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AN ALGEBRAIC APPROACH FOR EXTENDING HAMILTONIAN OPERATORS*

Tu Guizhang and Ma Wenxiu**
(Computing Centre, Academia Sinica, Beijing 100080)
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Abstract An algebraic approach for extending Hamiltonian operators is proposed. A relevant sufficient condition for generating new Lie algebras from known ones is presented. Some special cases are discussed and several illustrative examples are given.
Key Words Matrix differential operator; Lie algebra; Hamiltonian operator.
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1. Introduction

It is well known that many nonlinear evolution equations possess generalized Hamiltonian structures[1–4]. Hamiltonian operators play a crucial role in the algebraic and geometric theory of those Hamiltonian structures[5]. Based on Hamiltonian pairs, we can also construct, under certain conditions, a hierarchy of Hamiltonian equations possessing an infinite number of symmetries[6,7,8]. Therefore the search for new Hamiltonian operators and Hamiltonian pairs is one among the central topics in theory of Hamiltonian systems, there have been works[9,5,10] concerning the general theory of Hamiltonian operators. In the present paper, we propose an algebraic approach for extending Hamiltonian operators from lower orders to higher orders. We show that a large number of new Hamiltonian operators and new Hamiltonian pairs can be derived through this algebraic approach.

Let $u = (u_1(x,t), u_2(x,t), \cdots, u_q(x,t))$, $x, t \in R$, be a q-dimensional smooth function vector. The linear space of smooth functions $P[u] = P(x,t,u^{(m)}) = P(x,t,u,\cdots,u^{(m)})$, $m \geq 0$, is denoted by $\mathcal{A}$, $\mathcal{A}^q = \mathcal{A}\times\cdots\times\mathcal{A}(q \text{ times}) = \{(P_1, P_2, \cdots, P_q) | P_i \in \mathcal{A}, 1 \leq i \leq q\}$. Two functions P and Q of $\mathcal{A}$ are considered to be equivalent and denoted by $P \sim Q$ (mod D) if $P - Q = DR \equiv dR/dx$ holds for some $R \in \mathcal{A}$. The equivalent class that contains P is denoted by $\tilde{P} = \int P dx$, we call it a functional. The space of all functionals is represented by $\tilde{\mathcal{A}}$.

*The subject supported by the National Natural Science Foundation of China.
**Permanent Address: Institute of Mathematics, Fudan University, Shanghai 200433.

Definition 1 *A linear operator $J = J(x,t,u)$: $\mathcal{A}^q \to \mathcal{A}^q$ is called Hamiltonian if the bracket defined by*

$$\{\tilde{P},\tilde{Q}\} = \int \frac{\delta \tilde{P}}{\delta u}\Big(J\frac{\delta \tilde{Q}}{\delta u}\Big)^T dx, \quad \tilde{P},\tilde{Q} \in \tilde{\mathcal{A}}, \quad \frac{\delta}{\delta u} = \Big(\frac{\delta}{\delta u_1}, \frac{\delta}{\delta u_2}, \cdots, \frac{\delta}{\delta u_q}\Big) \tag{1.1}$$

is skew-symmetry

$$\{\tilde{P},\tilde{Q}\} = -\{\tilde{Q},\tilde{P}\}, \ \forall \tilde{P},\tilde{Q} \in \tilde{\mathcal{A}} \tag{1.2}$$

and satisfies the Jacobi identity

$$\{\{\tilde{P},\tilde{Q}\},\tilde{R}\} + \{\{\tilde{Q},\tilde{R}\},\tilde{P}\} + \{\{\tilde{R},\tilde{P}\},\tilde{Q}\} = 0, \quad \forall \tilde{P},\tilde{Q},\tilde{R} \in \tilde{\mathcal{A}} \tag{1.3}$$

In this case we call $\{\cdot,\cdot\}$ a Poisson bracket corresponding to the Hamiltonian operator J.

We observe that a matrix differential operator

$$J = (J_{ij})_{q\times q}, J_{ij} = \sum_{m=0}^{m(i,j)} P_m^{ij}[u]D^m, \quad D^m = \Big(\frac{d}{dx}\Big)^m, \quad P_m^{ij}[u] \in \mathcal{A} \tag{1.4}$$

may be considered as a linear operator $J : \mathcal{A}^q \to \mathcal{A}^q$, $P \mapsto JP^T$.

Definition 2[11] *If all the functions $P_m^{ij}[u], i,j = 1,2,\cdots,q, m = 0,1,\cdots,m(i,j)$, are linear with respect to u, then the operator J defined by (1.4) is called a u-linear operator; otherwise, J called a u-nonlinear operator.*

In this paper, we shall consider u-linear matrix differential operators with constant coefficients:

$$J = (J_{ij})_{q\times q}, \quad J_{ij} = \sum_{m=0}^{m(i,j)} \sum_{l=0}^{l(i,j)} \sum_{k=1}^{q} a_{ijlm}^k u_k^{(l)} D^m, \quad u_k^{(l)} = \Big(\frac{d}{dx}\Big)^l u_k \tag{1.5}$$

where the a_{ijlm}^k for all i,j,k,l,m are complex constants.

