there exists a proper large $t_B > t_0$, such that

$$Q(1-e^{-(\alpha+\theta)(t_\theta-t_0)})=1 \tag{3.46}$$

For characteristics $x = x_1(t,\bar{\beta})$ and $x = x_2(t,\bar{\eta})$, consider the equations as

$$x=\bar{\beta}-\int_0^t \mu(s-r)(\tau,\ x_1(\tau,\bar{\beta}))d\tau,\quad x=\bar{\eta}+\int_0^t \mu(s-r)(\tau,x_2(\tau,\bar{\eta}))d\tau \tag{3.47}$$

Thus, if $\bar{\beta}-\bar{\eta}$ is sufficiently large and $\bar{\beta}-\eta^*$ is sufficiently small, then

$$\frac{\bar{\beta}-\bar{\eta}}{2\mu(s_0(\bar{\beta})-r_0(\bar{\beta}))}\ge t_B$$

Therefore, by (3.38), as $t \ge t_0$ increases, along characteristics $x = x_1(t,\bar{\beta})$, $|r_x(x,t)|$ must go to infinite in a finite time t_B, that is, it will blow up in characteristic triangle with the base $\bar{\eta}\bar{\beta}$.

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