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THE SEMI-GLOBAL ISOMETRIC IMBEDDING IN R^3 OF TWO DIMENSIONAL RIEMANNIAN MANIFOLDS WITH GAUSSIAN CURVATURE CHANGING SIGN CLEANLY*

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Dedicated to the 70th birthday of Professor Zhou Yulin

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Abstract An abstract Riemannian metric $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ is given in $(u, v) \in [0, 2\pi] \times [-\delta, \delta]$ where E, F, G are smooth functions of (u, v) and periodic in u with period 2π. Moreover $K|_{v=0} = 0$, $K_v|_{v=0} \neq 0$, where K is the Gaussian curvature. We imbed it semiglobally as the graph of a smooth surface $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ of R^3 in the neighborhood of $v = 0$.

In this paper we show that, if $[K_v\Gamma_{11}^2]_{v=0} < 0$ and three compatibility conditions are satisfied, then there exists such an isometric imbedding.

1. Introduction

Let

$$ds^2 = E(u, v)du^2 + 2F(u, v)dudv + G(u, v)dv^2 \tag{1}$$

be a sufficiently smooth Riemannian metric in $(u, v) \in [0, 2\pi] \times I_\delta$, where $I_\delta = [-\delta, \delta]$ and E, F, G are periodic functions of u with period 2π.

Consider the isometric imbedding problem in the neighborhood of $\Lambda = [0, 2\pi] \times \{0\}$, i.e. realizing ds^2 in $[0, 2\pi] \times I_{\delta_1}$ $(0 < \delta_1 < \delta)$ as the graph of a smooth surface $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ in R^3 such that $ds^2 = dx^2 + dy^2 + dz^2$. It is well known that the above problem was solved by [1], [2] for the cases of Gaussian curvature $K(u, v) > 0$ or $K(u, v) < 0$ respectively. And it was solved by [3] for the case $K(p) = 0$, $DK(p) \neq 0$ in $I_\eta \times \{0\}$ (η is small). In this paper we solve the isometric imbedding problem in the neighborhood of Λ with $K|_\Lambda = 0$, $K_v|_\Lambda \neq 0$ and K_v has different sign with Γ_{11}^2 on $v = 0$. In case K_v has the same sign with Γ_{11}^2, the semi-global imbedding problem is still open. The reason is, in the later case to solve z reduced to Tricomi mixed type equation, it is difficult to treat for periodic case, while for the former case, it reduced to a Buseman mixed type equation and easily to be solved.

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2. Necessary Conditions for Imbedding

It is well known that

$$K = -\frac{1}{2}(EG - F^2)^{-1}(E_{vv} - 2F_{uv} + G_{uu}) + \frac{1}{2}\Gamma^1_{22}E_v + \frac{1}{2}\Gamma^2_{12}G_u + \frac{1}{2}\Gamma^1_{11}(G_u - 2E_v) + \frac{1}{2}\Gamma^2_{11}G_v \quad (2)$$

where $\Gamma^i_{jk}(1 \le i, j, k \le 2)$ are the Christoffel symbols, i.e.

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}(GE_u - 2FF_u + FE_v)/(EG - F^2) \\
\Gamma^2_{11} &= \frac{1}{2}(2EF_u - FE_u - EE_v)/(EG - F^2) \\
\Gamma^1_{12} &= \frac{1}{2}(GE_v - FG_u)/(EG - F^2) \\
\Gamma^2_{12} &= \frac{1}{2}(FG_u - FE_v)/(EG - F^2) \\
\Gamma^1_{22} &= \frac{1}{2}(2GF_v - GG_u - FG_v)/(EG - F^2) \\
\Gamma^2_{22} &= \frac{1}{2}(FG_u - 2FF_v + EG_v)/(EG - F^2)
\end{aligned} \quad (3)$$

Let $z(u, v)$ be an arbitrary smooth function of u, v and let the metric g be

$$g = ds^2 - dz^2 = (E - z_u^2)du^2 + 2(F - z_u z_v)dudv + (G - z_v^2)dv^2$$

Assume that g is flat. It means the Gaussian curvature $K_g = 0$. The condition for $K_g = 0$ is equivalent to[3]

$$(z_{uu} - \Gamma^1_{11}z_u - \Gamma^2_{11}z_v)(z_{vv} - \Gamma^1_{22}z_u - \Gamma^2_{22}z_v) - (z_{uv} - \Gamma^1_{12}z_u - \Gamma^2_{12}z_v)^2 - [EG - F^2 - (Gz_u^2 - 2Fz_u z_v + Ez_v^2)]K = 0 \quad (4)$$

Theorem 1 *If there exist smooth isometric imbedding functions $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ in the small neighborhood of Λ and periodic with period 2π in u and $z = O(v^2)$, then we have*

$$[K_v\Gamma^2_{11}]_\Lambda < 0 \quad (5)$$

$$\int_0^{2\pi} [|\Gamma^2_{11}|(EG - F^2)^{1/2}/E]_{v=0}du = 2\pi \quad (6)$$

$$\int_0^{2\pi} E(u, 0)^{1/2} \exp\left\{\sqrt{-1}\int_0^u [|\Gamma^2_{11}|(EG - F^2)^{1/2}/E]_{v=0}du\right\} = 0 \quad (7)$$

Proof Differentiating (4) with respect to v and taking $v = 0$, we have

$$-[\Gamma_{11}^2 z_{vv}^2]_{v=0} = [(EG - F^2)K_v]_{v=0} \neq 0$$

Hence

$$\Gamma_{11}^2(u,0) \neq 0, \quad z_{vv}(u,0) \neq 0$$

and (5) follows.

Since

$$EG - F^2 = (x_u y_v - y_u x_v)^2 + (y_u z_v - z_u y_v)^2 + (z_u x_v - x_u z_v)^2$$

we have

$$|x_u y_v - y_u x_v|_{v=0} = (EG - F^2)^{1/2}|_{v=0}$$

$$\begin{aligned}\Gamma_{11}^2 &= \Big[E\Big(F_u - \frac{1}{2}E_v\Big) - \frac{1}{2}FE_u\Big]/(EG - F^2) \\ &= [(x_u^2 + y_u^2 + z_u^2)(x_{uu}x_v + y_{uu}y_v + z_{uu}z_v) \\ &\quad -(x_u x_v + y_u y_v + z_u z_v)(x_u x_{uu} + y_u y_{uu} + z_u z_{uu})]/(EG - F^2) \\ &= [(x_u y_{uu} - y_u x_{uu})(x_u y_v - y_u x_v)/(x_u^2 + y_u^2)^{3/2} + o(1)]E^{3/2}/(EG - F^2) \\ &= K_\Lambda E^{3/2}/(EG - F^2)^{1/2}|_{v=0} + o(1)\end{aligned}$$

in the neighborhood of Λ, where K_Λ is the curvature of curve Λ-the image curve of Λ in the (x, y) place. Hence

$$[\Gamma_{11}^2(EG - F^2)^{1/2}/E]_{v=0} = K_\Lambda(u,0)E(u,0)^{1/2} = \frac{d\theta}{du}$$

where θ is the turning angle of Λ. The conditions for Λ being a smooth closed curve are

$$\oint d\theta = +2\pi$$

$$\oint (dx + \sqrt{-1}dy) = \oint e^{\sqrt{-1}\theta} ds = \oint e^{\sqrt{-1}\theta} E(u,0)^{1/2} du = 0$$

Hence we obtain the compatibility conditions (6) and (7). The theorem is proved.

