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$C^{1,\alpha}$-PARTIAL REGULARITY OF NONLINEAR PARABOLIC SYSTEMS*

Tan Zhong
(Dept. of Math., Jilin Univ., Changchun, 130023)
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Abstract We prove $C^{1,\alpha}$-partial regularity of weak solution of nonlinear parabolic systems

$$u_t^i - D_\alpha A_i^\alpha(x, t, u, Du) = B_i(x, t, u, Du), \quad i = 1, \cdots, N$$

under the main assumption that A_i^α and B_i satisfy the natural growth condition.
Key Words Nonlinear parabolic system; partial regularity; natural growth condition
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1. Introduction

In this paper we will extend some of the partial regularity results for nonlinear elliptic systems to parabolic case. Actually, we intend to show that the method developed in [1], [3] can be also used to study nonlinear parabolic systems.

Let Ω be an open set in R^n. $T > 0$ and $Q = \Omega \times [0, T]$, and let $z = (x, t)$, where $x \in \Omega$, $0 < t \leq T$, denote a point in Q and $\partial_p Q$ the parabolic boundary of Q. Let $u(z) = (u^1(z), \cdots, u^N(z))$ be a vector valued function defined in Q. Denote by Du the gradient of u, i.e., $Du = \{D_\alpha u^i\}_{i=1,\cdots,N;\alpha=1,\cdots,n}$.

Consider the nonlinear parabolic systems of the following type

$$u_t^i - D_\alpha A_i^\alpha(z, u, Du) = B_i(z, u, Du), \quad i = 1, \cdots, N \tag{1.1}$$

We suppose that A_i^α and B_i satisfy the natural growth condition:

$$A_i^\alpha(z, u, p)p_\alpha^i \geq \lambda|p|^2 - f^2, \quad f \in L^\sigma(Q) \tag{1.2}$$

$$|A_i^\alpha(z, u, p)| \leq C(|p| + f_i^\alpha), \quad f_i^\alpha \in L^\sigma(Q) \tag{1.3}$$

$$|B_i(z, u, p)| \leq a(|p|^2 + f_0), \quad f_0 \in L^\tau(Q) \tag{1.4}$$

or

$$|B_i(z, u, p)| \leq C(|p|^{2-\delta} + f_i), \quad f_i \in L^\tau(Q) \tag{1.4'}$$

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where $\lambda > 0$, $a \geq 0$ and δ are constants with $0 < \delta < \frac{n}{n+2}$. We denote

$$V_N(Q) = L^2(0,T;H^1(\Omega,R^N)) \cap L^\infty(Q,R^N)$$
$$W(Q) = L^2(0,T;H_0^1(\Omega,R^N)) \cap H^1(0,T;L^2(Q,R^N))$$

By a weak solution of (1.1) under the natural growth condition (1.2)–(1.4) (or (1.2), (1.3), (1.4)$'$) we mean a vector valued function $u \in V_N(Q)$ such that

$$\int_Q [A_i^\alpha(z,u,Du)D_\alpha\varphi^i - u^i\varphi_t^i]dz = \int_Q B_i(z,u,Du)\varphi^i dz \tag{1.1$'$}$$

for all $\varphi \in W(Q) \cap L^\infty(Q,R^N)$ with $\varphi(x,0) = 0$, $\varphi(x,T) = 0$, $\forall x \in \Omega$.

For $z_0 = (x_0,t_0) \in Q$, denote

$$B_R = B(x_0,R) = \{x \in R^n, |x - x_0| < R\}$$
$$I_R = I(t_0,R) = \{t \in R, t_0 - R^2 < t < t_0\}$$
$$Q_R = Q(z_0,R) = B(x_0,R) \times I(t_0,R)$$

We prove the main theorem:

Theorem 1.1 *Let $u \in V_N(Q)$ be a weak solution of system* (1.1). *Suppose that A_i^α and B_i satisfy*

(H_1) $|A_i^\alpha(z,u,p)| \leq C(|p|+1)$

(H_2) $\dfrac{\partial A_i^\alpha(z,u,p)}{\partial p_\beta^j}\xi_\alpha^i\xi_\beta^j \geq \lambda|\xi|^2, \quad \lambda > 0, \quad \forall\xi \in R^{nN}$

(H_3) $A_i^\alpha(z,u,p)(i = 1,\cdots,N; \alpha = 1,\cdots,n)$ *are of class C^1 with bounded continuous derivative*

$$\left|\frac{\partial A_i^\alpha}{\partial p_\beta^j}\right| \leq L$$

(H_4) $(1+|p|)^{-1}A_i^\alpha(z,u,p)$ *are Hölder continuous in (z,u) uniformly with respect to p, i.e.,*

$$|A_i^\alpha(z,u,p) - A_i^\alpha(y,v,p)| \leq c(1+|p|)\eta(|u|,|z-y|^2 + |u-v|^2)$$

where $\eta(s_1,s_2) \leq K(s_1)\min(s_2^{\gamma/2},L)$ for some γ, $0 < \gamma < 1$ and $L > 0$, $K(t)$ is an increasing function,

