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THE DIFFRACTION PROBLEM AND VERIGIN PROBLEM OF QUASILINEAR PARABOLIC EQUATION IN DIVERGENCE FORM FOR THE ONE-DIMENSIONAL CASE

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Abstract In this paper, we consider the flow of two immiscible fluids in a one-dimensional porous medium (the Verigin problem) and obtain a quasilinear parabolic equation in divergence form with the discontinuous coefficients. We prove first the existence and uniqueness of locally classical solution of the diffraction problem and then prove the existence of local solution of the Verigin problem.

Key Words Porous medium; discontinuous coefficient; diffraction
Classification 35K

0. Introduction

Since the 1940s, the after-production of petroleum by means of waterflooding has been used extensively to raise the production index. The relevant model for permeability can be idealized mathematically as a free boundary problem. Muskat supposed in 1937 a mathematical model for piston-type driving in [1]. Assuming that the flow moves horizontally and touches the boundary $\Gamma : x = h(t)$, and using the Darcy law and the mass conservation law, Verigin obtained in [2] the parabolic problem with respect to pressure p, called the Verigin problem later on.

The theory about the Verigin problem has been vigorously developed only for the one-dimensional case. The linear Verigin problem was studied by Kamynin in [3], [4], by Fulks and Guenther in [5], and by Evans in [6], [7]. Recently, research on quasilinear equations was set about by Meirmanov in [8] and Liang Jing in [9].

In general case, the free boundary is fixed first. And the problem with discontinuous coefficients to be considered first is called the diffraction problem. It was studied by Oleinik with Bernstein method, by Ladyženskaja with integral estimation, by Kamynin with potential method.

We are concerned with

$$\frac{\partial u_i}{\partial t}-\frac{\partial}{\partial x}\Big(k_i(x,t,u_i)\frac{\partial u_i}{\partial x}\Big)=f_i(x,t,u_i),\quad i=1,2$$

$$u_1=u_2, h'(t)=-g(u_1)k_1(x,t,u_1)\frac{\partial u_1}{\partial x}=-g(u_2)k_2(x,t,u_2)\frac{\partial u_2}{\partial x}, x=h(t)$$

The paper is divided into three sections. Section 1 discusses the uniform estimation for approximate solution under the smoothened coefficients. Section 2 proves the existence and uniqueness of local solution of diffraction problem and discusses the continuous dependence of solution on internal boundary perturbation. Section 3 proves the existence of local solution of the Verigin problem by means of Schauder fixed point theorem.

1. Uniform Estimation for Approximate Solution

Fix $h(t)$ and let $Q_1=\{(x,t):0<x<h(t), 0<t<T\}$, $Q_2=\{(x,t):h(t)<x<l, 0<t<T\}$, $Q_T=Q_1\cup Q_2$. We are concerned with the following diffraction problem

$$\begin{cases}\dfrac{\partial u_i}{\partial t}-\dfrac{\partial}{\partial x}\Big(k_i(x,t,u_i)\dfrac{\partial u_i}{\partial x}\Big)=f_i(x,t,u_i), & (x,t)\in Q_i,\ i=1,2\\ u_1=u_2,\ k_1(x,t,u_1)\dfrac{\partial u_1}{\partial x}=k_2(x,t,u_2)\dfrac{\partial u_2}{\partial x}, & x=h(t)\\ u(0,t)=\bar{u}_1(t), u(l,t)=\bar{u}_2(t), u(x,0)=\bar{u}_0(x), h(0)=b\end{cases}\tag{1.0}$$

and we assume

(I) $k_i(x,t,z)$, $f_i(x,t,z)\in C^3([0,l]\times[0,\infty)\times(-\infty,+\infty))$ and there exist constants $\gamma>0$, $b_1>0$, $b_2>0$ such that $k_i(x,t,z)\geq\gamma$, $f_i(x,t,z)z\leq b_1z^2+b_2$.

(II) $\bar{u}_1,\bar{u}_2\in C^1[0,T]$, $\bar{u}_0\in C[0,l]$, $k(x,0,\bar{u}_0(x))\bar{u}_{0x}\in C^1[0,l]$ with $\bar{u}_0(0)=\bar{u}_1(0)$, $\bar{u}_0(l)=\bar{u}_2(0)$ and denote $k(x,t,u)=k_i(x,t,u_i)$, $(x,t)\in Q_i$, $i=1,2$.

(III) $h(t)\in C^1[0,T]$.