Theorem 2 *The form* (1) *can be reduced to*

$$ds^2 = B(U,V)^2 dU^2 + dV^2 \tag{8}$$

where $B(U,V)$ is a sufficiently smooth function of (U,V) in the small neighborhood of Λ, and $B(U,V)$ is periodic in U with period 2π. Moreover we have

$$B(U,V) = E(U,0)^{1/2} + \frac{\Gamma_{11}^2(U,0)}{E(U,0)}[E(U,0)G(U,0) - F(U,0)^2]^{1/2}V + O(V^2) \tag{9}$$

Proof Comparing (1) with (8) we see that $U(u,v)$, $V(u,v)$, $B(u,v)$ must satisfy

$$B^2U_u^2+V_u^2=E,\quad B^2U_uU_v+V_uV_v=F,\quad B^2U_v^2+V_v^2=G$$

Eliminating B,U, we have

$$(E-V_u^2)(G-V_v^2)=(F-V_uV_v)^2$$

or

$$GV_u^2-2FV_uV_v+EV_v^2=EG-F^2 \tag{10}$$

Solve (10) with the initial stripe conditions

$$V|_{v=0}=0,\quad V_v|_{v=0}=[(EG-F^2)/E]^{1/2}|_{v=0} \tag{11}$$

It is easy to verify that (11) is not the characteristic stripe of (10). Hence by the theory of first order PDE, the solution of (10), (11) exists and is unique. Since (10) has periodic coefficients with respect to u, and the situation is similar to that in (11). Hence V is a periodic function of u with period 2π. Then solve U by

$$U_v=\frac{F-V_uV_v}{E-V_u^2}U_u \tag{12}$$

with the initial value

$$U|_{v=0}=u \tag{13}$$

Since

$$[E-U_u^2]_{v=0}=E(u,0)>0$$

there has $\delta_1>0$ such that $E-U_u^2>0$ in $[0,2\pi]\times I_{\delta_1}$. Therefore (12) with (13) has a unique solution. Since (12) has periodic coefficients and the initial value (13) satisfies

$$U(u+2\pi,0)=U(u,0)+2\pi$$

Hence we have

$$U(u+2\pi,v)=U(u,v)+2\pi$$

for $|v|<\delta_1$. Therefore U_u and U_v are periodic function of u. Consequently $(E-V_u^2)/U_u^2$ is a periodic function of u. Since $(E-V_u^2)/U_u^2|_{v=0}=E(u,0)>0$, there has a δ_2 such that $(E-V_u^2)/U_u^2>0$ in $[0,2\pi]\times I_{\delta_2}$.

Let $B=[(E-V_u^2)/U_u^2]^{1/2}$. It is easy to get the following expansions for U, V, B,

$$U=u+\frac{F(u,0)}{E(u,0)}v+O(v^2) \tag{14}$$

$$V=\Big[\frac{E(u,0)G(u,0)-F(u,0)^2}{E(u,0)}\Big]^{1/2}v+O(v^2) \tag{15}$$

$$B=E(u,0)^{1/2}+\Big\{\frac{E_v(u,0)}{2E(u,0)^{1/2}}-E(u,0)^{1/2}\Big[\frac{F(u,0)}{E(u,0)}\Big]_u\Big\}v+O(v^2) \tag{16}$$

Let

$$B = B_0(U) + B_1(U)V + O(V^2)$$
$$= B_0\Big(u + \frac{F(u,0)}{E(u,0)}v + O(v^2)\Big) + B_1\Big(u + \frac{F(u,0)}{E(u,0)}v + O(v^2)\Big)$$
$$\cdot\Big\{\Big[\frac{E(u,0)G(u,0) - F(u,0)^2}{E(u,0)}\Big]^{1/2} v + O(v^2)\Big\} + O(v^2) \tag{17}$$

Determine B_0, B_1 by comparing the coefficients of (16) with (17) we obtain (9). By the periodic properties of U, V, B we obtain that B is periodic with period 2π of U. The theorem is proved completely.

When ds^2 is represented by (8), (4) is reduced to

$$\Big(z_{UU} - \frac{B_U}{B}z_U + BB_V z_V\Big)z_{VV} - \Big(z_{UV} - \frac{B_V}{B}z_U\Big)^2 - [B^2(1 - z_V^2) - z_U^2]K = 0 \tag{18}$$

3. Solving Imbedding Function z

In the following we discuss how to find a periodic smooth solution of (18) in the neighborhood of Λ such that $z = O(V^2)$. The method is a variant of [4].

Substituting $z = V^2[z_2(U) + Z]$ into (18) and dividing it by $2BB_V V^2 z_2$ we obtain

$$\Big\{\frac{1}{z_2}\Big[\Big(\frac{z_2''}{2BB_V} - \frac{B_U z_2'}{2B^2 B_V}\Big)V + z_2 + \Big(\frac{1}{2BB_V}Z_{UU} - \frac{B_U}{2B^2B_V}Z_U\Big)V + Z + \frac{1}{2}VZ_V\Big]$$
$$\cdot\Big(\frac{2z_2}{V} + 2\frac{Z}{V} + 4Z_V + VZ_{VV}\Big) - \frac{2z_2}{V}\Big\}$$
$$-\frac{1}{2z_2BB_V}\Big[\Big(2 - \frac{B_V}{B}V\Big)z_2' + \Big(2 - \frac{B_V}{B}V\Big)Z_U + VZ_{UV}\Big]^2$$
$$-\frac{BK - 4B_V V z_2^2}{2z_2 B_V V^2} + \frac{1}{2z_2BB_V}\cdot[B^2(2z_2 + 2Z + VZ_V)^2 + V^2(z_2' + Z_U)^2]K = 0$$

Applying (9) under the assumption (5) we have

$$\frac{K_V}{BB_V}\Big|_{V=0} = \Big[\frac{K_v}{V_v B B_v}\Big]_{v=0} = \Big[\frac{EK_v}{(EG - F^2)\Gamma_{11}^2}\Big]_{v=0} > 0$$

Hence we can take $z_2 = \frac{1}{2}[B(U,0)K_V(U,0)/B_V(U,0)]^{1/2}$ and the above expression becomes

$$V(1 + f_{22}V)Z_{VV} + (5 + f_2V)Z_V + (4 + f_0V)\frac{Z}{V}$$
$$+\frac{1}{BB_V}Z_{UU} + f_1Z_U + f_{12}VZ_{UV} + \frac{1}{2z_2}\Big(2\frac{Z}{V} + 4Z_V + VZ_{VV}\Big)(2Z + VZ_V)$$
$$+\boldsymbol{F}(Z, VZ_V, V^2Z_{VV}, Z_U, Z_{UU}, VZ_{UV}) + f(U,V) = 0$$

where f_{ij}, $f_i (0 \le i,j \le 2)$ and f are bounded smooth functions of (U,V). And $\boldsymbol{F}$ is a bounded function of homogeneous quadratic terms of its arguments.