(H_5) $|B_i(z,u,p)| \leq a(|p|^2 + b)$, $2aM < \lambda$, $\sup\limits_Q|u| = M$

or

$$|B_i(z,u,p)| \leq C(|p|^{2-\delta} + b)$$

Then the first derivatives $D_\alpha u^i$ of u are Hölder continuous in an open set $Q_0 \subset Q$ with meas $(Q \setminus Q_0) = 0$.

In proving the theorem stated above, we need the following Lemma which can be found in [5].

Lemma 1.1 *Suppose that* (1.2)–(1.4) *and* $2aM < \lambda$ (*or* (1.2), (1.3), (1.4)$'$) *hold with* $\sup_Q |u| = M$, $\sigma > 2$ *and* $\tau > 1$. *Then there exists a* $p > 2$ *such that* $D_\alpha u^i \in L^p_{\mathrm{loc}}(Q)$, *and for every* $Q(z_0, 4R) \subset Q$ *we have*

$$\Big(\int_{Q_R} |Du|^p dz\Big)^{\frac{1}{p}} \le C\Big[\Big(\int_{Q_{4R}} |Du|^2 dz\Big)^{\frac{1}{2}} + \Big(\int_{Q_{4R}} F^p dz\Big)^{\frac{1}{p}}\Big]$$

where $R \le R_0$, R_0 *and* C *are constants independent of* u, *and*

$$F = |f| + \sum_{i,\alpha} |f_i^\alpha| + \sum_i |f_i|^{\frac{1}{2}} \quad (\textit{or } |f_0|^{\frac{1}{2}})$$

Remark 1.1 Suppose that (H_1), (H_2), and (H_5) in Theorem 1.1 hold. Then the inequality in Lemma 1.1 becomes

$$\Big(\int_{Q_R} (1 + |Du|)^p dz\Big)^{\frac{1}{p}} \le C\Big(\int_{Q_{4R}} (1 + |Du|^2) dz\Big)^{\frac{1}{2}}$$

2. Caccioppoli's Second Inequality

Denote

$$u_\sigma(t) = \frac{1}{|B_\sigma|}\int_{B_\sigma} u dx, \quad B_\sigma = B_\sigma(x_0)$$

$$u_R = \frac{1}{|Q_R|}\int_{Q_R} u dz, \qquad Q_R = Q_R(z_0)$$

We have

Theorem 2.1 *Let* $u \in V_N(Q)$ *be a weak solution of system* (1.1). *Suppose that the conditions in Theorem* 1.1 *hold. Then for every* $z_0 \in Q$, *every* $p_0 \in R^{nN}$ *and every* r, R *with* $0 < r < \frac{R}{4}$ *and* $Q_R \subset Q$ *we have*

$$\int_{Q_r(z_0)} |Du - p_0|^2 dz \le C\Big\{\frac{1}{(R-r)^2}\int_{Q_{R/4}(z_0)} |u - u_{R/4}(t) - p_0(x - x_0)|^2 dz + R^{n+2+2\alpha} h(z_0, R)\Big\} \tag{2.1}$$

where $h(z_0, R) = h(1 + |u_R| + |(Du)_R| + \phi(z_0, R)^{1/2})$, $\alpha = \frac{\delta}{2}\Big(1 - \frac{2}{p}\Big)$.

Proof Without loss of generality we may assume $x_0 = 0$. Let $0 < r \le \rho < \sigma \le \frac{R}{4}$. Choose $\chi(x) \in C_0^\infty(B(0,\sigma))$ with $0 \le \chi \le 1$, $\chi \equiv 1$ in $B(0,\rho)$, $|D\chi| \le \frac{C}{(\sigma - \rho)}$ and satisfying

(i) $\chi(x) = \chi(-x)$

(ii) $\sup_{B_\sigma} \chi(x) \le C \int_{B_\sigma} \chi(x)dx$

Define

$$u_{\chi,\sigma}(t) = \int_{B_\sigma} \chi^2 u dx \Big/ \int_{B_\sigma} \chi^2 dx, \quad U(x,t) = u(x,t) - p_0 x$$

Then from (i) (ii) it follows that

$$U_{\chi,\sigma}(t) = u_{\chi,\sigma}(t) \tag{2.2}$$

$$\int_{Q_\sigma(z_0)} |U(x,t) - U_{\chi,\sigma}(t)|^2 dz \le C \int_{Q_\sigma(z_0)} |U(x,t) - U_\sigma(t)|^2 dz \tag{2.3}$$

i.e.