2. An Algebraic Approach

Let $J = J(u) : \mathcal{A}^q \to \mathcal{A}^q$ be a u-linear Hamiltonian operators as defined by (1.5) where $u = (u_1(x,t), u_2(x,t), \cdots, u_q(x,t))$. In the following we shall construct a new Hamiltonian operator $\bar{J} = \bar{J}(\bar{u}) : \bar{\mathcal{A}}^{qn} \to \bar{\mathcal{A}}^{qn}$, where $\bar{u} = (\bar{u}^1, \bar{u}^2, \cdots, \bar{u}^n)$, $\bar{\mathcal{A}}^{qn} = \bar{\mathcal{A}}^q \times \cdots \times \bar{\mathcal{A}}^q$ (n times), and $\bar{u}^i = (u_{(i-1)q+1}(x,t), u_{(i-1)q+2}(x,t), \cdots, u_{iq}(x,t)), 1 \le i \le n$, $\bar{\mathcal{A}}^q = \bar{\mathcal{A}} \times \cdots \times \bar{\mathcal{A}}$ (q times) in which $\bar{\mathcal{A}}$ denotes the linear space of smooth functions $P[\bar{u}] = P(x,t,\bar{u}^{(m)}), m \ge 0$.

By taking integration by parts and making use of the u-linearity of J we can write the inner product

$$(JP,Q) = \int \sum_{i,j=1}^{q} Q_i J_{ij} P_j dx \text{ where } P = (P_1, P_2, \cdots, P_q), Q = (Q_1, Q_2, \cdots, Q_q) \in \mathcal{A}^q \tag{2.1}$$

in the form of $(u,R) = \int \sum_{i=1}^{q} u_i R_i dx$ for some $R = (R_1, R_2, \cdots, R_q) \in \mathcal{A}^q$. It is not difficult to prove that R is uniquely determined by P, Q and J. Thus we can write $R = [P,Q]_J$ and we have

$$(J(u)P,Q) = (u, [P,Q]_J), \quad P,Q \in \mathcal{A}^q \tag{2.2}$$

For example, for $J = 2uD + u_x$, we have $[P,Q]_J = QP_x - PQ_x$. It is known that J is Hamiltonian if and only if $\mathcal{A}^q$ is a Lie algebra with respect to the product $[\cdot,\cdot]_J$.[12]

Given a set of complex constant matrices

$$C^i = (c^i_{jk})_{n\times n}, \quad 1 \le i \le n \tag{2.3}$$

for $P = (P^1, P^2, \cdots, P^n)$, $Q = (Q^1, Q^2, \cdots, Q^n) \in \bar{\mathcal{A}}^{qn}$, we define the product $[P,Q]$ of P, Q as follows

$$[P,Q] = ([P,Q]^1, [P,Q]^2, \cdots, [P,Q]^n) \tag{2.4a}$$

$$[P,Q]^i = \sum_{j,k=1}^{n} c^i_{jk} [P^j, Q^k]_J, \quad 1 \le i \le n \tag{2.4b}$$

Then we have

$$\begin{aligned}(\bar{u}, [P,Q]) &= \int \sum_{i=1}^{n} \bar{u}^i ([P,Q]^i)^T dx \\ &= \int \sum_{i=1}^{n} \bar{u}^i \sum_{j,k=1}^{n} c^i_{jk} ([P^j, Q^k]_J)^T dx = \sum_{i,j,k=1}^{n} c^i_{jk} (\bar{u}^i, [P^j, Q^k]_J) dx \\ &= \sum_{i,j,k=1}^{n} c^i_{jk} (J(\bar{u}^i) P^j, Q^k) = \sum_{j,k=1}^{n} \Big(\sum_{i=1}^{n} c^i_{jk} J(\bar{u}^i) P^j, Q^k \Big)\end{aligned}$$

Setting

$$\bar{J} = \bar{J}(\bar{u}) = (\bar{J}_{kj})_{nq\times nq} = (\bar{J}_{kj})_{n\times n} = \Big(\sum_{i=1}^{n} c^j_{jk} J(\bar{u}^i) \Big)_{n\times n} \tag{2.5}$$

we obtain

$$(\bar{u}, [P,Q]) = (\bar{J}(\bar{u}) P, Q), \quad P,Q \in \bar{\mathcal{A}}^{qn} \tag{2.6}$$

From (2.6), we obtain easily that $(\bar{\mathcal{A}}^{qn}, [\cdot,\cdot])$ is a Lie algebra if and only if $\bar{J} = \bar{J}(\bar{u})$ is a Hamiltonian operator of order nq.

We summarize the above-mentioned results as follows.