Let

$$V = \frac{\varepsilon^2}{2} Y, Z = \varepsilon w$$

$$X = c\int_0^U [B(U,0)B_V(U,0)]^{1/2} dU$$

where

$$c = 2\pi / \int_0^{2\pi} [B(U,0)B_V(U,0)]^{1/2} dU$$

We have

$$\begin{aligned} L(w) &= (1+\varepsilon^2 Y f_{22}) Y w_{YY} + 5(1+\varepsilon^2 Y f_2) w_Y + 4(1+\varepsilon^2 Y f_0)\frac{w}{Y} + \varepsilon^2(w_{XX} + f_1 w_X) \\ &\quad + \varepsilon^2 Y f_{12} w_{XY} + \frac{\varepsilon}{2z_2}\Big(2\frac{w}{Y} + 4w_Y + Y w_{YY}\Big)(2w + Y w_Y) \\ &\quad + \varepsilon^3 \tilde{F}(w, Y w_Y, Y^2 w_{YY}, w_X, w_{XX}, Y w_{XX}) + \varepsilon f(X, \varepsilon^2 Y) \\ &= (1+\varepsilon F_{22}) Y w_{YY} + (5+\varepsilon F_2) w_Y + (4+\varepsilon F_0)\frac{w}{Y} + \varepsilon^2(1+\varepsilon F_{11}) w_{YY} \\ &\quad + \varepsilon^2 F_1 w_X + \varepsilon^2 Y F_{12} w_{XY} + \varepsilon f(X, \varepsilon^2 Y) = 0 \end{aligned} \tag{19}$$

where $\tilde{F}$ has the similar property to F. And $F_{ij}(0 \le i, j \le 2)$ are bounded smooth linear functions with respect to w, Yw_Y, $Y^2 w_{YY}$, w_X, w_{XX}, Yw_{XY}. The linearized approximation of (19) is

$$0 = L(u + w^*) \approx L(w^*) + L'(w^*)u$$

where $w^* = w - u$, and

$$\begin{aligned} L'(w)u &= \lim_{t\to 0}[L(w+tu) - L(w)]/t \\ &= (1+\varepsilon\tilde{F}_{22}) Y u_{YY} + 5(1+\varepsilon\tilde{F}_2) u_Y + 4(1+\varepsilon\tilde{F}_0)\frac{u}{Y} \\ &\quad + \varepsilon^2(1+\varepsilon\tilde{F}_{11}) u_{XX} + \varepsilon^2 \tilde{F}_1 u_X + \varepsilon^2 Y \tilde{F}_{12} u_{XY} \end{aligned} \tag{20}$$

where $\tilde{F}_{ij}$ have the similar expressions to $F_{ij}(0 \le i, j \le 2)$.

We are going to study the linearized equation as follows:

$$L'(w)u = g(X, Y), \quad (X, Y) \in \bar{G} \equiv [0, 2\pi] \times [-2, 2] \tag{21}$$

where g is a given smooth function. Consider a boundary value problem of (21)

$$u(X, 2) = 0,\ X \in [0, 2\pi];\ u(0, Y) = u(2\pi, Y), |Y| \le 2 \tag{22}$$

Let the vector V be

$$V = (V_1, V_2, V_3)^t = e^{Y/8}(u_Y, u/Y, \varepsilon u_X)^t$$

Equation (21) is reduced to

$$RV = A\frac{\partial V}{\partial Y} + B\frac{\partial V}{\partial X} + CV = (g,0,0)^t \tag{23}$$

where

$$A = \begin{bmatrix} Y(1+\varepsilon\tilde{F}_{22}) & 0 & 0 \\ 0 & 4Y & 0 \\ 0 & 0 & -(1+\varepsilon\tilde{F}_{11}) \end{bmatrix}$$

$$B = \begin{bmatrix} \varepsilon\tilde{F}_{12} & 0 & \varepsilon(1+\varepsilon\tilde{F}_{11}) \\ 0 & 0 & 0 \\ \varepsilon(1+\varepsilon\tilde{F}_{11}) & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 5(1+\varepsilon\tilde{F}_2) - \frac{Y}{8}(1+\varepsilon\tilde{F}_{22}) & 4(1+\varepsilon\tilde{F}_0) & \varepsilon\tilde{F}_1 \\ -4 & 4\left(1-\frac{Y}{8}\right) & 0 \\ 0 & 0 & \frac{1}{8}\left(1+\varepsilon\tilde{F}_{11}\right) \end{bmatrix}$$

Equation (22) is reduced to

$$V_2|_{Y=2} = V_3|_{Y=2} = 0, \quad z \in [0,2\pi]; V(0,Y) = V(2\pi,Y), |Y| \le 2 \tag{24}$$

Conversely, and solution $V \in C^1(\bar{G})$ of (23), (24) satisfies $-V_{3Y} + V_{1X} + \frac{1}{8}V_3 = 0$, or $\varepsilon(e^{-Y/8}V_1)_X = (e^{-Y/8}V_3)_Y$. Hence there exists function $u(X,Y)$ such that

$$u_X = \frac{1}{\varepsilon}e^{-Y/8}V_3, \ u_Y = e^{-Y/8}V_1, \ u_X|_{Y=2} = 0, \ u|_{Y=2} = \text{const}$$

Taking the above constant to be zero, we have

$$u_Y = e^{-Y/8}V_1 = e^{-Y/8}[YV_{2Y} + (1-Y/8)V_2] = (e^{-Y/8}YV_2)_Y$$
$$u = e^{-Y/8}YV_2 + C(X)$$

Letting $Y = 2$ we obtain $C(X) = 0$. Hence u is a solution of (21) and (22).

In the following we denote various constants by $C_1, C_2, \cdots$; and scalar product and norm in $L^2(G)$ space by $(\ ,\ )$, $\|\cdot\|$; and the norm in $H_k(G)$ (functions having k's order generalized derivatives belong to $L^2(G)$) by $\|\cdot\|_k$; and the norm in $C^k(G)$ (continuous function with its k's order derivatives in G) by $|\cdot|_{C^k(G)}$.

Assume $|w|_{C^3(G)} \le 1$. We can choose ε_0 small such that when $0 < \varepsilon \le \varepsilon_0$, (21) is symmetric positive[5] since

$$\gamma + C + C^t - A_Y - B_X$$

$$\gamma|_{\varepsilon=0} = \begin{bmatrix} 9-Y/4 & 0 & 0 \\ 0 & 4-Y & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \geq I/4, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and (22) are the admissible boundary conditions of (21). Hence we have the energy inequality

$$(V, RV) = \left(V, \frac{\gamma+\gamma^t}{2}V\right) + (V, \beta B)|_{Y=-2}^{2} \geq C_1\|V\|^2$$

where β is the boundary operator of R. Hence

$$\|V\|^2 \leq \frac{1}{C_1}(V, RV) \leq \frac{1}{C_1}\|V\|\,\|RV\| = C_2\|V\|\,\|g\|$$

$$\|V\| \leq C_2\|g\|$$

It is obvious that

$$\left\{D_0 = I,\ D_1 = \frac{\partial}{\partial X}, D_2 = \alpha_2(Y)\frac{\partial}{\partial Y},\ D_3 = \alpha_3(Y)(Y-2)\frac{\partial}{\partial Y}\right\}$$

forms a complete system of tangential differential operators of (21) on G if $\alpha_2(Y) + \alpha_3(Y) = 1$ and $\alpha_2(Y) = 1\ \left(Y \leq \frac{1}{2}\right)$, $\alpha_3(Y) = 1 (Y \geq 1)$. Take functions $\alpha_2(Y)$, $\alpha_3(Y)$ to be sufficiently smooth.

