J. Partial Diff. Eqs., *Vol.6*, *No.1* (1993), 39–61

LONG-TIME ASYMPTOTIC BEHAVIOR OF LAX-FRIEDRICHS SCHEME

Ying Lungan and Zhou Tie
(Department of Mathematics, Peking University, Beijing, 100871)
Dedicated to the 70th birthday of Professor Zhou Yulin
(Received Jan. 30, 1992, revised Jun. 11, 1992)

Abstract In this paper we investigate the asymptotic stability of the discrete shocks of the Lax-Friedrichs scheme for hyperbolic systems of conservation laws. For single equations, we show that the discrete shocks of the Lax-Friedrichs scheme are asymptotically stable in the sense of l^2 and l^1. For the systems of conservation laws, if the summation of initial perturbations equals to zero, we show the l^2 stability and l^1 boundedness.

Key Words Lax-Friedrichs scheme; discrete travelling waves; asymptotic stability; hyperbolic conservation laws; energy method

Classifications 39A11, 35L65

1. Introduction

We consider hyperbolic systems of conservation laws

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0 \tag{1.1}$$

Let $x_j = j\Delta x$, $t_n = n\Delta t$; Δx and Δt are respectively the space and time step sizes. Denote the approximation of $u(x_j, t_n)$ by u_j^n, the Lax-Friedrichs scheme (L-F scheme) is:

$$u_j^{n+1}-\frac{1}{2}(u_{j+1}^n+u_{j-1}^n)+\frac{\lambda}{2}(f(u_{j+1}^n)-f(u_{j-1}^n))=0 \tag{1.2}$$

where $\lambda = \Delta t/\Delta x$. Or, in general, we have the following scheme:

$$u_j^{n+1}-u_j^n+\frac{\lambda}{2}(f(u_{j+1}^n)-f(u_{j-1}^n))=\frac{\alpha}{2}(u_{j+1}^n-2u_j^n+u_{j-1}^n) \tag{1.3}$$

where $0 < \alpha \le 1$. When $\alpha = 1$, (1.3) is just (1.2).

The L-F scheme has been playing important roles both in the theory and numerical computations of hyperbolic conservation laws. In 1950's Oleinik [1] studied the existence for global solutions of single conservation laws using this scheme. In 1980's, Diperna [2] and Ding Xiaxi, Chen Guiqiang, Luo Peizhu [3] also used it to prove the existence of weak solutions with large amplitude for some 2×2 systems. The L-F scheme also played an important role in the development of the difference methods. It is a representative for monotone schemes. About the monotone schemes, there have been systematic theories [4]–[6].

On the asymptotic stability of the difference equations, Jennings first investigated the monotone schemes [7]. But the work is only restricted to the strictly monotone schemes-that is, if we denote the scheme as

$$u_j^{n+1} = G(u_{j-r}^n, u_{j-r+1}^n, \cdots, u_{j+t}^n) \tag{1.4}$$

then the first order derivatives of G about each of its arguments must be positive. Obviously, the L-F scheme does not satisfy this condition. Moreover, there are some mistakes among the stability theorem in [7]. As much as we know, Ralston pointed out the mistakes in [7] and made a correction in his unpublished work. Engquist and Osher, in their paper [8], quoted part of Ralston's results, but unfortunately, there are still some mistakes in this part of paper [8]. We shall show this at the end of this section.

For the L-F scheme approximating systems, Chern [9] has proved that the solutions of the scheme is asymptotically stable provided that the initial value is a constant state when $|x|$ is sufficiently large. Liu and Xin [10] have proved that, for scheme (1.3), if $0 < \alpha < 1$; the solutions of Riemann problem are single or multiple shocks; and if the summation of the initial perturbations equals to zero, then the scheme solutions are asymptotically stable. Besides, on other schemes, Majda and Ralston [11] have proved the existence of the travelling wave solutions for a class of schemes using the center manifold theorem. Smyrlis [12] has proved the asymptotic stability for the stationary discrete shocks of the Lax-Wendroff scheme. Szepessy [13] proved the asymptotic stability for a kind of implicit finite element schemes approximating systems. Yu [14] proved that under some conditions the Lax-Wendroff scheme can not have the travelling wave solutions.

The aim of the present paper is to study the asymptotic stability of the L-F scheme (1.2). We shall prove that its solution on the odd grid nodes and on the even grid nodes tends to two travelling waves respectively. We first consider scalar equations. Although on the grid with double space and time step size, the L-F scheme is strictly monotone, then following Ralston's unpublished work, one can get a kind of stability, in this paper we will show that the energy integration method gives a better result. The more important is, our method can be applied to systems of equations. We shall combine the method in [10] with the method we use for scalar equations to get the result on systems under the similar conditions in [10].

The organization of this paper is as follows: at the end of Section 1 we show that the asymptotically results in [7] and [8] can not be generally true. In Section 2 we shall prove that when the initial value is a small perturbation of a travelling wave the solution is l^2 asymptotically stable. In Section 3, we shall prove our l^1-stability for large perturbation. Finally, Section 4 contains the results for systems of equations.

Now, we discuss the stability results in [7] and [8].