Theorem 1 *Let a Hamiltonian operator $J = J(u)$: $\mathcal{A}^q \to \mathcal{A}^q$ be defined by* (1.5). *Then $\bar{\mathcal{A}}^{qn} (n \geq 1)$ with the product* (2.4) *forms a Lie algebra iff the operator $\bar{J} = \bar{J}(\bar{u}) : \bar{\mathcal{A}}^{qn} \to \bar{\mathcal{A}}^{qn}$ defined by* (2.5) *is a Hamiltonian operator.*

The necessity part of this theorem can provide an algebraic approach to extend Hamiltonian operators from lower orders to higher orders. The steps for construction are as follows:

1. choose a special Hamiltonian operator $J = J(u) : \mathcal{A}^q \to \mathcal{A}^q$;
2. calculate the Lie product $[P,Q]_J$ for $P, Q \in \mathcal{A}^q$ by using the equation (2.2);
3. determine a set of constants $\{c^i_{jk} | i,j,k = 1,2,\cdots,n\}$ such that $\bar{\mathcal{A}}^{qn}$ forms a Lie algebra with respect to the product $[P,Q]$ defined by (2.4);
4. generate a higher order Hamiltonian operator $\bar{J} = \bar{J}(\bar{u}) : \bar{\mathcal{A}}^{qn} \to \bar{\mathcal{A}}^{qn}$ by using the formula (2.5).

Through these steps, we can generate a hierarchy of higher order Hamiltonian operators $\{(\bar{J}_{kj})_{nq\times nq}\}_{n=1}^{\infty}$, starting from a lower order Hamiltonian operator $J = (J_{ij})_{q\times q}$.

3. A Sufficient Condition

This section gives a sufficient condition for $(\bar{\mathcal{A}}^{qn}, [\cdot,\cdot])$ being a Lie algebra.

Let $E = \{e_1, e_2, \cdots, e_n\}$ be the set of n symbols and let $Y = L_C(E)$ denote the complex linear space spanned by E. Given a set of complex constants $\{c^i_{jk} | i,j,k = 1,2,\cdots,n\}$, we can define a bilinear operation $*$ on Y as follows

$$e_j * e_k = \sum_{i=1}^{n} c^i_{jk} e_i, \quad j,k = 1,2,\cdots,n \tag{3.1}$$

Now $< Y, * >$ forms an algebra.

Theorem 2 *If the algebra $< Y, * >$ is commutative and the multiplication $*$ satisfies the condition*

$$a * (b * c) = b * (c * a), \quad \forall a,b,c \in Y \tag{3.2}$$

then $\bar{\mathcal{A}}^{qn}$ forms an infinite dimensional Lie algebra with respect to the product $[\cdot,\cdot]$ defined by (2.4).

Proof Let $P = (P^1, P^2, \cdots, P^n)$, $Q = (Q^1, Q^2, \cdots, Q^n)$, $R = (R^1, R^2, \cdots, R^n)$ be three vectors of $\bar{\mathcal{A}}^{qn}$. By (2.4), we have

$$[P,Q]^i = \sum_{j,k=1}^{n} c^i_{jk} [P^j, Q^k]_J, \quad i = 1,2,\cdots,n$$

and

$$
\begin{aligned}
[[P,Q],R]^l &= \sum_{r,s=1}^{n} c^l_{rs}[[P,Q]^r,R^s]_J \\
&= \sum_{r,s=1}^{n} c^l_{rs}\sum_{j,k=1}^{n} c^r_{jk}[[P^j,Q^k]_J,R^s]_J \\
&= \sum_{j,k,s=1}^{n}\Big(\sum_{r=1}^{n} c^l_{rs}c^r_{jk}\Big)[[P^j,Q^k]_J,R^s]_J, \quad l=1,2,\cdots,n
\end{aligned}
$$

Because the multiplication $*$ is commutative and satisfies (3.2), we have

$$
\sum_{r=1}^{n} c^l_{rs}c^r_{jk} = \sum_{r=1}^{n} c^l_{rj}c^r_{ks}, \quad l,j,k,s=1,2,\cdots,n \tag{3.3}
$$

Thus

$$
\begin{aligned}
&[[P,Q],\ R]^l + \mathrm{cycle}(P,Q,R) \\
&= \sum_{j,k,s=1}^{n}\Big(\sum_{r=1}^{n} c^l_{rs}c^r_{jk}\Big)\{[[P^j,Q^k]_J,R^s]_J + \mathrm{cycle}(P^j,Q^k,R^s)\} \\
&= 0, \quad l=1,2,\cdots,n
\end{aligned} \tag{3.4}
$$

In addition, since $< Y, * >$ is commutative, we see that $c^i_{jk} = c^i_{kj}$, $i,j,k=1,2,\cdots,n$. Therefore

$$
[P,Q]^i = \sum_{j,k=1}^{n} c^i_{jk}[P^j,Q^k]_J = -\sum_{j,k=1}^{n} c^i_{kj}[Q^k,P^j]_J = -[Q,P]^i, \quad i=1,2,\cdots,n \tag{3.5}
$$

The equalities (3.4), (3.5) show that $(\bar{A}^{qn},[\cdot,\cdot])$ is indeed a Lie algebra.