A is nonsingular when $1 \leq Y \leq 2$, $0 < \varepsilon \leq \varepsilon_0$. Hence we have

$$\frac{\partial}{\partial Y} = D_2 + \alpha_3 A^{-1}(R - BD_1 - C)$$

$$D_1 R = D_X R = A_X\frac{\partial}{\partial Y} + B_X\frac{\partial}{\partial X} + C_X + RD_X$$

$$RD_1 = D_1 R - A_X[D_2 + \alpha_3 A^{-1}(R - DB_1 - C)] - B_X D_1 - C_X$$

$$= -\sum_{\tau=0}^{3} P_{1\tau}D_\tau + (D_1 - t_1)R$$

where

$$P_{10} = \alpha_3 A_X A^{-1}C + C_X,\ P_{11} = -\alpha_3 A_X A^{-1}B + B_X$$

$$P_{12} = A_X,\ P_{13} = 0,\ t_1 = \alpha_3 A_X A^{-1}$$

Similarly we have

$$RD_\sigma = -\sum_{\tau=0}^{3} P_{\sigma\tau}D_\tau + (D_\sigma - t_\sigma)R \quad (\sigma = 1,2,3) \tag{25}$$

Denote $R|_{\varepsilon=0} = R^0$, $A|_{\varepsilon=0} = A^0$.

$$(R^0 + \eta_\sigma A_Y^0)D_\sigma = -s_\sigma^0 + (D_\sigma - t_\sigma^0)R^0 \quad (s_\sigma^0 = P_{\sigma0}^0) \tag{26}$$

where $\eta_\sigma = \begin{cases} 0. & (\sigma = 1), \\ 1 & (\sigma = 2,3). \end{cases}$ Hence the $\varepsilon = 0$ part of $RD_\sigma + \sum_{\tau=1}^{3} P_{\sigma\tau} D_\tau$ is

$$\begin{pmatrix} R^0 & 0 & 0 \\ 0 & R^0 + \partial A_Y^0 & 0 \\ 0 & 0 & R^0 + \partial A_Y^0 \end{pmatrix}$$

which is positive symmetric. Hence we can choose ε_1 small such that $RD_\sigma + \sum_{\tau=1}^{3} P_{\sigma\tau} D_\tau$ is positive symmetric when $0 \le \varepsilon \le \varepsilon_1$. And the term $(D_\sigma V, P_{\sigma 0} V)$ is estimated by

$$|(D_\sigma V, P_{\sigma 0} V)| \le C_1 \|V\| \, |||V|||_1$$

where we denote

$$|||V|||_s^2 = \sum_{0 \le l \le s} \|D_{\sigma_1} \cdots D_{\sigma_l} V\|^2$$

Hence the energy inequality for $D_0 V$ with $D_\sigma V_2|_{Y=2} = D_\sigma V_3|_{Y=2} = 0$ $(\sigma = 1,2,3)$ gives

$$\begin{aligned} |||V|||_1 &\le \|V\| + \sum_{\sigma=1}^{3} \|D_\sigma V\| \le \|V\| + \sum_\sigma \|(D_\sigma - t_\sigma) R V\| + C_3 \|V\| \\ &\le C_4 \|g\|_1 + (1 + C_3) C_2 \|g\| \le C_5 \|g\|_1 \end{aligned}$$

Next we study the higher order reduced system. From (25) we get

$$\begin{aligned} RD_{\sigma_1} D_{\sigma_2} &= -\sum_{\tau=0}^{3} P_{\sigma_1 \tau} D_\tau D_{\sigma_2} + (D_{\sigma_1} - t_{\sigma_1}) R D_{\sigma_2} \\ &= -\sum_{\tau=0}^{3} P_{\sigma_1 \tau} D_\tau D_{\sigma_2} - \sum_{\tau=0}^{3} (D_{\sigma_1} - t_{\sigma_1})(P_{\sigma_2 \tau} D_\tau) + (D_{\sigma_1} - t_{\sigma_1})(D_{\sigma_2} - t_{\sigma_2}) R \end{aligned}$$

$$\begin{aligned} RD_{\sigma_1} \cdots D_{\sigma_s} = &-\sum_\tau P_{\sigma_1 \tau} D_\tau D_{\sigma_2} \cdots D_{\sigma_s} - \sum_\tau (D_{\sigma_1} - t_{\sigma_1})(P_{\sigma_2 \tau} D_\tau D_{\sigma_3} \cdots D_{\sigma_s}) \\ &- \sum (D_{\sigma_1} - t_{\sigma_1}) \cdots (D_{\sigma_{p-1}} - t_{\sigma_{p-1}})(P_{\sigma_p \tau} D_{\sigma_{p+1}} \cdots D_{\sigma_s}) - \cdots \\ &- \sum (D_{\sigma_1} - t_{\sigma_1}) \cdots (D_{\sigma_{s-1}} - t_{\sigma_{s-1}})(P_{\sigma_s \tau} D_\tau) + \prod_{i=1}^{s} (D_{\sigma_i} - t_{\sigma_i}) R \\ = &-\sum_{p,\tau} P_{\sigma_p, \tau} D_{\sigma_1} \cdots D_{\sigma_{p-1}} D_\tau D_{\sigma_{p+1}} \cdots D_{\sigma_s} \\ &- \sum D^{q_1} t_{\tau_1} \cdots D^{q_{l-1}} t_{\tau_{l-1}} \cdot D^{q_l} P_{\sigma_l \tau} D_\tau D_{\sigma_{l+1}} \cdots D_{\sigma_s} + \prod_{i=1}^{s} (D_{\sigma_i} - t_{\sigma_i}^0) R^0 \end{aligned}$$

And from (26) we have

$$[R^0+(\eta_{\sigma_1}+\cdots+\eta_{\sigma_s})A_Y^0]D_{\sigma_1}\cdots D_{\sigma_s}$$
$$=-\sum_p(D_{\sigma_1}-t^0_{\sigma_1})\cdots(D_{\sigma_{p-1}}-t^0_{\sigma_{p-1}})s^0_{\sigma_p}D_{\sigma_{p+1}}\cdots D_{\sigma_s}+\prod_{i=1}^{s}(D_{\sigma_i}-t^0_{\sigma_i})R^0$$

Hence the $\varepsilon=0$ part of

$$RD_{\sigma_1}\cdots D_{\sigma_s}+\sum_p P_{\sigma_p\tau}D_{\sigma_1}\cdots D_{\sigma_{p-1}}D_\tau D_{\sigma_{p+1}}\cdots D_{\sigma_s}$$
$$+\sum D^{q_1}t_{\tau_1}\cdot D^{q_{l-1}}t_{\tau_{l-1}}D^{q_l}P_{\sigma_l\tau}D_\tau D_{\sigma_{l+1}}\cdot D_{\sigma_s}$$

is

$$[R^0+(\eta_{\sigma_1}+\cdots+\eta_{\sigma_s})A_Y^0]D_{\sigma_1}\cdots D_{\sigma_s}$$
$$+\sum(D_{\sigma_1}-t^0_{\sigma_1})\cdot(D_{\sigma_{p-1}}-t^0_{\sigma_{p-1}})\cdot s^0_{\sigma_p}D_{\sigma_{p+1}}\cdots D_{\sigma_s}$$

Consequently, the reduced system of order s for $(RV)^0$ is symmetric positive.