$$\int_{Q_\sigma(z_0)} |u(x,t) - u_{\chi,\sigma}(t) - p_0 x|^2 dz \le C \int_{Q_\sigma(z_0)} |u(x,t) - u_\sigma(t) - p_0 x|^2 dz \tag{2.4}$$

Let $s \in (t_0 - \rho^2, t_0)$. Choose $\tau_m \in C_0^\infty\left(t_0 - \sigma^2, s + \frac{1}{m}\right)$ satisfying $\tau_m = 1$ in $[t_0 - \rho^2, s]$, $0 \le \tau_m \le 1$, $0 \le \tau_m'(t) \le C/(\sigma - \rho)^2$ when $t \in (t_0 - R^2, t_0 - \rho^2)$ and $\tau_m'(t) < 0$ when $t \in \left(s, s + \frac{1}{m}\right)$, $\frac{1}{m} < t_0 - s$. Choose $g_l \in C_0^\infty\left(-\frac{1}{l}, \frac{1}{l}\right)$ with $g_l(t) = g_l(-t) \ge 0$ and $\int_{-\infty}^{\infty} g_l(t)dt = 1$. Denote $u_l = u * g_l$, and define

$$\phi = \chi^2 \tau_m^2 [u_l - u_{\chi,\sigma}^l(t) - p_0 x] \tag{2.5}$$

$$\psi = (1 - \chi^2 \tau_m^2)[u_l - u_{\chi,\sigma}^l(t) - p_0 x] \tag{2.6}$$

where $p_0 \in R^{nN}$, $u_{\chi,\sigma}^l(t) = [u_{\chi,\sigma}(t)] * g_l = (u * g_l)_{\chi,\sigma}(t)$. From (2.5) (2.6) we have

$$\phi + \psi = u_l - u_{\chi,\sigma}^l(t) - p_0 x$$

$$D\phi + D\psi = Du_l - p_0$$

Choose test function ϕ_l in $(1.1)'$ (spt $\phi_l \subset Q$ for l sufficiently large) we have

$$\int_{Q_\sigma} A_i^\alpha(z, u, Du) D_\alpha \phi_l^i dz = \int_{Q_\sigma} B_i(z, u, Du)\phi_l^i dz + \int_{Q_\sigma} u^i (\phi_l^i)_t' dz \tag{2.7}$$

Notice that

$$\begin{aligned}
\int_{Q_\sigma} u^i\, (\phi_l^i)_t' dz &= \int_{Q_\sigma} u_l^i \phi_t^i dz \\
&= \int_{Q_\sigma} [u_l - u_{\chi,\sigma}^l(t) - p_0 x][\chi^2 \tau_m^2 (u_l - u_{\chi,\sigma}^l(t) - p_0 x)]_t' dz \\
&\quad - \int_{Q_\sigma} [u_{\chi,\sigma}^l(t) + p_0 x]_t' [\chi^2 \tau_m^2 (u_l - u_{\chi,\sigma}^l(t) - p_0 x)] dz
\end{aligned} \tag{2.8}$$

in which the last term vanishes as shown by an easy calculation. Therefore we have

$$\int_{Q_\sigma} u^i(\phi_l^i)'_t dz = \frac{1}{2}\int_{Q_\sigma} |u_l - u^l_{\chi,\sigma}(t) - p_0 x|^2 (\chi^2 \tau_m^2)'_t dz = (*)$$

Let $l \to \infty$ (Note that ϕ, ψ are independent of l below, and safisfy $D\phi + D\psi = Du - p_0$) for $z_0 \in Q$, $u_0 \in R^N$ we have

$$\int_{Q_\sigma} A_i^\alpha(z, u, Du) D_\alpha \phi^i dz$$

$$= -\int_{Q_\sigma} [A_i^\alpha(z_0, u_0, Du) - A_i^\alpha(z, u, Du)] D_\alpha \phi^i dz + \int_{Q_\sigma} A_i^\alpha(z_0, u_0, Du) D_\alpha \phi^i dz$$

$$= -\mathrm{I} + \int_{Q_\sigma} A_i^\alpha(z_0, u_0, Du - D\psi) D_\alpha \phi^i dz$$

$$+ \int_{Q_\sigma}\int_0^1 \frac{\partial A_i^\alpha(z_0, u_0, Du - \theta D\psi)}{\partial p_\beta^j} d\theta\, D_\beta \psi^j D_\alpha \phi^i dz$$

$$= -\mathrm{I} + \int_{Q_\sigma} A_i^\alpha(z_0, u_0, p_0 + D\phi) D_\alpha \phi^i dz - \mathrm{II}$$