According to Theorem 2, once we have a commutative algebra $< Y, * >$ which satisfies (3.2), we can obtain a hierarchy of new Hamiltonian operators $\Big\{(\bar{J}_{kj})_{n\times n} = \Big(\sum_{i=1}^{n} c^i_{jk}J(\bar{u}^i)\Big)_{n\times n}\Big\}_{n=1}^{\infty}$ from an old Hamiltonian operator $J=J(u)$.

4. The General Solution of Two Dimensional Case

When $n=2$, the commutative algebra $< Y, * >$ only possesses six different structural constants. In this case, we can obtain the general solution of (3.2) or (3.3).

Theorem 3 *Let $< Y, * >$ be a two dimensional commutative algebra, then the multiplication $*$ satisfies (3.2) or (3.3) iff the structural constants $C^i = (c^i_{jk})_{2\times 2}$, $i=1,2$, are one of the following forms*

$$
C^1 = \begin{bmatrix} \xi & 0 \\ 0 & 0 \end{bmatrix}, C^2 = \begin{bmatrix} 0 & 0 \\ 0 & \eta \end{bmatrix} \tag{4.1}
$$

$$C^1 = \begin{bmatrix} \alpha\xi + \eta & 0 \\ 0 & 0 \end{bmatrix}, C^2 = \begin{bmatrix} \xi & \eta \\ \eta & -\alpha\eta \end{bmatrix} \tag{4.2}$$

$$C^1 = \begin{bmatrix} \beta\xi + \alpha\eta & \xi \\ \xi & \eta \end{bmatrix}, C^2 = \begin{bmatrix} \alpha\xi & \alpha\eta \\ \alpha\eta & \xi - \beta\eta \end{bmatrix} \tag{4.3}$$

where ξ, η, α, β are arbitrary constants.

Proof We know easily that (3.2) or (3.3) is equivalent to

$$\begin{cases} c_{11}^l c_{22}^1 + c_{21}^l c_{22}^2 = c_{12}^l c_{21}^1 + c_{22}^l c_{21}^2 = c_{12}^l c_{12}^1 + c_{22}^l c_{12}^2, & l = 1,2 \\ c_{12}^l c_{11}^1 + c_{22}^l c_{11}^2 = c_{11}^l c_{12}^1 + c_{21}^l c_{12}^2 = c_{11}^l c_{21}^1 + c_{21}^l c_{21}^2, & l = 1,2 \end{cases}$$

Since two right equalities in the above equations hold automatically, we see that the equation (3.2) or (3.3) is equivalent to

$$\begin{cases} c_{11}^l c_{22}^1 + c_{21}^l c_{22}^2 = c_{12}^l c_{21}^1 + c_{22}^l c_{21}^2, & l = 1,2 \quad (4.4) \\ c_{12}^l c_{11}^1 + c_{22}^l c_{11}^2 = c_{11}^l c_{12}^1 + c_{21}^l c_{12}^2, & l = 1,2 \quad (4.5) \end{cases}$$

We observe that the equation (4.4) with $l = 2$ is the same as the equation (4.5) with $l = 1$. Thus (4.4), (4.5) are easily seen to be equivalent to

$$\begin{cases} c_{11}^1 c_{22}^1 + c_{21}^1 c_{22}^2 = c_{12}^1 c_{21}^1 + c_{22}^1 c_{21}^2 & (4.6) \\ c_{11}^2 c_{22}^1 = c_{12}^2 c_{21}^1 & (4.7) \\ c_{12}^2 c_{11}^1 + c_{22}^2 c_{11}^2 = c_{11}^2 c_{12}^1 + c_{21}^2 c_{12}^2 & (4.8) \end{cases}$$

Obviously, the equation (4.7) amounts to

$$\det \begin{bmatrix} c_{12}^1 & c_{22}^1 \\ c_{11}^2 & c_{12}^2 \end{bmatrix} = 0$$

and Equations (4.6) and (4.8) are equivalent to

$$\det \begin{bmatrix} c_{12}^1 & c_{22}^1 \\ c_{11}^1 - c_{12}^2 & c_{12}^1 - c_{22}^2 \end{bmatrix} = \det \begin{bmatrix} c_{11}^2 & c_{12}^2 \\ c_{11}^1 - c_{12}^2 & c_{12}^1 - c_{22}^2 \end{bmatrix} = 0$$

Thus the set of constants $\{c_{ij}^k | i, j, k = 1, 2\}$ satisfies (3.3) if and only if

$$\text{rank} \begin{bmatrix} c_{12}^1 & c_{22}^1 \\ c_{11}^2 & c_{12}^2 \\ c_{11}^1 - c_{12}^2 & c_{12}^1 - c_{22}^2 \end{bmatrix} \leq 1$$

If $(c_{12}^1, c_{22}^1) = (c_{11}^2, c_{12}^2) = 0$, then setting $c_{11}^1 = \xi$, $c_{22}^2 = \eta(\xi, \eta = \text{const.}$, the same below.), C^1 and C^2 take the form of the case (1).