Assume that $|w|_{C^4(G)}\le 1$. We can choose $\varepsilon_s>0$ $(s=2,3,\cdots,s,\varepsilon_0\ge\varepsilon_1\ge\cdots\ge\varepsilon_s)$ such that the reduced system of order s is symmetric positive. By energy inequality we have

$$|||V|||_2\le C_1[\|(D_{\sigma_1}-t_{\sigma_1})(D_{\sigma_2}-t_{\sigma_2})RV\|+|||V|||_1]\le C_1\|g\|_2$$

When $s>2$,

$$|||V|||_s\le C_{s1}\Big(\Big\|\prod_{i=1}^{s}(D_{\sigma_i}-t_{\sigma_i})RV\Big\|+|||V|||_{s-1}$$
$$+\sum_{\substack{q_1+\cdots+q_l\ge 1\\ q_1+\cdots q_l+r-l+1\le s}}\|D^{q_1}t_{\tau_1}\cdots D^{q_{l-1}}t_{\tau_{l-1}}D^{q_l}P_{\sigma_l\tau}]_0^\varepsilon D_\tau D_{\sigma_{l+1}}\cdots D_{\sigma_r}V\|$$
$$=C_{s1}(I+|||V|||_{s-1}+II) \tag{27}$$

where $4[H]_0^\varepsilon=H(\varepsilon)-H(0)$.

$$II\le\varepsilon C_{s2}\sum_{\substack{p_1+\cdots+p_l\ge 1\\ p_1+\cdots+p_l+r-l+1\le s}}\|D^{p_1}F_1\cdots D^{p_l}F_lD_\tau D_{\sigma_{l+1}}\cdots D_{\sigma_r}V\|$$

where $F_i(1\le i\le l)$ are $\partial^\alpha w$ (differentiating with respect to X,Y) of order $|\alpha|\le 3$. By use of Hölder inequality we have

$$\Big\|\prod_1^l D^{p_j}F_jD^{r-l}V\Big\|\le\prod_{j=1}^{l}\Big[\int|D^{p_j}F_j|^{2s/p_j}\Big]^{p_j/(2s)}$$
$$\Big[\int|D^{s-\sum_1^l p_j}V|^{2s/(s-\sum_1^l p_j)}\Big]^{(s-\sum_1^l p_j)/(2s)}$$

By use of Gagliardo-Nirenberg inequality we have

$$\left[\int |D^p v|^{2s/p}\right]^{p/(2s)} \leq \left[\int |D^2 v|^2\right]^{p/(2s)} |v|_{C^0}^{1-p/s} = \|D^s v\|^{p/s} |v|_{C^0}^{1-p/s}$$

Hence

$$II \leq \varepsilon C_{s2} \prod_1^l (\|D^s F_j\|^{p_j/s} |F_j|^{1-p_j/s}) \cdot \|V\|_s^{1-\sum_1^l p_j/s} |V|_{C^0}^{\sum_1^l p_j/s}$$

$$\leq \varepsilon C_{s3} (\|w\|_{s+3} \|V\|_{C^0})^{\sum_1^l p_j/s} \|V\|_s^{1-\sum_1^l p_j/s}$$

$$\leq \varepsilon C_{s4} (\|w\|_{s+3} |V|_{c^0} + \|V\|_s) \tag{28}$$

Similarly,

$$I = \left\| \prod_1^s (D_{\sigma_i} - t_{\sigma_i}) g \right\| \leq \|g\|_s + \sum_{\substack{q_1+\cdots+q_l+r\leq s \\ q_1+\cdots+q_l\geq 1}} \|D^{q_1} t_{\sigma_1} \cdots D^{q_l} t_{\sigma_l} D^r g\|$$

$$\leq C_{s5} (\|g\|_s + \|w\|_{s+3} |g|_{C^0}) \tag{29}$$

Substituting (28) and (29) into (27) we get

$$|||V|||_s \leq C_{s6} (\|g\|_s + \varepsilon \|V\|_s + |||V|||_{s-1} + \|w\|_{s+3} (|g|_{C^0} + |V|_{C^0})$$

or

$$|||V|||_s \leq C_{s7} (\|g\|_s + \varepsilon \|V\|_s + \|w\|_{s+3} (|g|_{C^0} + |V|_{C^0}) \tag{30}$$

Denote the differentiation with respect to X, Y of order $|\alpha| = s$ by ∂^s. Since $\frac{\partial}{\partial X} = D_1$, $\frac{\partial}{\partial Y} = D_2 + \alpha_3 A^{-1}(R - RD_1 - C)$, these expressions can be put together as

$$\partial^1 = \sum E_\tau(X, Y) D_\tau + \varepsilon \sum F_\tau D_\tau + F_4 R$$

where $F_\tau (0 \leq \tau \leq 4)$ depend on the derivatives $\partial^1_\alpha w$ of order $|\alpha| \leq 3$ only. Hence

$$\partial^2 = \partial^1 (E_\tau(X, Y) D_\tau) + \partial^1 \left(\varepsilon \sum F_\tau D_\tau + F_4 R\right)$$

$$= E_\tau D_\tau \partial^1 + E_\tau D_\tau + \partial^1 \left(\varepsilon \sum F_\tau D_\tau + F_4 R\right)$$

$$= \sum_{|l|\leq 2} E^l_{\tau_1 \tau_2} D_{\tau_1} D_{\tau_2} + \sum_{|\alpha|\leq 1} E_\alpha(X, Y) \partial^\alpha \left(\varepsilon \sum F_\tau D_\tau + F_4 R\right)$$

$$\partial^s = \sum_{|\alpha|\leq s} E^s_{\tau_1 \cdots \tau_s} D_{\tau_1 \cdots \tau_s} + \sum_{|\alpha|\leq s-1} E_\alpha(X, Y) \partial^\alpha \left(\varepsilon \sum F_\tau D_\tau + F_4 R\right) \tag{31}$$

When $1 \leq s \leq 2$, since $\|w\|_{C^4} \leq 1$ and $|||V|||_s \leq C_2 \|g\|_s$, (31) gives

$$\|V\|_s \leq C_1 (|||V|||_s + \varepsilon \|V\|_{s-1} + \|V\|_{s-1} + \|RV\|_{s-1}) \leq C_3 (\|g\|_s + \varepsilon \|V\|_s + \|V\|_{s-1})$$

Combining the above inequality with $\|V\| \le C_0\|g\|$ we get

$$\|V\| \le C_0\|g\|, \|V\|_1 \le C_3\|g\|_1, \|V\|_2 \le C_4\|g\|_2 \tag{32}$$

When $s > 2$, by use of (31) and (30) we have

$$\begin{aligned}\|V\|_s &\le C_{s1}(\||V\||_s + \varepsilon\|V\|_s + \varepsilon\|V\|_{C^0}\|w\|_{s+3} + \|RV\|_{s-1} + \|RV\|_{C^0}\|w\|_{s+2}) \\ &\le C_{s2}(\|g\|_s + \varepsilon\|V\|_s + \|w\|_{s+3}(|g|_{C^0} + |V|_{C^0}))\end{aligned}$$

By use of imbedding theorem we have

$$|g|_{C^0} \le C_1\|g\|_2, \quad |V|_{C^0} \le C_2\|V\|_2 \le C_3\|g\|_2$$

Combining it with (32) we have

$$\|V\|_s \le C_{s3}(\|g\|_s + \zeta\|g\|_2\|w\|_{s+3}) \tag{33}$$

where $\zeta = \begin{cases} 0 & (0 \le s \le 2), \\ 1 & (s \ge 3). \end{cases}$

Theorem 3 *Let $w \in C^\infty(\bar{G})$, $G = [0, 2\pi] \times (-2, 2)$, $|w|_{C^4(G)} \le 1$. Then $\forall g \in C^\infty(\bar{G})$ and $\forall s \ge 0$, $\exists \varepsilon_s > 0$ and $C_s > 0$, such that when $0 < \varepsilon \le \varepsilon_s$, (21) and (22) have a unique solution u satisfying*

$$\begin{aligned}&u \in H_s(G), \quad u/Y \in H_s(G) \\ &\|u\|_s + \|u/Y\|_s \le C_s(\|g\|_s + \zeta\|g\|_2\|w\|_{s+3})\end{aligned} \tag{34}$$

Proof Substituting $V = e^{Y/8}(u_Y, u/Y, \varepsilon u_X)$ into (33), we get

$$\|u/Y\|_s \le C_{s4}(\|g\|_s + \zeta\|g\|_2\|w\|_{s+3}), \quad s \ge 3$$

Equation (34) follows from the above inequalities combining with

$$\partial_X^{s_1}\partial_Y^{s_2}u = Y\partial_X^{s_1}\partial_Y^{s_2}(u/Y) + s_2\partial_X^{s_1}\partial_Y^{s_2-1}(u/Y)$$

The theorem is proved.