$$= -\mathrm{I} - \mathrm{II} + \int_{Q_\sigma} [A_i^\alpha(z_0, u_0, p_0 + D\phi) - A_i^\alpha(z_0, u_0, p_0)] D_\alpha \phi^i dz$$

$$\geq -\mathrm{I} - \mathrm{II} + \int_{Q_\sigma} [A_i^\alpha(z_0, u_0, p_0 + D\phi) - A_i^\alpha(z_0, u_0, p_0)] D_\alpha \phi^i dz$$

$$\geq -\mathrm{I} - \mathrm{II} + \lambda \int_{Q_\sigma} |D\phi|^2 dz \tag{2.9}$$

and

$$(*) \overset{l\to\infty}{\longrightarrow} \int_{Q_\sigma} |u - u_{\chi,\sigma}(t) - p_0 x|^2 \chi^2 \tau_m \tau'_m dz$$

$$= \int_{t_0-\sigma^2}^{t_0-\rho^2} dt \int_{B_\sigma} |u - u_{\chi,\sigma}(t) - p_0 x|^2 \chi^2 \tau_m \tau'_m dx$$

$$+ \int_s^{s+\frac{1}{m}} dt \int_{B_\sigma} |u - u_{\chi,\sigma}(t) - p_0 \chi|^2 \chi^2 \tau_m \tau'_m dx$$

combining (2.7), (2.8), (2.9) with $(*)$ we have

$$\lambda \int_{Q_\sigma} |D\,\phi|^2 dz - \int_s^{s+\frac{1}{m}} dt \int_{B_\sigma} |u - u_{\chi,\sigma}(t) - p_0 x|^2 \chi^2 \tau_m \tau'_m dx$$

$$\leq \mathrm{I} + \mathrm{II} + \int_{Q_\sigma} B_i(z, u, Du) \phi^i dz$$

$$+ \int_{t_0-\sigma^2}^{t_0-\rho^2} dt \int_{B_\sigma} |u - u_{\chi,\sigma}(t) - p_o x|^2 \chi^2 \tau_m \tau'_m dx \tag{2.10}$$

We first estimate the terms in the right-hand side. It is obvious that

$$
\begin{aligned}
\mathrm{I} &= \int_{Q_\sigma} [A_i^\alpha(z_0, u, Du) - A_i^\alpha(z, u, Du)] D_\alpha \phi^i dz \\
&\le C \int_{Q_\sigma} (1 + |Du|) \eta(|u|, |z - z_0|^2 + |u - u_0|^2) |D\phi| dz \\
&\le C(\varepsilon) \int_{Q_\sigma} (1 + |Du|^2) \eta dz + \varepsilon \int_{Q_\sigma} |D\phi|^2 dz
\end{aligned}
$$

using the condition (H_3) in Theorem 1.1, we have

$$
\begin{aligned}
\mathrm{II} &= \int_{Q_\sigma} \int_0^1 \frac{\partial A_i^\alpha(z_0, u_0, Du - \theta D\psi)}{\partial p_\beta^j} d\theta D_\beta \psi^j D_\alpha \phi^i dz \\
&\le C \int_{Q_\sigma} |D\psi||D\phi| dz
\end{aligned}
$$

Notice that $|D\psi|$, $|D\phi| \le C|Du - p_0| + \dfrac{C}{(\sigma - \rho)}|u - u_{\chi,\sigma}(t) - p_0 x|$ and supp $D\psi \subset Q_\sigma \setminus Q_\rho$, we have

$$
\mathrm{II} \le C \left\{ \frac{1}{(\sigma - \rho)^2} \int_{Q_\sigma} |u - u_{\chi,\sigma}(t) - p_0 x|^2 dz + \int_{Q_\sigma \setminus Q_\rho} |Du - p_0|^2 dz \right\} \tag{2.11}
$$

By the condition (H_5) in Theorem 1.1 we have

$$
\int_{Q_\sigma} B_i(z, u, Du) \phi^i dz \le C \int_{Q_\sigma} (1 + |Du|^2) |\phi| dz \tag{2.12}
$$

we know that

$$
0 \le \tau_m'(t) \le C/(\sigma - \rho)^2 \qquad \text{in } (t_0 - \sigma^2, t_0 - \rho^2) .
$$

and hence

$$
\begin{aligned}
&\int_{t_0 - \sigma^2}^{t_0 - \rho^2} dt \int_{B_\sigma} |u - u_{\chi,\sigma}(t) - p_0 x|^2 \chi^2 \tau_m \tau_m' dx \\
&\qquad \le \frac{C}{(\sigma - \rho)^2} \int_{Q_\sigma} |u - u_{\chi,\sigma}(t) - p_0 x|^2 dz
\end{aligned} \tag{2.13}
$$