If $(c_{12}^1, c_{22}^1) = 0$, but $(c_{11}^2, c_{12}^2) =: (\xi, \eta) \neq 0$, then $(c_{11}^1 - c_{12}^2, c_{12}^1 - c_{22}^2) = \alpha(\xi, \eta)$, and C^1, C^2 take the form as shown in the case (2).

If $(c_{12}^1, c_{22}^1) =: (\xi, \eta) \neq 0$, then $(c_{11}^2, c_{12}^2) = \alpha(\xi, \eta)$, $(c_{11}^1 - c_{12}^2, c_{12}^1 - c_{22}^2) = \beta(\xi, \eta)$. Thus the corresponding matrices C^1, C^2 take the form of the case (3).

According to (2.4), we can easily calculate three Lie products, corresponding respectively to (4.1), (4.2) and (4.3), as follows:

$$\begin{cases} [P,Q]^1 = \xi[P^1,Q^1]_J \\ [P,Q]^2 = \eta[P^2,Q^2]_J \end{cases} \tag{4.9}$$

$$\begin{cases} [P,Q]^1 = (\alpha\xi + \eta)[P^1,Q^1]_J \\ [P,Q]^2 = \xi[P^1,Q^1]_J + \eta([P^1,Q^2]_J + [P^2,Q^1]_J) - \alpha\eta[P^2,Q^2]_J \end{cases} \tag{4.10}$$

$$\begin{cases} [P,Q]^1 = (\beta\xi + \alpha\eta)[P^1,Q^1]_J + \xi([P^1,Q^2]_J + [P^2,Q^1]_J) + \eta(P^2,Q^2]_J \\ [P,Q]^2 = \alpha\xi[P^1,Q^1]_J + \alpha\eta([P^1,Q^2]_J + [P^2,Q^1]_J) + (\xi - \beta\eta)[P^2,Q^2]_J \end{cases} \tag{4.11}$$

By Theorem 1, the relevant three kinds of Hamiltonian operators are as follows:

$$\bar{J}_1 = \begin{bmatrix} \xi J(\bar{u}^1) & 0 \\ 0 & \eta J(\bar{u}^2) \end{bmatrix}$$

$$\bar{J}_2 = \begin{bmatrix} (\alpha\xi + \eta)J(\bar{u}^1) + \xi J(\bar{u}^2) & \eta J(\bar{u}^2) \\ \eta J(\bar{u}^2) & -\alpha\eta J(\bar{u}^2) \end{bmatrix}$$

$$\bar{J}_3 = \begin{bmatrix} (\beta\xi + \alpha\eta)J(\bar{u}^1) + \alpha\xi J(\bar{u}^2) & \xi J(\bar{u}^1) + \alpha\eta J(\bar{u}^2) \\ \xi J(\bar{u}^1) + \alpha\eta J(\bar{u}^2) & \eta J(\bar{u}^1) + (\xi - \beta\eta)J(\bar{u}^2) \end{bmatrix}$$

We observe that three Hamiltonian operators contain arbitrary constants ξ and η:

$$\bar{J}_1 = \xi \begin{bmatrix} J(\bar{u}^1) & 0 \\ 0 & 0 \end{bmatrix} + \eta \begin{bmatrix} 0 & 0 \\ 0 & J(\bar{u}^2) \end{bmatrix} = \xi\bar{J}_1^1 + \eta\bar{J}_1^2$$

$$\bar{J}_2 = \xi \begin{bmatrix} \alpha J(\bar{u}^1) + J(\bar{u}^2) & 0 \\ 0 & 0 \end{bmatrix} + \eta \begin{bmatrix} J(\bar{u}^1) & J(\bar{u}^2) \\ J(\bar{u}^2) & -\alpha J(\bar{u}^2) \end{bmatrix} = \xi\bar{J}_2^1 + \eta\bar{J}_2^2$$

$$\bar{J}_3 = \xi \begin{bmatrix} \beta J(\bar{u}^1) + \alpha J(\bar{u}^2) & J(\bar{u}^1) \\ J(\bar{u}^1) & J(\bar{u}^2) \end{bmatrix} + \eta \begin{bmatrix} \alpha J(\bar{u}^1) & \alpha J(\bar{u}^2) \\ \alpha J(\bar{u}^2) & J(\bar{u}^1) - \beta J(\bar{u}^2) \end{bmatrix} = \xi\bar{J}_3^1 + \eta\bar{J}_3^2$$

From general theory of Hamiltonian operators, we thus obtain three Hamiltonian pairs $\bar{J}_i^1, J_i^2, i = 1,2,3$.

5. Special Commutative Algebras

Let $Y = C[x]_n$ denote the set of complex coefficient polynomials in x with degrees less than n. In this case, we may choose $e_i = x^{i-1}, i = 1,2,\cdots,n$. Let $r \in Z$ be an integer, we introduce in Y the multiplication $\overset{r}{*}$ as follows

$$a \overset{r}{*} b = [x^{-r}a(x)b(x)]_0^{n-1}, \quad a,b \in C[x]_n \tag{5.1}$$

where the notation $[c(x)]_0^{n-1}$ denotes the part of a Laurent polynomial $c(x)$ with degrees $0,1,\cdots,n-1$.