Smoothen the coefficients and let

$$k_\varepsilon(x,t,u)=k_1(x,t,u)(1-H_\varepsilon(x-h(t))+k_2(x,t,u)H_\varepsilon(x-h(t))$$

$$f_\varepsilon(x,t,u)=f_1(x,t,u)(1-H_\varepsilon(x-h(t))+f_2(x,t,u)H_\varepsilon(x-h(t))$$

where $H_\varepsilon(x)=\begin{cases}0 & \text{for } x\leq-\varepsilon\\ 1 & \text{for } x\geq\varepsilon\end{cases}$, $H_\varepsilon(x)\in C^3(-\infty,+\infty)$. Smoothen initial-boundary value and fix arbitrarily $\alpha\in(0,1)$. There exist $\bar{u}_{0\varepsilon}\in C^{2+\alpha}[0,l]$, $\bar{u}_{i\varepsilon}\in C^{1+\alpha/2}[0,T]$, being smoothened forms of $\bar{u}_0$ and $\bar{u}_i$ respectively and satisfying the compatibility conditions of orders 0 and 1, such that

$$\|k_\varepsilon(x,0,\bar{u}_{0\varepsilon}(x))\bar{u}_{0\varepsilon x}\|_{\alpha,[0,l]}\leq\|k(x,0,\bar{u}_0(x))\bar{u}_{0x}\|_{\alpha,[0,l]}$$

$$\|\bar{u}_{i\varepsilon}\|_{1,[0,T]}\leq C+\|\bar{u}_i\|_{1,[0,T]},\quad \|\bar{u}_{0\varepsilon}\|_{0,[0,l]}\leq 2\|\bar{u}_0\|_{0,[0,l]},\quad i=1,2$$

$$\|\bar{u}_{0\varepsilon}\|_{2,[0,b/2]}+\|\bar{u}_{0\varepsilon}\|_{2,[(b+l)/2,l]}\leq 2\|\bar{u}_0\|_{2,[0,b/2]}+2\|\bar{u}_0\|_{2,[(b+l)/2,l]}$$

Here, we smoothen $\bar{u}_0(x)$ first, then determine the value of $\bar{u}'_{i\varepsilon}(0)$ and take C depending on $|u'_{i\varepsilon}(0)|$, and finally we smoothen $\bar{u}_i$.

Lemma 1.1 *The quasilinear parabolic problem*

$$\begin{cases} u_{\varepsilon t} - \dfrac{\partial}{\partial x}\left(k_\varepsilon(x,t,u_\varepsilon)\dfrac{\partial u_\varepsilon}{\partial x}\right) = f_\varepsilon(x,t,u_\varepsilon) \\ u_\varepsilon(0,t) = \bar{u}_{1\varepsilon}(t), u_\varepsilon(l,t) = \bar{u}_{2\varepsilon}(t), u_\varepsilon(x,0) = \bar{u}_{0,\varepsilon}(x) \end{cases} \tag{1.1}$$

has a solution $u_\varepsilon \in C^{2+\alpha,1+\alpha/2}(\bar{Q}_T) \cap C^{3+\alpha,1+(1+\alpha)/2}(Q_T)$.

For this classical result, see Chap. V in [10].

In what follows we shall give a series of estimations independent of ε.

(1) Maximum norm estimation and Hölder norm estimation

On the basis of assumption (I), it follows from Theorem 2.9 of Chap. I and Theorem 10.1 of Chap. III in [10] that

$$\|u_\varepsilon\|_{0,\bar{Q}_T} \le M_0 \tag{1.2}$$

and

$$\|u_\varepsilon\|_{\beta,\bar{Q}_T} \le M_1 \tag{1.3}$$

where $0 < \beta < 1$ and M_0, M_1, β are independent of ε.

(2) Local estimation for $u_{\varepsilon x}$

First, introducing the barrier functions

$$\omega_1(x,t) = \pm\frac{1}{K_1}\ln(1+K_2x) + \bar{u}_{1\varepsilon}(t), \quad \omega_2(x,t) = \pm\frac{1}{K_1}\ln(1+K_2(1-x)) + \bar{u}_{2\varepsilon}(t)$$

yields the estimation of $u_{\varepsilon x}$ on $x=0$ and $x=1$, i.e. the estimation of $u_{\varepsilon x}$ on the parabolic boundary $\partial_p Q_T$. Applying the transformation

$$y = x - h(t), \quad t' = t \quad \text{(denoted by } t \text{ still)}$$

Next, letting $v = k_\varepsilon \dfrac{\partial u}{\partial y}$ yields the equation with respect to v

$$\frac{\partial}{\partial t}\left(\frac{v}{k_\varepsilon}\right) - \frac{\partial}{\partial y}\left(\frac{v}{k_\varepsilon}\right)h'(t) - v_{yy} = \frac{\partial}{\partial y}f_\varepsilon \tag{1.4}$$