From (2.10)-(2.13) and the estimate of I, we have

$$
\begin{aligned}
&\int_{t_0 - \rho^2}^{s} dt \int_{B_\sigma} |D\phi|^2 dx - \int_s^{s + \frac{1}{m}} \tau_m \tau_m' dt \int_{B_\sigma} |u - u_{\chi,\sigma}(t) - p_0 x|^2 \chi^2 dx \\
&\quad \le C \left\{ \frac{1}{(\sigma - \rho)^2} \int_{Q_\sigma} |u - u_{\chi,\sigma}(t) - p_0 x|^2 dz + \int_{Q_\sigma \setminus Q_\rho} |Du - p_0|^2 dz \right\} \\
&\quad\quad + C \left\{ \int_{Q_\sigma} (1 + |Du|^2) \eta dz + \int_{Q_\sigma} (1 + |Du|^2) |\phi| dz \right\}
\end{aligned} \tag{2.14}
$$

Let $s \to t_0$ and take into account $\tau'_m < 0$ in $\left(s, s+\frac{1}{m}\right)$ and $|D\phi| = |Du - \rho_0|$ in $(t_0 - \rho^2, s) \cap Q_\rho$, we have

$$\int_{Q_\rho} |Du - p_0|^2 dz \le C\left\{\frac{1}{(\sigma-\rho)^2}\int_{Q_\sigma} |u - u_{X,\sigma}(t) - p_0 x|^2 dz + \int_{Q_\sigma \setminus Q_\rho} |Du - p_0|^2 dz\right\} + A + B$$

where

$$A = C\int_{Q_\sigma} (1 + |Du|^2)\eta(|u|, |z - z_0|^2 + |u - u_0|^2)dz$$

$$B = C\int_{Q_\sigma} (1 + |Du|^2)|\phi|dz$$

Choosing $u_0 = u_R = u_{z,R} = \fint_{Q_R(x_0)} udz$ and using Remark 2.1 we have

$$A \le C\sigma^{n+2}\Big(\fint_{Q_\sigma} (1 + |Du|)^p dz\Big)^{\frac{2}{p}}\Big(\fint_{Q_\sigma} \eta(|u|, |z - z_0|^2 + |u - u_R|^2)dz\Big)^{1-\frac{2}{p}}$$

$$\le CR^{n+2}\fint_{Q_R} (1 + |Du|^2)dz\eta\Big(|u_R|, R^2 + \fint_{Q_R} |u - u_R|^2 dz\Big)^{1-\frac{2}{p}}$$

Using the following inequality

$$\fint_{Q_R} |u - u_R|^2 dz \le \fint_{Q_R} |u - u_{X,R}(t)|^2 dz \le CR^2\fint_{Q_R} |Du|^2 dz$$

we have

$$A \le R^{n+2+2\alpha}h(|u_R| + |(Du)_R| + \phi(z_0, R)^{1/2})$$

where $\alpha = \delta\left(\frac{1}{2} - \frac{1}{p}\right)$ and $h(t)$ is an increasing function.

Without loss of generality we can suppose $1 - \frac{2}{p} \le \frac{1}{2}$. Choosing $p_0 = (Du)_R$, we obtain

$$B \le CR^{n+2+2\alpha}\fint_{Q_R} (1 + |Du|^2)dz\Big[\fint_{Q_R} |Du|^2 dz + |p_0|^{\frac{1}{1-\frac{2}{p}}}\Big]^{1-\frac{2}{p}}$$

$$\le R^{n+2+2\alpha}h(|(Du)_R| + \phi(z_0, R)^{1/2})$$

where $\alpha = 1 - \frac{2}{p}$.

From the estimates of A and B we have

$$\int_{Q_\rho} |Du - \rho_0|^2 dz \le C\left\{\frac{1}{(\sigma-\rho)^2}\int_{Q_\sigma} |u - u_\sigma(t) - p_0 x|^2 dz + \int_{Q_\sigma \setminus Q_\rho} |Du - p_0|^2 dz\right\}$$
$$+ R^{n+2+2\alpha}h(z_0, R)$$

where $h(z_0,R)=h(|u_R|+|(Du)_R|+\phi(z_0,R)^{1/2})$ is an increasing function. Now we note that

$$\int_{Q_\sigma}|u-u_\sigma(t)-p_0x|^2dz\le\int_{Q_{R/4}}|u-u_{R/4}(t)-p_0x|^2dz \tag{2.15}$$

for $\sigma<\frac{R}{4}$.

Using hole-filling technique and Lemma 3.3 of Chap. V in [2] we finally get (2.1).