We proceed to find a sufficiently smooth solution of (11) in the region $(X, Y) \in (0, 2\pi] \times [-1, 1]$. Taking a constant $\theta > 8$, we have $\eta_n = 1 - [\theta^{-1} + \cdots + \theta^{-(n-1)}] > \frac{1}{2}$. Denote $G_n = \{(X, Y)| 0 \le X \le 2\pi, |Y| < 2\eta_n\}$, then $\bar{G}_n \subset G$. Let $v \in C^{s^*}(\bar{G}_n)$ (s^* is an integer to be determined later) be of period 2π in X, then

$$v(X, Y) = \sum_{j=-\infty}^{\infty} a_j(Y)\exp(\sqrt{-1}jX)$$

with coefficients $a_j(Y) \in C^{s^*}(-2\eta_n \le Y \le 2\eta_n)$. Denote the mollifier by

$$J_n v = \sum_{|j|\le\theta_n} (J(\theta_n Y)\theta_n a_j(Y)) \exp(\sqrt{-1}jX)$$

with $\theta_n = \theta^{\tau^n}\left(\frac{3}{4} < \tau < 2\right)$ and $J(Y) \in C^\infty(R^1)$ satisfying $\int J(Y)dY = 1$, $\int Y^p J(Y)dY = 0$, $p = 1, 2, \cdots, s$ ($s \le s^*$, s^* to be determined later), and supp $J \subset (-1, 1)$. It is well known that for any $v \in C^{s^*}(\bar{G}_n)$,

$$\|J_n v\|_{H_{s_1}}(G_{n+1}) \le C(s_1, s_2)\theta_n^{s_1 - s_2}\|v\|_{H_{s_2}}(G_n), \quad s_2 \le s_1 \le s \tag{35}$$

$$\|(I - J_n)v\|_{H_{s_2}}(G_{n+1}) \le C(s_1, s_2)\theta_n^{s_2 - s_1}\|v\|_{H_{s_1}}(G_n), \ s_2 \le s_1 \le s \tag{36}$$

The constant in (35), (36) will not change when θ_n is increasing. Consider the boundary value problem of linearized equation of (21):

$$L'(w_n)u_n = -L(w_n) \tag{37}$$

$$u_n(X, 2\eta_n) = 0, \quad X \in [0, 2\pi]; \ u_n(0, Y) = u_n(2\pi, Y), \ |Y| \le 2\eta_n \tag{38}$$

and

$$w_0 = 0, \quad w_{n+1} = w_n + Y J_n \frac{u_n}{Y} \tag{39}$$

By Theorem 3, (37) with (38) admits a unique solution in $H_s(G_n)$ for given s if $0 < \varepsilon \le \varepsilon_s$. Moreover by use of (34), the solution satisfies

$$\|u_k\|_{H_s(G_k)} + \|u_k/Y\|_{H_s(G_k)} \le C_s[\|g_k\|_{H_s(G_k)} + \zeta\|w_k\|_{H_{s+3}(G_k)}\|g_k\|_{H_2(G_k)}], \quad 0 \le k \le n \tag{40}$$

when $|w_k|_{C^4(\bar{G}_K)} \le 1$ $(0 \le k \le n)$, where $g_k = -L(w_k)$ and C_s is independent of k and w_k.

Lemma 4 *Let*

$$|w_k|_{C^4(\bar{G}_k)} \le 1, \quad |w_k/Y|_{C^2(\bar{G}_k)} \le 1, \ 0 \le k \le n \tag{41}$$

Then we have

$$\|g_k\|_{H_s(G_k)} \le C_s^*[\|w_k\|_{H_{s+2}(G_k)} + \|w_k/Y\|_{H_s(G_k)} + \|g_0\|_{H_s(G)}] \tag{42}$$

$$\|w_{k+1}\|_{H_{s+3}(G_{k+1})} + \|w_k/Y\|_{H_{s+3}(G_{k+1})} \le C_s^{*k+1}\theta_{k+1}^{\beta}\|g_0\|_{H_s(G)} \tag{43}$$

for some constant $\beta \ge 9$ *and* $0 \le k \le n$.

Proof Equation (42) follows from

$$\begin{aligned}\|g_k\|_{H_s(G_k)} &\le \|L(w_k) - L(w_0)\|_{H_s(G_k)} + \|L(w_0)\|_{H_s(G)} \\ &\le C_{s1}[\|w_k\|_{H_{s+2}(G_k)}|w_k/Y|_{C^0(G_k)} + \|w_k/Y\|_{H_s(G_k)}\|w_k\|_{C^2(G_k)} \\ &\quad + \|g_0\|_{H_s(G)}]\end{aligned}$$

and (41). From (41), (42) we have

$$\|g_k\|_{H_2(G_k)} \le C_2^*[\|w_k\|_{H_4(G_k)} + \|w_k/Y\|_{H_2(G_k)} + \|g_0\|_{H_2(G)}] \le C_{22} \tag{44}$$

Substituting (44) into (40) we have

$$\|u_k\|_{H_s(G_k)} + \|u_k/Y\|_{H_s(G_k)} \le C_{s3}[\|g_k\|_{H_s(G_k)} + \zeta\|w_k\|_{H_{s+3}(G_k)}] \tag{45}$$

By use of (39), (35), (45), (42) we have

$$\begin{aligned}
&\|w_{k+1}\|_{H_{s+3}(G_{k+1})} + \|w_{k+1}/Y\|_{H_{s+3}(G_{k+1})} \\
&\quad \le \|w_k\|_{H_{s+3}(G_{k+1})} + \|w_k/Y\|_{H_{s+3}(G_{k+1})} + C_{s4}\Big\|J_k\frac{u_k}{Y}\Big\|_{H_{s+3}(G_{k+1})} \\
&\quad \le \|w_k\|_{H_{s+3}(G_{k+1})} + \|w_k/Y\|_{H_{s+3}(G_{k+1})} + C_{s5}\theta_k^3\|u_k/Y\|_{H_s(G_k)} \\
&\quad \le C_{s1}\theta_k^3[\|w_k\|_{H_{s+3}(G_k)} + \|w_k/Y\|_{H_{s+3}(G_k)} + \|g_k\|_{H_s(G_k)}] \\
&\quad \le C_{s2}\theta_k^3[\|w_k\|_{H_{s+3}(G_k)} + \|w_k/Y\|_{H_{s+3}(G_k)} + \|g_0\|_{H_s(G)}]
\end{aligned}$$

By induction on k we obtain

$$\begin{aligned}
&\|w_{k+1}\|_{H_{s+3}(G_{k+1})} + \|w_k/Y\|_{H_{s+3}(G_{k+1})} \\
&\qquad \le (k+1)C_{s2}^{k+1}\theta_k^3\cdots\theta_0^3\|g_0\|_{H_s(G)},\ 0 \le k \le n
\end{aligned}$$

in the last step of the above inequality we make use of $w_0 = 0$. We have

$$\theta_k^3\cdots\theta_0^3 = \theta^{3(\tau^k+\tau^{k+1}+\cdots+1)} = \theta^{3(\tau^{k+1}-1)/(\tau-1)} \le \theta_{k+1}^{\beta},\ \beta \ge 9$$

since $\frac{4}{3} < \tau < 2$. (43) follows by replacing eC_{s2} by C_s^*. The lemma is proved completely.