Simplifying it yields

$$\begin{aligned} v_t - k_\varepsilon v_{yy} - v_y h'(t) - \frac{k_{\varepsilon u}}{k_\varepsilon}vv_y + \frac{v}{k_\varepsilon}[(k_2-k_1)h'(t)H'_\varepsilon(y) + A(y,t)] \\ = k_\varepsilon[(f_2-f_1)H'_\varepsilon(y) + B(y,t)] \end{aligned} \tag{1.5}$$

where

$$A(y,t) = -k_{2t}H_\varepsilon(y) - k_{1t}(1-H_\varepsilon) - k_{\varepsilon u}f_\varepsilon - k_\varepsilon f_{\varepsilon u}$$

$$B(y,t) = f_{2y}H_\varepsilon(y) + f_{1y}(1-H_\varepsilon(y))$$

Construct an auxiliary function $W = C_0 + \beta t + \left(1 + \int_0^y H_\varepsilon(z)dz\right)^{-\alpha}$, where C_0 is the maximum norm of v on the boundary. Take suitable α, β, it follows from the comparison principle that there exists $t_1 > 0$, $\varepsilon_0 > 0$ such that

$$v(y,t) \le w(y,t) \le C_0 + 1 + \left(1 + \int_0^y H_\varepsilon(z)dz\right)^{-\alpha}, \quad t \le t_1, \quad \varepsilon \le \varepsilon_0$$

Therefore we get

$$\|u_{\varepsilon x}\|_{0,\bar{Q}_{t_1}} \le M_2 \tag{1.6}$$

where M_2 depends on M_0 and d, and d is the shortest distance between the internal and external boundaries.

(3) Hölder norm estimation for $k_\varepsilon(x,t,u_\varepsilon)\dfrac{\partial u_\varepsilon}{\partial x}$

We shall first prove that $\dfrac{\partial}{\partial x}\left(k_\varepsilon \dfrac{\partial u_\varepsilon}{\partial x}\right)$ is uniformly bounded near $x = 0$.

Obviously, we have

$$f_\varepsilon(x,t,u_\varepsilon) = f_1(x,t,u_\varepsilon), k_\varepsilon(x,t,u_\varepsilon) = k_1(x,t,u_\varepsilon), \ 0 \le x \le \frac{d}{2}, \ 0 \le t \le t_1$$

Let $W(x,t) = k_\varepsilon \dfrac{\partial u_\varepsilon}{\partial x}$, then W satisfies

$$W_{xx} = E(x,t,u_\varepsilon)W_t + F(x,t,u_\varepsilon,W)W_x + G(x,t,u_\varepsilon)W + D(x,t,u_\varepsilon) \tag{1.7}$$

where

$$E = k_1^{-1}(x,t,u_\varepsilon), \quad F = -k_{1u}k_1^{-2}W$$

$$G = -k_1^{-2}(k_{1t} + k_{1u}f_1 + k_1 f_{1u}), \quad D = f_{1x}(x,t,u_\varepsilon)$$

Construct an auxiliary function $W = f(\omega) = -2M' + 3eM'\int_0^\omega e^{-s^2}ds$, where M' stands for the maximum norm of W. It follows from Beinstein estimation method that

$$|u_{\varepsilon t}| \le M_3, \quad |u_{\varepsilon xx}| \le M_3, \quad 0 \le x \le \frac{d}{4}, \quad 0 \le t \le t_1$$

Hence, $|u_{\varepsilon x}(0,t_1) - u_{\varepsilon x}(0,t_2)| \le 4M_3|t_1 - t_2|^{1/2}$, where M_3 depends on d and is independent of ε.

The similar discussion works near $x = 1$.

Therefore, by smoothening the initial-value, it follows that $\left\|k_\varepsilon \dfrac{\partial u_\varepsilon}{\partial x}\right\|_{\alpha,\partial_p Q_{t_1}}$ is uniformly bounded and denoted by M_3 still.