Theorem 4 *When $1-n \le r \le 0$ or $n-1 \le r \le 2n-2$, the algebra $< C[x]_n, \overset{r}{*} >$ is commutative and satisfies the condition (3.2), the multiplication $\overset{r}{*}$ being defined by (5.1).*

Proof Set

$$x^j \overset{r}{*} x^k = \sum_{i=0}^{n-1} c_{(j+1)(k+1)}^{i+1} x^i, \quad j,k = 0,1,\cdots,n-1$$

By the definition of multiplication $\overset{r}{*}$, we have

$$x^j \overset{r}{*} x^k = [x^{j+k-r}]_0^{n-1} = \sum_{i=0}^{n-1} \delta_{i,j+k-r} x^i, \quad j,k = 0,1,\cdots,n-1$$

Theorefore

$$c_{jk}^i = \delta_{i+1,j+k-r}, \quad i,j,k = 1,2,\cdots,n \tag{5.2}$$

Evidently, the multiplication $\overset{r}{*}$ defined in this way is commutative. We now turn to prove the property (3.2). Choose $a = x^i, b = x^j, c = x^k$, $i,j,k = 0,1,\cdots,n-1$. Then

$$\begin{aligned} a \overset{r}{*} (b \overset{r}{*} c) &= x^i \overset{r}{*} (x^j \overset{r}{*} x^k) \\ &= x^i \overset{r}{*} \Big(\sum_{s=0}^{n-1} \delta_{s,j+k-r} x^s \Big) = \sum_{t=0}^{n-1} \Big(\sum_{s=0}^{n-1} \delta_{t,i+s-r} \delta_{s,j+k-r} \Big) x^t \\ &= \begin{cases} x^{i+j+k-2r}, & \text{when } 0 \le i+j+k-2r \le n-1 \text{ and } 0 \le j+k-r \le n-1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

and

$$b \overset{r}{*} (c \overset{r}{*} a) = x^j \overset{r}{*} (x^k \overset{r}{*} x^i)$$

$$= x^j \overset{r}{*} \Big(\sum_{s=0}^{n-1} \delta_{s,k+i-r} x^s \Big) = \sum_{t=0}^{n-1} \Big(\sum_{s=0}^{n-1} \delta_{t,j+s-r} \delta_{s,k+i-r} \Big) x^t$$

$$= \begin{cases} x^{i+j+k-2r}, & \text{when } 0 \le i+j+k-2r \le n-1 \text{ and } 0 \le k+i-r \le n-1 \\ 0, & \text{otherwise} \end{cases}$$

Suppose first that $1-n \le r \le 0$. Then we have $i+j+k-2r \ge 0$, $j+k-r \ge 0$. If $i+j+k-2r \le n-1$, we have

$$j+k-r \le i+j+k-r \le n-1+r \le n-1$$

$$k+i-r \le i+j+k-r \le n-1+r \le n-1$$

Therefore

$$a \overset{r}{*} (b \overset{r}{*} c) = \begin{cases} x^{i+j+k-2r}, & \text{when } i+j+k-2r \le n-1 \\ 0, & \text{otherwise} \end{cases} \tag{5.3}$$

and

$$b \overset{r}{*} (c \overset{r}{*} a) = \begin{cases} x^{i+j+k-2r}, & \text{when } i+j+k-2r \le n-1 \\ 0, & \text{otherwise} \end{cases} \tag{5.4}$$

Thus

$$a \overset{r}{*} (b \overset{r}{*} c) = b \overset{r}{*} (c \overset{r}{*} a)$$

Suppose next that $n-1 \le r \le 2n-2$. Then we have $i+j+k-2r \le n-1$, $k+i-r \le n-1$. If $i+j+k-2r \ge 0$, we have

$$j+k-r \ge i+j+k-r-(n-1) \ge i+j+k-2r \ge 0$$

$$k+i-r \ge i+j+k-r-(n-1) \ge i+j+k-2r \ge 0$$

Therefore

$$a \overset{r}{*} (b \overset{r}{*} c) = \begin{cases} x^{i+j+k-2r}, & \text{when } i+j+k-2r \ge 0 \\ 0, & \text{otherwise} \end{cases} \tag{5.5}$$

and

$$b \overset{r}{*} (c \overset{r}{*} a) = \begin{cases} x^{i+j+k-2r}, & \text{when } i+j+k-2r \ge 0 \\ 0, & \text{otherwise} \end{cases} \tag{5.6}$$

Thus

$$a \overset{r}{*} (b \overset{r}{*} c) = b \overset{r}{*} (c \overset{r}{*} a)$$

By the bilinearity of the operation $\overset{r}{*}$, we see that the multiplication $\overset{r}{*}$ satisfies the condition (3.2).