Lemma 5 *Let (41) be true. Then there exists a constant $\chi > 8/(2-\tau)$ such that for any $s^* > \beta + 2 + \chi\tau$ the inequality*

$$\|g_{k+1}\|_{L^2(G_{k+1})} \le \theta_{k+1}^{-\chi}\|g_0\|_{s^*} \quad (k = 0, 1, \cdots, n) \tag{46}$$

holds when $\theta \ge \theta^*$ and $0 < \varepsilon < \varepsilon_{s^*}(\theta)$ for some constants θ^* and $\varepsilon_{s^*}(\theta)$

Proof We have

$$\begin{aligned}
-g_{k+1} = L(w_{k+1}) &= L\Big(w_k + YJ_k\frac{u_k}{Y}\Big) = L(w_k) + L'(w_k)YJ_k\frac{u_k}{Y} + Q\Big(w_k, J_k\frac{u_k}{Y}\Big) \\
&= L'(w_k)\Big[Y(J_k - I)\frac{u_k}{Y}\Big] + Q\Big(w_k, J_k\frac{u_k}{Y}\Big)
\end{aligned} \tag{47}$$

where Q is the quadratic form of

$$J_k\frac{u_k}{Y}, \Big(J_k\frac{u_k}{Y}\Big)_X, \quad \Big(J_k\frac{u_k}{Y}\Big)_Y, \Big(J_k\frac{u_k}{Y}\Big)_{XX}, \Big(J_k\frac{u_k}{Y}\Big)_{XY}, \Big(J_k\frac{u_k}{Y}\Big)_{YY}$$

with bounded coefficients because of the condition (41).

Using (22), (41), (36), (42) and (43) we have

$$\begin{aligned}\Big\|L'(w_k)\Big[Y(J_k-I)\frac{u_k}{Y}\Big]\Big\|_{L^2(G_{k+1})} &\le C_1\Big\|(J_k-I)\frac{u_k}{Y}\Big\|_{H_2(G_{k+1})}\\ &\le C_{s^*1}\theta^{2-s^*}\Big\|\frac{u_k}{Y}\Big\|_{H_{s^*}(G_k)} \quad (\forall s^*>2)\\ &\le C_{s^*2}[\|w_k\|_{H_{s^*+3}(G_k)}+\|w_k/Y\|_{H_{s^*+3}(G_k)}+\|g_0\|_{H_s(G)}]\\ &\le C^k_{s^*1}\theta_k^{2-s^*+\beta}\|g_0\|_{H_s(G)} \qquad (48)\end{aligned}$$

Using Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\Big\|Q\Big(w_k,J_k\frac{u_k}{Y}\Big)\Big\|_{L^2(G_{k+1})} &\le C_1\sum_{|\alpha|\le 2}\Big[\int_{G_{k+1}}\Big|\partial^\alpha\Big(J_k\frac{u_k}{Y}\Big)\Big|^4 dXdY\Big]^{1/2}\\ &\le C_2\Big\|J_k\frac{u_k}{Y}\Big\|^2_{H_4(G_{k+1})}\Big|J_k\frac{u_k}{Y}\Big|_{C^0(G_{k+1})} \qquad (49)\end{aligned}$$

Using (35) and (40), we obtain

$$\Big\|J_k\frac{u_k}{Y}\Big\|_{H_4(G_{k+1})}\le C_3\theta_k^4\Big\|\frac{u_k}{Y}\Big\|_{L^2(G_k)}\le C_4\theta_k^4\|g_k\|_{L^2(G_k)} \qquad (50)$$

Using imbedding theorem, (40), (42) and (41), we have

$$\begin{aligned}\Big|J_k\frac{u_k}{Y}\Big|_{C^0(G_{k+1})} &\le C_1\Big\|J_k\frac{u_k}{Y}\Big\|_{H_2(G_{k+1})}\le C_1\Big\|\frac{u_k}{Y}\Big\|_{H_2(G_k)}\le C_2\|g_k\|_{H_2(G_k)}\\ &\le C_5[\|w_k\|_{H_4(G_k)}+\|w_k/Y\|_{H_2(G_k)}+\|g_0\|_{H_2(G)}]\le C_6 \qquad (51)\end{aligned}$$

Combining (47)–(51), we have

$$\|g_{k+1}\|_{L^2(G_{k+1})}\le C^k_{s^*}\theta_k^{2-s^*+\beta}\|g_0\|_{s^*}+C_0\theta_k^8\|g_k\|^2_{L^2(G_k)} \qquad (52)$$

Let

$$\max(C_0,1)\theta_k^\chi\|g_k\|_{L^2(G_k)}=d_k$$

where χ is a constant to be determined. From (52) we have

$$d_{k+1}\le d_k^2+\frac{1}{4}\|g_0\|_{s^*} \qquad (53)$$

if

$$8+\chi(\tau-2)<0,\quad s^*>2+\beta+\chi\tau \qquad (54)$$

and

$$\max(C_0,1)C^k_{s^*}\le\theta^{\tau^k(s^*-2-\beta-\chi\tau)} \qquad (55)$$

are valid. These constants χ, s^* and θ can be chosen consecutively such that (54) and (55) hold. Since $g_0=-L(w_0)=-\varepsilon f(X,\varepsilon^2Y)$, we can choose a small $\varepsilon_{s^*}(\theta)$ such that when $0<\varepsilon\le\varepsilon_{s^*}(\theta)$

$$\|g_0\|_{s^*}=\|\varepsilon f(X,\varepsilon^2Y)\|_{s^*(G)}\le 1$$

and

$$\max(C_0,1)^2\theta_0^2\|g_0\|_{L^2(G)} \le \frac{1}{4}$$

hold. Hence we have $d_0^2 \le \|g_0\|_{s^*}/4 \le \frac{1}{4}$. Inserting this inequality into (53) we get

$$d_1 \le \frac{1}{2}\|g_0\|_{s^*},\ d_2 \le \frac{1}{4}\|g_0\|_{s^*}^2 + \frac{1}{4}\|g_0\|_{s^*} \le \frac{1}{2}\|g_0\|_{s^*}, \cdots, d_{k+1} \le \frac{1}{2}\|g_0\|_{s^*},\ 0 \le k \le n$$

i.e. (46) is true. This proves the lemma.