Let $m\to\infty$ in (2.14) we get

$$\int_{Q_\sigma}|u(x,s)-u_{\chi,\sigma}(s)\ -p_0x|^2\chi^2dx\le C\left\{\frac{1}{(\sigma-\rho)^2}\int_{Q_\sigma}|u-u_{\chi,\sigma}(t)-p_0x|^2dz\right.$$
$$\left.+\int_{Q_\sigma\setminus Q_\rho}|Du-p_0|^2dz\right\}+R^{n+2+2\alpha}h(z_0,R) \tag{2.16}$$

Choosing $\rho=\frac{R}{8}$, $\sigma=\frac{R}{4}$ and noting

$$\frac{C}{R^2}\int_{Q_{R/4}}|u-u_{\chi,\frac{R}{4}}(t)-p_0x|^2dz\le C\int_{Q_{R/4}}|Du-p_0|^2dz$$

let s run over $(t_0-\rho^2,t_0)$, we have

Theorem 2.2 *Suppose that the conditions in Theorem 1.1 hold. Let $u\in V_N(Q)$ be a weak solution of systems (1.1). Then for every $R<R_0\wedge$ dist (z_0,∂_pQ), we have*

$$\sup_{t\in I_{R/8}}\int_{B_{R/8}}|u(x,t)-u_{\chi,\frac{R}{4}}(t)-p_0x|^2dx$$
$$\le C\left\{\int_{Q_{\frac{R}{4}}}|Du-p_0|^2dz\right\}+R^{n+2+2\alpha}h(z_0,R) \tag{2.17}$$

where χ is a cut-off function for $(B_{R/8},B_{R/4})$.

Denote $2^+=\frac{2n}{n+2}$.

Let $r=\frac{R}{8}$ in (2.1), we have

$$\int_{Q_{R/4}}|u-u_{R/4}(t)-p_0(x-x_0)|^2dz$$
$$=\int_{Q_{R/4}}U(x,t)-U_{R/4}(t)|^2dz\le\int_{Q_{R/4}}|U(x,t)-U_{R/2}(t)|^2dz$$
$$=\int_{Q_{R/4}}|u-u_{\chi,\frac{R}{2}}(t)-p_0(x-x_0)|^2dz$$

where χ is a cut-off function for $(B_{R/4}, B_{R/2})$, from (2.1) we have

$$\int_{Q_{R/8}} |Du - p_0|^2 dz \le \frac{C}{R^2}\int_{Q_{R/4}} \int |U - U_{\chi,\frac{R}{2}}(t)|^2 dz + R^{2\alpha}h(z_0,R)$$

$$\le \frac{C}{R^2}\int_{I_{R/4}} \Big[\Big(\int_{B_{R/4}} |U - U_{\chi,\frac{R}{2}}(t)|^2 dx\Big)^{1-\frac{2^+}{2}} \Big(\int_{B_{R/4}} |U - U_{\chi,\frac{R}{2}}(t)|^2 dx\Big)^{\frac{2^+}{2}}\Big] dt$$

$$+R^{2\alpha}h(z_0,R)$$

$$\le \frac{C}{R^2}\sup_{t\in I_{R/4}} \Big(\int_{B_{R/4}} |U - U_{\chi,\frac{R}{2}}(t)|^2 dx\Big)^{1-\frac{2^+}{2}} \int_{I_{R/4}} \Big(\int_{B_{\frac{R}{2}}} |U - U_{\chi,\frac{R}{2}}(t)|^2 dx\Big)^{\frac{2^+}{2}} dt$$

$$+R^{2\alpha}h(z_0,R)$$

$$\le \frac{C}{R^2}\Big(R^2\int_{Q_{R/2}} |Du - p_0|^2 dz + R^{2+2\alpha}h(z_0,R)\Big)^{1-\frac{2^+}{2}} \int_{I_{R/4}} R^{2^+}\Big(\int_{B_{R/2}} |Du$$

$$-p_0|^{2^+} dx\Big) dt + R^{2\alpha}h(z_0,R)$$

Using Young's inequality we have

$$\int_{Q_{R/8}} |Du - p_0|^2 dz \le C\Big(\int_{Q_R} |Du - p_0|^{2^+} dz\Big)^{\frac{2}{2^+}} + \theta\int_{Q_R} |Du - p_0|^2 dz + R^{2\alpha}h(z_0,R)$$

$$\text{with } \theta < 1 \tag{2.18}$$

Finally we use Prop. 1.3 in [4] and have the following theorem.

Theorem 2.3 (Reverse Hölder inequality) *Let $u \in V_N(Q)$ be a weak solution of system (1.1). Suppose that the conditions in Theorem 1.1 hold. Then there exists a $q > 2$ $(q < p)$ such that*

$$\Big(\int_{Q_{R/8}} |Du - (Du)_{R/8}|^q dz\Big)^{\frac{2}{q}} \le C\int_{Q_R} |Du - (Du)_R|^2 dz + R^{2\alpha}h(z_0,R) \tag{2.19}$$

where α is similar to that in Theorem 2.1.