Denote $Q'_{t_1} = \{(y,t) : -h(t) < y < l - h(t), 0 < t < t_1\}$, on which we discuss v. Let

$$\omega(y,t) = \pm v(y,t), \omega_k = (\omega - k)^+, Q(\rho,\tau) = K_\rho \times (t_0, t_0 + \tau) \in Q'_{t_1}$$

where $\operatorname{vrai}\max\limits_{Q(\rho,\tau)} \omega(y,t) - k \le 1$, $K_\rho = (y_0 - \rho, y_0 + \rho)$, y_0 and t_0 are arbitrary fixed by $Q(\rho,\tau) = K_\rho \times (t_0, t_0 + \tau) \in Q'_T$. Take η as a truncation function such that $0 \le \eta \le 1$, with $\eta = 0$ on the lateral face of $Q(\rho,\tau)$. Multiplying both sides of Equation (1.4) by the test function $\omega_k\eta^2$ and integrating by parts yield

$$\iint_{Q(\rho,\tau)} \left(\frac{\omega}{k_\varepsilon}\right)_t \omega_k\eta^2 + \iint_{Q(\rho,\tau)} \omega_{ky}^2\eta^2$$
$$\leq C\Big[\iint_{Q(\rho,\tau)} \omega_k^2(\eta_y^2+\eta|\eta_t|) + \Big(\int_{t_0}^{t_0+\tau} \text{mes}^2 A_{k,\rho}(t)dt\Big)^{\frac{1}{2}}\Big] \quad (1.8)$$

where $A_{k,\rho}(t) = \{y \in K_\rho,\ \omega > k\}$. Taking suitable η we have

$$\max_{t_0\leq t\leq t_0+\tau} \int_{K_{\rho-\sigma_1\rho}} \frac{\omega_k^2(y,t)}{k_\varepsilon^*(y,t_0)}dy$$
$$\leq \int_{K_\rho} \frac{\omega_k^2(y,t_0)}{k_\varepsilon^*(y,t_0)}dy + C\Big[(\sigma_1 p)^{-2}\iint_{Q(\rho,\tau)} \omega_k^2 + C\int_{t_0}^{t_0+\tau} \text{mes}^2 A_{k,\rho}(t)dt)^{\frac{1}{2}}\Big] \quad (1.9)$$

$$|\omega_k^2|_{Q(\rho-\sigma_1\rho,\ \tau-\sigma_2\tau)}$$
$$\triangleq \max_{t_0<t<t_0+\tau-\sigma_2\tau} \int_{K_{\rho-\sigma_1\rho}} \omega_k^2 dy + \iint_{Q(\rho-\sigma_1\rho,\tau-\sigma_2\tau)} \omega_{k_y}^2 dydt$$
$$\leq C\Big\{[(\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}]\|\omega_k\|^2_{L^2(Q(\rho,\tau))} + \Big(\int_{t_0}^{t_0+\tau} \text{mes}^2 A_{k,\rho}(t)dt\Big)^{\frac{1}{2}}\Big\} \quad (1.10)$$

where $k^*(y,t) = k_\varepsilon(y+h(t),\ t, u(y+h(t),t))$. Here, (1.9), (1.10) correspond to (7.1), (7.2) respectively in Section 7 of Chap. II of [10]. According to Section 7 of Chap. II and Theorem 8 of Section 8 in [10], we have

Theorem 1.1 *Let the boundary Q'_{t_1} satisfy the uniform external cone condition, and let $v(y,t) \in V_2^{1,0}(Q'_{t_1})$ satisfy (1.9), (1.10) and be Hölder uniformly continuous on the boundary, then $v(y,t)$ is Hölder uniformly continuous on $\bar{Q}'_{t_1}$.*

Therefore, there exists M_4 and β (for convenience the Hölder coefficient is denoted by β still), independent of ε, such that

$$\Big\|k_\varepsilon(x,t,u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x}\Big\|_{\beta,\bar{Q}_{t_1}} \leq M_4 \quad (1.11)$$

(4) Multiplying Equation (1.4) by a suitable test function and integrating by parts yield that there exists M_5, independent of ε, such that

$$\iint_{Q_{t_1}} u_{\varepsilon t}^2 + \iint_{Q_{t_1}} \Big[\frac{\partial}{\partial x}\Big(k_\varepsilon\frac{\partial u_\varepsilon}{\partial x}\Big)\Big]^2 \leq M_5 \quad (1.12)$$