Theorem 6 *When E, F, G are sufficiently smooth ($\in C^{30}$) and periodic on u with period 2π in $[0,2\pi]\times[-\delta,\delta]$, and (5) is valid, there exists a constant $\delta_2 (0 < \delta_2 \le \delta)$ such that (4) has a smooth solution $z(u,v)$ ($\in C^{13}$) and periodic on u with period 2π in $[0,2\pi]\times[-\delta_2,\delta_2]$. And $z(u,v) = 0(v^2)$.*

Proof The theorem is true if we find a smooth and periodic solution $z(U,V)$ of (8) with $z = O(V^2)$. The function $z(U,V)$ exists if we can prove (41) by Theorems 3, 4, 5. The sufficient condition for (41) is true when the following inequality

$$\|w_k\|_{H_6(G_k)} + \|w_k/Y\|_{H_6(G_k)} \le \Gamma \tag{56}$$

holds for every k, where Γ is a suitable constant. We prove (56) by induction. (56) is true for $k=0$ since $w_0 = 0$. Assume (56) is true for $0 \le k \le n$. Using (39), (43) and (40), $\forall 6 \le s \le s^*$ we have

$$\begin{aligned}
\|w_{n+1}\|_{H_s(G_{n+1})} + \|w_{n+1}/Y\|_{H_s(G_{n+1})} &\le C_1 \sum_{k=0}^{n} \left\| J_k \frac{u_k}{Y} \right\|_{H_s(G_{k+1})} \\
&\le C_1 \sum_0^n \left\|\frac{u_k}{Y}\right\|_{H_s(G_k)} \le C_2 \sum_0^n \left\|\frac{u_k}{Y}\right\|_{H_{s^*}(G_k)}^{s/s^*} \left\|\frac{u_k}{Y}\right\|_{L^2(G_k)}^{1-s/s^*} \\
&\le C_3 \sum_0^n (C_s^{*k}\theta_k^{\beta}\|g_0\|_{s^*})^{s/s^*}(\theta_k^{-\chi}\|g_0\|_{s^*})^{1-s/s^*} \\
&\le C_3 \sum_0^n (c_s^{*k}\theta_k^{\beta s/s^* - \chi(1-s/s^*)})\|g_0\|_{s^*}
\end{aligned} \tag{57}$$

Take $\beta = 9$, $\tau = \frac{4}{3} + \sigma$, where σ is a small positive constant. $\chi = 12 + O(\sigma) > 8(2-\tau)$, $s^* = 28 > 2 + \beta + \chi\tau$, $s = 15 < \frac{4}{7}s^*$, and take θ suitable large. Then the right-hand side of (57) converges. Hence $\le C_1\|g_0\|_{s^*}$. Take $\varepsilon_{s^*(\theta)}$ suitable small, then when $0 < \varepsilon \le e_{s^*(\theta)}$, we have $C_1\|g_0\|_{s^*} \le \Gamma$, i.e. (56) is true for $k = n+1$. By induction, (56) is always true, so is (41). Hence w_n and w_n/Y converge uniformly to functions w and w/Y in $H_s(\overline{G}_\infty)$ respectively. From $H_s(\overline{G}_\infty) = H_{15}(\overline{G}_\infty) \subset C^{13}(\overline{G}_\infty)$, we have w, $w/Y \subset C^{13}(\bar{G}_\infty)$. From $L(w_n) \to 0$ we have $L(w) = 0$ in $C^{13}(\bar{G}_\infty) \supset C^{13}([0,2\pi]\times[-1,1])$. And $z = V^2[z^2(U) + \varepsilon w]$ is a solution of (18).

By Theorem 2, $B(U,V) \in C^{30}([0,2\pi]\times[-\delta_1,\delta_1])$ since $E, F, G \in C^{30}([0,2\pi]\times[-\delta,\delta])$ and $K \in C^{28}([0,2\pi]\times[-\delta,\delta])$, we can take $s^* = 28$. The theorem is proved completely.

4. Finding Imbedding Functions x, y

Theorem 7 *When E, F, $G \in C^{30}([0,2\pi]\times[-\delta,\delta])$ are periodic functions of u with period 2π, and (5), (6), (7) are valid, there exist smooth isometric imbedding functions $x(u,v)$, $y(u,v)$, $z(u,v)$ satisfying $ds^2 = Edu^2 + 2Fdudv + Gdv^2 = dx^2 + dy^2 + dz^2$ in $u, v \in \{[0,2\pi]\times[-\delta_2,\delta_2]\}$. And x, y, z are periodic functions of u with period 2π.*

Proof By use of Theorem 2, we have $ds^2 = B(U,V)^2dU^2 + dV^2$, where $B(U,V)$ satisfies (9). Find a $z = z(U,V)$ by Theorem 6 such that $z \in C^{13}([0,2\pi]\times[-\delta_2,\delta_2])$ with $z = O(V^2)$ and $g = ds^2 - dz^2$ is flat. By use of Theorem 2 again we have

$$\begin{aligned} g = ds^2 - dz^2 &= B(U,V)^2dU^2 + dV^2 - dz^2 \\ &= (B^2 - z_U^2)dU^2 - 2z_Uz_V\,dU\,dV + (1 - z_V^2)dV^2 \\ &= \tilde{B}(U,V)d\tilde{U}^2 + d\tilde{V}^2 \end{aligned}$$

From (14), (15), (16) we have

$$\begin{aligned} \tilde{U} &= U - [z_Uz_V/(B^2 - z_U^2)]_{V=0}V + O(V^2) = U + O(V^2) \\ \tilde{V} &= \{[(B^2 - z_U^2)(1 - z_U^2) - z_U^2z_V^2]/(B^2 - z_U^2)\}^{1/2}|_{V=0}V + O(V^2) = V + O(V^2) \\ \tilde{B} &= (B^2 - z_U^2)^{1/2}|_{V=0} + \{(BB_V - z_Uz_{UV})/(B^2 - z_U^2)^{1/2} \\ &\quad -(B^2 - z_U^2)^{1/2}[-z_Uz_V/(B^2 - z_U^2)]_U\}_{V=0}V + O(V^2) \\ &= B(U,0) + B_V(U,0)V + O(V^2) \\ &= E(U,0)^{1/2} + \frac{\Gamma_{11}^2(U,0)}{E(U,0)}[E(U,0)G(U,0) - F(U,0)^2]^{1/2}V + O(V^2) \\ &= \alpha(U) - \beta(U)V + O(V^2) \\ &= \alpha(\tilde{U}) - \beta(\tilde{U})\tilde{V} + O(\tilde{V}^2) \end{aligned}$$

where

$$\alpha(U) = E(U,0)^{1/2}$$

$$\beta(U) = \frac{\Gamma_{11}^2(U,0)}{E(U,0)}[E(U,0)G(U,0) - F(U,0)^2]^{1/2}$$

Since g is flat, we have $K_g = 0$, or

$$K_g = -\tilde{B}_{\tilde{V}\tilde{V}}/\tilde{B} = 0$$

Therefore we have

$$\tilde{B} = \alpha(\tilde{U}) - \beta(\tilde{U})\tilde{V}$$

Denoting $U^* = \int_0^{\tilde{U}} \beta(\tau)d\tau$, we have

$$\begin{aligned} ds^2 - dz^2 &= [\alpha(\tilde{U}) - \beta(\tilde{U})\tilde{V}]^2d\tilde{U}^2 + d\tilde{V}^2 \\ &= [\alpha(\tilde{U})/\beta(\tilde{U}) - \tilde{V}]^2dU^{*2} + d\tilde{V}^2 \\ &= dx^2 + dy^2 \end{aligned}$$

where

$$x = \tilde{V}\cos U^* + \int_0^{\tilde{U}} \alpha(\tilde{U})\sin U^* d\tilde{U}$$

$$y = \tilde{V}\sin U^* - \int_0^{\tilde{U}} \alpha(\tilde{U})\cos U^* d\tilde{U}$$

Compatibility conditions (6), (7) garantee that x, y are the smooth functions of $\tilde{U}, \tilde{V}$ with period 2π of $\tilde{U}$.

The theorem is proved completely.

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