3. Partial Regularity

In this section we will prove Theorem 1.1. First we have the following proposition:

Proposition 3.1 *Let $u \in V_N(Q)$ be a weak solution of system (1.1) with $\sup_Q |u| = M$. Suppose that the conditions in Theorem 1.1 hold. Then there exists an $\alpha \in (0,1)$, such that for every $z_0 \in R^{n+1}$ and $0 < \rho < R < \min(1, \text{dist }(z_0, \partial_p Q))$, we have*

$$\int_{Q_\rho} |Du - (Du)_\rho|^2 dz \le C\Big[\Big(\frac{\rho}{R}\Big)^{n+4} + \omega(z_0,R)\Big]\int_{Q_R} |Du - (Du)_R|^2 dz$$

$$+R^{n+2+2\alpha}H(z_0,R) \tag{3.1}$$

where

$$\omega(z_0,R)=\omega[C(n)(1+|u_R|+|(Du)_R|+\phi(z_0,R)^{1/2},\phi(z_0,R)]$$
$$H(z_0,R)=H(|u_R|+|(Du)_R|+\phi(z_0,R)^{1/2})$$

with $\omega(s_1,s_2)$ *is an increasing function in* s_1, *and going to zero as* $s_2\to 0$ *uniformly for* s_1 *in a bounded set,* $H(s)$ *is an increasing function of* s, *and* $\phi(z_0,R)=\int_{Q_R}|Du-(Du)_R|^2dz$.

Proof When no confusion exists we will omit the subindex z_0 in $u_{z_0,R}$ and $(Du)_{z_0,R}$. Denote

$$A^{\alpha\beta}_{ij0}=A^{\alpha}_{ip^j_\beta}(z_0,u_R,(Du)_{R/8})$$

$$\tilde{A}^{\alpha\beta}_{ij}=\int_0^1 A^{\beta}_{ip^j_\beta}(z_0,u_R,(Du)_{R/8}+t(Du-(Du)_{R/8}))dt$$

Then system (1.1) can be rewritten as

$$u^i_t-D_\alpha\Big[A^{\alpha\beta}_{ij0}D_\beta u^j\Big]=-D_\alpha\Big\{\Big[A^{\alpha\beta}_{ij0}-\tilde{A}^{\alpha\beta}_{ij}\Big]\Big[D_\beta u^j-(D_\beta u^j)_{R/8}\Big]\Big\}$$
$$-D_\alpha\Big\{A^\alpha_i(z_0,u_R,Du)-A^\alpha_i(z,u,Du)\Big\}+B_i(z,u,Du) \tag{3.2}$$

let v be the solution of the Dirichlet problem

$$\begin{cases} v^i_t-D_\alpha(A^{\alpha\beta}_{ij0}D_\beta v^j)=0 & \text{in } Q_{R/8}(z_0)\\ v-u=0 & \text{on } \partial_p Q_{R/8}\end{cases} \tag{3.3}$$

For all $\rho<\dfrac{R}{8}$ we have

$$\int_{Q_\rho}|Dv-(Dv)_\rho|^2dz\le C\Big(\frac{\rho}{R}\Big)^{n+4}\int_{Q_{R/8}}|Dv-(Dv)_{R/8}|^2dz \tag{3.4}$$

Let $w=u-v$. We know that

$$\int_{Q_\rho}|Du-(Du)_\rho|^2dz\le C\Big(\frac{\rho}{R}\Big)^{n+4}\int_{Q_R}|Du-(Du)_R|^2dz+C\int_{Q_{R/8}}|Dw|^2dz$$

Obviously $w\in W(Q_{R/8})$, and for all $\varphi\in W(Q_{R/8})\cap L^\infty(Q_{R/8},R^N)$ with $\varphi(x,t_0)=0$, w satisfies

$$\int_{Q_{R/8}}A^{\alpha\beta}_{ij0}D_\beta w^jD_\alpha\varphi^idz-\int_{Q_{R/8}}w^i\varphi^i_tdz$$
$$=\int_{Q_{R/8}}[A^{\alpha\beta}_{ij0}-\tilde{A}^{\alpha\beta}_{ij}][D_\beta u^j-(D_\beta u^j)_{R/8}]D_\alpha\varphi^idz$$
$$+\int_{Q_{R/8}}[A^\alpha_i(z_0,u_R,Du)-A^\alpha_i(z,u,Du)]D_\alpha\varphi^idz+\int_{Q_{R/8}}B_i(z,u,Du)\varphi^idz \tag{3.5}$$