Remark 1 When $1-n \le r < \frac{1}{2}(1-n)$ or $\frac{3}{2}(n-1) < r \le 2n-2$, it follows from (5.3) or (5.5) respectively that $< C[x]_n, \overset{r}{*} >$ is the algebra with null double product[13,14].

Remark 2 When $0 < r < n-1$, the multiplication $\overset{r}{*}$ doesn't satisfy the condition (3.2).

We can obtain from (5.2) the following matrix expressions concerning the structural constants $C^i = (c^i_{jk})_{n\times n}, 1 \le i \le n$, of the algebra $< C[x]_n, \overset{r}{*} >$:

(1) When $r = 1-n$,

$$C^1 = C^2 = \cdots = C^{n-1} = 0, C^n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

(2) When $r = 2-n$,

$$C^1 = C^2 = \cdots = C^{n-2} = 0, C^{n-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, C^n = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

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(n) When $r = 0$,

$$C^1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, C^2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \cdots, C^n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

(n+1) When $r = n-1$,

$$C^1 = \begin{bmatrix} 0\,0 & \cdots & 0\,1 \\ 0\,0 & \cdots & 1\,0 \\ \vdots\vdots & \ddots & \vdots\vdots \\ 0\,1 & \cdots & 0\,0 \\ 1\,0 & \cdots & 0\,0 \end{bmatrix}, C^2 = \begin{bmatrix} 0\,0\,0 & \cdots & 0\,0 \\ 0\,0\,0 & \cdots & 0\,1 \\ \vdots\vdots\vdots & \ddots & \vdots\vdots \\ 0\,0\,1 & \cdots & 0\,0 \\ 0\,1\,0 & \cdots & 0\,0 \end{bmatrix}, \cdots, C^n = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

(n+2) When $r = n$,

$$C^1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \cdots, C^{n-1} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}, C^n = 0$$

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(2n) When $r = 2n-2$,

$$C^1 = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}, C^2 = C^3 = \cdots = C^n = 0$$

From these expressions, we find easily the following $2n$ new Hamiltonian operators $\bar{J}(\bar{u})$:

$$\begin{bmatrix} J(\bar{u}^n) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} J(\bar{u}^{n-1}) & J(\bar{u}^n) & 0 & \cdots & 0 \\ J(\bar{u}^n) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \cdots, \begin{bmatrix} J(\bar{u}^1) & J(\bar{u}^2) & \cdots & J(\bar{u}^n) \\ J(\bar{u}^2) & J(\bar{u}^3) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J(\bar{u}^n) & 0 & \cdots & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \cdots & 0 & J(\bar{u}^1) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & J(\bar{u}^{n-2}) & J(\bar{u}^{n-1}) \\ J(\bar{u}^1) & \cdots & J(\bar{u}^{n-1}) & J(\bar{u}^n) \end{bmatrix}, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & J(\bar{u}^1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & J(\bar{u}^1) & \cdots & J(\bar{u}^{n-1}) \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & J(\bar{u}^1) \end{bmatrix}$$

6. Examples of Hamiltonian Operators

In this section, we give some illustrative examples.

Example 1 Let the Hamiltonian operator J be given by $J(u) = 2uD + u_x$, where $u = u(x,t)$ is a scalar smooth function and set $\bar{u} = (\bar{u}^1, \bar{u}^2) = (u_1, u_2)$. From the results of Section 4, we obtain three kinds of Hamiltonian operators as follows:

$$\bar{J}_1(\bar{u}) = \begin{bmatrix} \xi(2u_1D + u_{1x}) & 0 \\ 0 & \eta(2u_2D + u_{2x}) \end{bmatrix}$$

$$\bar{J}_2(\bar{u}) = \begin{bmatrix} (\alpha\xi + \eta)(2u_1D + u_{1x}) + \xi(2u_2D + u_{2x}) & \eta(2u_2D + u_{2x}) \\ \eta(2u_2D + u_{2x}) & -\alpha\eta(2u_2D + u_{2x}) \end{bmatrix}$$

$$\bar{J}_3(\bar{u}) = \begin{bmatrix} (\beta\xi + \alpha\eta)(2u_1D + u_{1x}) + \alpha\xi(2u_2D + u_{2x}) & \xi(2u_1D + u_{1x}) + \alpha\eta(2u_2D + u_{2x}) \\ \xi(2u_1D + u_{1x}) + \alpha\eta(2u_2D + u_{2x}) & \eta(2u_1D + u_{1x}) + (\xi - \beta\eta)(2u_2D + u_{2x}) \end{bmatrix}$$

where ξ, η, α, β are arbitrary constants.