Lemma 1.3 *In the region Q_1 or Q_2, u_ε has the derivatives occuring in the equation and having their internal norms independent of ε.*

2. Existence and Uniqueness of Solution of the Diffraction Problem

From the uniform estimation in Section 1 and the discussion about compactness, it follows that there exists a subsequence of u_ε (denoted by u_ε still) and $u \in C(\bar{Q}_{t_1})$, $v \in C(\bar{Q}_{t_1})$ such that

$$u_\varepsilon \xrightarrow{C(\bar{Q}_{t_1})} u, \quad k_\varepsilon \frac{\partial u_\varepsilon}{\partial x} \xrightarrow{C(\bar{Q}_{t_1})} v \text{ as } \varepsilon \to 0 \text{ and } \frac{\partial u_\varepsilon}{\partial t} \to \frac{\partial u}{\partial t}, \quad \frac{\partial^2 u_\varepsilon}{\partial x^2} \to \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u_\varepsilon}{\partial x} \to \frac{\partial u}{\partial x}$$

in any internal closed region of Q_i. Hence, $v = k\frac{\partial u}{\partial x}$ and u is just the classical solution of problem (1.0), satisfying

$$\|u\|_{0,\bar{Q}_{t_1}} \le M_0 \tag{2.1}$$

$$\|u\|_{\beta,\bar{Q}_{t_1}} \le M_1 \tag{2.2}$$

$$\left\|k(x,t,u)\frac{\partial u}{\partial x}\right\|_{0,\bar{Q}_{t_1}} \le C(M_0)M_2 \tag{2.3}$$

$$\left\|k(x,t,u)\frac{\partial u}{\partial x}\right\|_{\beta,\bar{Q}_{t_1}} \le M_4 \tag{2.4}$$

$$\iint_{Q_{t_1}} \left[u_t^2 + \left(\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)\right)^2\right] dxdt \le M_5 \tag{2.5}$$

Assume that u^1, u^2 are distinct classical solutions of problem (1.0). Let $W = u^1 - u^2$, then $W \in W_2^{1,0}(Q_{t_1}) \cap C(\bar{Q}_{t_1})$ and $W|_{\partial_p Q_{t_1}} = 0$. Multiplying Equation (1.0) and integrating by parts on Q_τ yields

$$\iint_{Q_\tau} u_t^i W\, dxdt + \iint_{Q_\tau} k(x,t,u^i)\frac{\partial u^i}{\partial x}\frac{\partial W}{\partial x} dxdt = \iint_{Q_\tau} f(x,t,u^i)W\, dxdt \tag{2.6$_i$}$$

where $i = 1,2$. Subtracting $(2.6)_2$ from $(2.6)_1$ and arranging yield

$$\frac{d}{d\tau}\iint_{Q_\tau} \frac{1}{2}W^2 + \gamma\iint_{Q_\tau}\left(\frac{\partial W}{\partial x}\right)^2 \le C\iint_{Q_\tau} W^2$$

It follows from Gronwall inequality and $W|_{t=0} = 0$ that $W|_{t=\tau} = 0$. Therefore $u^1 \equiv u^2$ and the diffraction problem (1.0) has a unique local solution.

Theorem 2.1 *Under assumptions* (I)–(III), *there exists a unique locally classical solution for problem* (1.0), *satisfying estimations* (2.1)–(2.5).

By means of integral estimation and by reference to the method of estimating weak maximum norm, we have the continuous dependence of the solution on internal boundary perturbation.

Theorem 2.2 *Let $x = h_i(t)$ be internal boundary and $h_i(t) \in C^1[0,t_1]$, $i = 1,2$, then the solutions u^1, u^2 of corresponding diffraction problem satisfy*

$$\max_{0\le t\le t_1}\int_0^l (u^1-u^2)^2 dx + \iint_{Q_{t_1}}\left[\frac{\partial(u^1-u^2)}{\partial x}\right]^2 dxdt \le C_1\int_0^{t_1}[h_1'(t)-h_2'(t)]^2 dt \tag{2.7}$$

$$\|u^1-u^2\|_{0,\bar Q_{t_1}} \le C_2\|h_1-h_2\|_{1,[0,t_1]} \tag{2.8}$$

where C_1, C_2, M_0, M_2 depend on $\|h_i(t)\|_{1,[0,t_1]}$.