From Lemma 7 in [5] we have

$$\int_{Q_{R/8}} |Dw|^2 dz \le C \int_{Q_{R/8}} |A_{ij0}^{\alpha\beta} - \tilde{A}_{ij}^{\alpha\beta}|^2 |Du - (Du)_{R/8}|^2 dz$$
$$+C \int_{Q_{R/8}} |A_i^\alpha(z_0, u_R, Du) - A_i^\alpha(z, u, Du)|^2 dz$$
$$+ \int_{Q_{R/8}} |B_i(z, u, Du)||w| dz$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III} \tag{3.6}$$

From the assumption (H_3) in Theorem 1.1 it follows that there exists a nonnegative bounded and continuous function $\omega(s_1, s_2)$ such that:

a) $\omega(s_1, s_2)$ is increasing in s_1 for fixed s_2 and in s_2 for fixed s_1

b) $\omega(s_1, s_2)$ is concave in s_2 for fixed s_1,

c) $\omega(s_1, 0) = 0$,

d) for every $(z, u, p), (y, v, p) \in Q \times R^N \times R^{nN}$ with $|u| + |p| \le M$ and for every $i, j = 1, \cdots, N$; $\alpha, \beta = 1, \cdots, n$, it holds that

$$|A_{ip_\beta^j}^\alpha(z, u, p) - A_{ip_\beta^j}^\alpha(y, v, q)| \le \omega(M, |z - y|^2 + |u - v|^2 + |p - q|^2)$$

Therefore, by using reverse Hölder inequality (2.19) and the boundedness of ω we have

$$\mathrm{I} \le \Big(\int_{Q_{R/8}} |Du - (Du)_{R/8}|^\pi dz\Big)^{\frac{2}{\pi}} \Big(\int_{Q_{R/8}} \omega dz\Big)^{1-\frac{2}{\pi}}$$
$$\le C\Big[\int_{Q_R} |Du - (Du)_R|^2 dz + R^{n+2+2\alpha} h(z_0, R)\Big]\Big(\int_{Q_{R/8}} \omega dz\Big)^{1-\frac{2}{\pi}}$$
$$\le \omega[C(n)(1 + |u_R| + |(Du)_R| + \phi(z_0, R)^{1/2}), \phi(z_0, R)]^{1-\frac{2}{\pi}} \int_{Q_R} |Du - (Du)_R|^2 dz$$
$$+ R^{n+2+2\alpha} H(z_0, R) \tag{3.7}$$

where $\alpha = \frac{\delta}{2}\Big(1 - \frac{2}{p}\Big)$, and

$$\mathrm{II} \le \int_{Q_{R/8}} (1 + |Du|)^2 \eta dz \le C R^{n+2} \Big(\int_{Q_{R/8}} (1 + |Du|)^p dz\Big)^{\frac{2}{p}} \Big(\int_{Q_{R/8}} \eta dz\Big)^{1-\frac{2}{p}}$$
$$\le R^{n+2+2\alpha} H(z_0, R) \tag{3.8}$$

Using the assumption (H_5) in Theorem 1.1 and the boundedness of ω, letting $1 - \frac{2}{p} \le \frac{1}{2}$ and noting

$$\int_{Q_R} |Dw|^2 dz \le \int_{Q_R} (1 + |Du|^2) dz$$

We get

$$\mathrm{III} \leq C \int_{Q_{R/8}} (1+|Du|^2)|w|dz \leq C\Big(\int_{Q_{R/8}} (1+|Du|^2)^{\frac{p}{2}} dz\Big)^{\frac{2}{p}} \Big(\int_{Q_{R/8}} |w|^{\frac{1}{1-\frac{2}{p}}} dz\Big)^{1-\frac{2}{p}}$$

$$\leq C \int_{Q_R} (1+|Du|^2)dz \Big(R^2 \int_{Q_R} |Dw|^2 dz\Big)^{1-\frac{2}{p}}$$

$$\leq R^{n+2+2\alpha} H(z_0, R), \quad \alpha = \Big(1 - \frac{2}{p}\Big) \tag{3.9}$$

From (3.6) and the estimates of I, II, III, it follows that

$$\int_{Q_{R/8}} |Dw|^2 dz \leq \omega[C(n)(1+|u_R| + |(Du)_R| + \phi(z_0,R)^{1/2}),\ \phi(z_0,R)]^{1-\frac{2}{\pi}}$$

$$\cdot \int_{Q_R} |Du - (Du)_R|^2 dz + R^{n+2+2\alpha} H(z_0,R), \quad \alpha = \frac{\delta}{2}\Big(1-\frac{2}{p}\Big) \tag{3.10}$$

From (3.5) and (3.10) we see that (3.1) holds for $\rho < \frac{R}{8}$, (3.1) is obvious for $\frac{R}{8} \leq \rho < R$.

Finally, similar to [4] and Chap. VI in [2], we can prove Theorem 1.1.

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