Example 2 Choose the same Hamiltonian operator $J(u) = 2uD + u_x$. Set $\bar{u} = (\bar{u}^1, \bar{u}^2, \bar{u}^3) = (u_1, u_2, u_3)$; $P = (P^1, P^2, P^3)$, $Q = (Q^1, Q^2, Q^3) \in \bar{A}^3$. We define

$$[P,Q]^1 = [P^1, Q^3]_J + [P^2, Q^2]_J + [P^3, Q^1]_J$$
$$[P,Q]^2 = [P^2, Q^3]_J + [P^3, Q^2]_J$$
$$[P,Q]^3 = [P^3, Q^3]_J$$

This is just the Lie product corresponding to the algebra $< C[x]_3, \overset{2}{*} >$. Therefore we obtain the following Hamiltonian operator discussed in [14]

$$\bar{J}(\bar{u}) = \begin{bmatrix} 0 & 0 & 2u_1D + u_{1x} \\ 0 & 2u_1D + u_{1x} & 2u_2D + u_{2x} \\ 2u_1D + u_{1x} & 2u_2D + u_{2x} & 2u_3D + u_{3x} \end{bmatrix}$$

Similarly, define the Lie product $[\cdot,\cdot]$ of $\bar{A}^3$ as follows

$$[P,Q]^1 = [P^1,Q^1]_J$$

$$[P,Q]^2 = [P^1,Q^2]_J + [P^2,Q^1]_J$$

$$[P,Q]^3 = [P^1,Q^3]_J + [P^2,Q^2]_J + [P^3,Q^1]_J$$

We can obtain another Hamiltonian operator

$$\bar{J}(\bar{u}) = \begin{bmatrix} 2u_1D + u_{1x} & 2u_2D + u_{2x} & 2u_3D + u_{3x} \\ 2u_2D + u_{2x} & 2u_3D + u_{3x} & 0 \\ 2u_3D + u_{3x} & 0 & 0 \end{bmatrix}$$

which corresponds to the algebra $< C[x]_3, \overset{0}{*} >$.

Example 3 Choose one Hamiltonian operator given in [13]

$$J(u) = \begin{bmatrix} u_{1x} + 2u_1D & u_{1x} + (u_1+u_2)D \\ u_{2x} + (u_1+u_2)D & u_{2x} + 2u_2D \end{bmatrix}$$

Set $\bar{u} = (\bar{u}^1, \bar{u}^2) = (u_1, u_2, u_3, u_4)$; $P = (P^1, P^2)$, $Q = (Q^1, Q^2) \in \bar{A}^2 \times \bar{A}^2$. Define

$$[P,Q]^1 = [P^1,Q^1]_J$$

$$[P,Q]^2 = [P^1,Q^2]_J + [P^2,Q^1]_J$$

which is just the Lie product corresponding to the algebra $< C[x]_2, \overset{0}{*} >$. Then we obtain the following Hamiltonian operator

$$\bar{J}(\bar{u}) = \begin{bmatrix} u_{1x} + 2u_1D & u_{1x} + (u_1+u_2)D & u_{3x} + 2u_3D & u_{3x} + (u_3+u_4)D \\ u_{2x} + (u_1+u_2)D & u_{2x} + 2u_2D & u_{4x} + (u_3+u_4)D & u_{4x} + 2u_4D \\ u_{3x} + 2u_3D & u_{3x} + (u_3+u_4)D & 0 & 0 \\ u_{4x} + (u_3+u_4)D & u_{4x} + 2u_4D & 0 & 0 \end{bmatrix}$$

Example 4 Choose a scalar Hamiltonian operator $J(u) = 2uD + u_x$ again. Set $\bar{u} = (\bar{u}^1, \bar{u}^2, \bar{u}^3, \bar{u}^4) = (u_1, u_2, u_3, u_4)$; $P = (P^1, P^2, P^3, P^4)$, $Q = (Q^1, Q^2, Q^3, Q^4) \in \bar{A}^4$. Define

$$[P,Q]^1 = [P^1,Q^4]_J + [P^2,Q^3]_J + [P^3,Q^2]_J + [P^4,Q^1]_J$$

$$[P,Q]^2 = [P^2,Q^4]_J + [P^4,Q^2]_J$$

$$[P,Q]^3 = [P^3,Q^4]_J + [P^4,Q^3]_J$$

$$[P,Q]^4 = [P^4,Q^4]_J$$

It's not difficult to show that $(\bar{A}^4, [\cdot,\cdot])$ forms an infinite Lie algebra. By Theorem 1, we can obtain the following Hamiltonian operator

$$J(\bar{u}) = \begin{bmatrix} 0 & 0 & 0 & 2u_1D + u_{1x} \\ 0 & 0 & 2u_1D + u_{1x} & 2u_2D + u_{2x} \\ 0 & 2u_1D + u_{1x} & 0 & 2u_3D + u_{3x} \\ 2u_1D + u_{1x} & 2u_2D + u_{2x} & 2u_3D + u_{3x} & 2u_4D + u_{4x} \end{bmatrix}$$

Through these examples, we see that, from a lower order Hamiltonian operator, a hierarchy of higher order Hamiltonian operators can be easily obtained by using the algebraic approach in Section 2 and no additional verification is needed. As a comparison, if we try to verify directly from the definition that they are indeed Hamiltonian operators, then even if we know the forms of those higher order operators, we have still to make a large amount of calculation.

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