3. Existence of Local Solution of the One-dimensional Verigin Problem

We are concerned with the following Verigin problem

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \dfrac{\partial}{\partial x}\Big(k_i(x,t,u_i)\dfrac{\partial u_i}{\partial x}\Big) = f_i(x,t,u_i) \quad (x,t)\in Q_i,\ i=1,2 \\ u_1 = u_2, h'(t) = -g(u_1)k_1(x,t,u_1)\dfrac{\partial u_1}{\partial x} = -g(u_2)k_2(x,t,u_2)\dfrac{\partial u_2}{\partial x}, \quad x = h(t) \\ u(0,t) = \bar u_1(t), u(l,t) = \bar u_2(t), u(x,0) = \bar u_0(x), h(0) = b \end{cases} \tag{3.0}$$

under assumptions (I), (II) and

(IV) $\qquad g(z) \in C^1(-\infty,+\infty)$

Consider the closed convex set $A = \{h(t)\in C^1[0,t_0],\ h(0) = b,\ \frac{b}{2}\le h(t)\le \frac{l+b}{2},\ |h'(t)|\le M\}$ in Banach space $C^1[0,t_0]$, where t_0 is to be determined and $M = 1 + \left|g(\bar u_0(b))k_1(b,0,\bar u_0(b))\frac{\partial \bar u_0}{\partial x}(b)\right|$. It follows from the results in Section 2 that for any $h(t)\in A$ there exists a unique classical solution u with smoothness to a certain degree. Define the operator T

$$T(h) = b - \int_0^{t_0} g(u_1)k_1(h(t),t,u_1(h(t),t))\frac{\partial u_1}{\partial x}(h(t),t)dt$$

Obviously, the definition makes sense and it follows from (2.4) that

$$\begin{aligned} &\|T(h)\|_{C^{1+\beta/2}[0,t_0]} \le C(M)\Big\|k_1(x,t,u_1)\frac{\partial u_1}{\partial x}\Big\|_{\beta,\bar Q_T}\|g(u_1)\|_{\beta,\bar Q_T} \le M^*(b,M) \\ &\left|\frac{dT(h)}{dt}(t) - \frac{dT(h)}{dt}(0)\right| = \left|g(u_1)k_1\frac{\partial u_1}{\partial x}(h(t),t) - g(u_1)k_1\frac{\partial u_1}{\partial x}(b,0)\right| \le M^* t^{\beta/2} \end{aligned} \tag{3.1}$$

Take $t_2 = {M^*}^{-2/\beta}$. When $t_0 \le t_2$, $\left|\frac{dT(h)}{dt}\right| \le M$. Again, since $|T(h) - b| \le tM^*$, take $t_3 = \min\left(\frac{b}{2M^*}, \frac{l-b}{2M^*}\right)$. When $t_0\le t_3$, $\frac{b}{2}\le T(h)\le \frac{l+b}{2}$. Therefore, we take

$t_0 = \min(t_1, t_2, t_3)$. The operator T will map A onto A. By Eq. (3.1), T is compact in A.

In what follows we discuss the continuity of the operator.

Denote by u, u_n the solutions of problem (1.0), corresponding to h, h_n respectively, where $h_n \xrightarrow{A} h$. Due to compactness, there exists a subsequence of h_n (the subsequence is denoted by h_n still) such that for the corresponding solution u_n, we have

$$u_n(x,t) \xrightarrow{C(\bar{Q}_{t_0})} v(x,t), \quad k^n(x,t,u_n)\frac{\partial u_n}{\partial x} \xrightarrow{C(\bar{Q}_{t_0})} W(x,t)$$

where $k^n = \begin{cases} k_1 & \text{for } x < h_n(t) \\ k_2 & \text{for } x > h_n(t) \end{cases}$. By Theorem 2.2, obviously we have $v(x,t) \equiv u(x,t)$. It can be proved by Eqs. (2.2), (2.4) that $W \equiv k(x,t,u)\dfrac{\partial u}{\partial x}$. Hence we have $g(u_n)k^n(x,t,u_n)\dfrac{\partial u_n}{\partial x} \xrightarrow{C(\bar{Q}_{t_0})} g(u)k\dfrac{\partial u}{\partial x}$, thereby we have $T(h_n) \xrightarrow{A} T(h)$, i.e. the operator T is continuous. By means of Schauder fixed point theorem, we have

Theorem 3.1 *Under the assumptions* (I), (II), (IV), *problem* (3.0) *has a classical solution in* $[0, t_0]$, *where* t_0 *depends only on* $\|h(t)\|_{C^1[0,t_0]}$.

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