Suppose (1.1) is a scalar equation. For convenience, we assume $f'' > 0$, u_r, u_l and

s be constants satisfying the Rankine-Hugoniot relation,

$$f(u_r) - f(u_l) = s(u_r - u_l)$$

and $u_l > u_r$, i.e., the entropy condition. Then, Equation (1.1) has the following weak solution,

$$u(x,t) = \begin{cases} u_l, & \text{for } x < x_0 + st \\ u_r, & \text{for } x > x_0 + st \end{cases}$$

where x_0 is an arbitrary constant. Corresponding to difference equation (1.2) we have the following viscous conservation law

$$u_t + (f(u))_x = \mu u_{xx}, \quad \mu > 0 \tag{1.5}$$

which has travelling wave solution $u = u(x - st)$ satisfying

$$\lim_{\xi \to +\infty} u(\xi) = u_r, \quad \lim_{\xi \to -\infty} u(\xi) = u_l$$

The convergence is at exponential rate, and moreover $u'(\xi) < 0$. Hence, the travelling wave solutions of the differential equation have the following property,

$$u(x, t + \Delta t) = u(x - s\Delta t, t) \tag{1.6}$$

Since the solutions of a difference equation are only defined on the grid nodes, (1.6) does not always make sense. If we assume $\eta = s\lambda$ is a rational p/q, we can construct a refined grid,

$$\mathcal{L}_\eta = \{m\eta + n | \eta = s\lambda; m, n \in \boldsymbol{Z}\}$$

then, if $j \in \mathcal{L}_\eta$, corresponding to (1.6), we have

$$u^n_{j-\eta} = G(u^n_{j-r}, \cdots, u^n_{j+t}) \tag{1.7}$$

Suppose G is a strictly monotone scheme, the existence of the solution for (1.7) was proved in [7]. Moreover the u^n_j also tends to u_l or u_r at exponential rate, and it is also strictly monotone about j.

In [7] and [8], the stability theorem reads: assume the initial value u^0_j satisfies $u^0_j \in (u_r, u_l)$, and there is a travelling wave ϕ^n_j such that $u^0_j - \phi^0_j \in l^1(\mathcal{L}_\eta)$, then when $n \to \infty$, u^n_j tends to a travelling wave in $l^1(\mathcal{L}_\eta)$.

In general situations, the above asymptoticity can not be true. To show this we can take an example as follows: let $\eta = 1/2$, ϕ^n_j is a travelling wave, we define u^n_j as:

$$u^n_j = \begin{cases} \phi^n_j, & \text{for } j \in \boldsymbol{Z}, \\ \phi^n_{j+\frac{1}{2}+j_0}, & \text{for } j = k + \dfrac{1}{2} \quad k \in \boldsymbol{Z} \end{cases}$$

where j_0 is an arbitrary integer. Since in the difference equation (1.4) ($r = t = 1$), the integral nodes and the semi-nodes are independent of each other, u^n_j is still a solution of (1.4). But no matter how large the n becomes, u_j^n is always a combination of two travelling waves, which means it can not converge to a certain travelling wave. Hence the l^1 stability on $\mathcal{L}_\eta$ can not be generally true.

In [11], instead of the refined grid, an expanded grid was introduced to make (1.6) have sense. Let $\Delta t_q = q\Delta t$, if we use Δt_q as the time step size, then, corresponding to (1.7), we have

$$u_{j-p}^{nq} = G^q(u_{j-qr}^{nq}, \cdots, u_{j+qt}^{nq}) \tag{1.8}$$

Here, $j-p$ is already an integer. From now on, we shall investigate the asymptotic stability on the original or the expanded grid.

2. l^2 Asymptotic Behavior, Small Perturbation

In this section we investigate the l^2 asymptotic stability of the L-F scheme approximating scalar conservation laws (1.1).

Using (1.2) iterate twice, we get a strictly monotone scheme

$$u_j^{n+1} = \frac{1}{4}(u_{j+1}^n + 2u_j^n + u_{j-1}^n) - \frac{\lambda}{4}[f(u_{j+1}^n) - f(u_{j-1}^n)] - \frac{\lambda}{2}[f(u_{j+\frac{1}{2}}^{n+\frac{1}{2}}) - f(u_{j-\frac{1}{2}}^{n+\frac{1}{2}})] \tag{2.1}$$

where

$$u_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(u_{j+1}^n + u_j^n) - \frac{\lambda}{2}[f(u_{j+1}^n) - f(u_j^n)]$$

$$u_{j-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(u_{j-1}^n + u_j^n) - \frac{\lambda}{2}[f(u_j^n) - f(u_{j-1}^n)]$$

It is clear that the results about the existence of travelling wave solutions for the strictly monotone schemes can be applied to scheme (2.1). Thus, we can define the travelling waves of the L-F scheme (1.2) at the odd (even) nodes by (2.1).

About the initial values, we have the following lemma.

Lemma 2.1 *Let $u_j^0 (j \in \boldsymbol{Z})$ be an initial value and satisfy*

$$\sum_{j\in\boldsymbol{Z}} |u_j^0 - \psi_j^0| \le \delta$$

where ψ^0 is a travelling wave of scheme (2.1). Then, there must exist another travelling wave ϕ^0 such that

$$\sum_{j\in\boldsymbol{Z}} (u_j^0 - \phi_j^0) = 0$$

and

$$\sum_{j\in\boldsymbol{Z}} |u_j^0 - \phi_j^0| \le 2\delta$$

The proof of Lemma 2.1 is a consequence of the fact that a travelling wave continuously depends on its value at a point (see [7]). We omit it here.

Our main result in this section is

Theorem 2.1 *If $f'' > 0$, $u_l > u_r$, $\lambda \max |f'| \le c_0 < 1$, and, if u_j^0 satisfies*

(i) *there exists a travelling wave v_j^0 such that*

$$\sum_j |u_j^0 - v_j^0| \le b_2 \tag{2.2}$$

where $b_2 > 0$ is a sufficiently small quantity,

(ii)

$$|u_j^0 - u_l(u_r)| = O(|j|^{-\mu}) \quad |j| \to \infty, \quad \mu > \frac{3}{2}$$

then there is a travelling wave ϕ_j^n of scheme (2.1), satisfying

$$\sum_{j+n=odd} |u_j^n - \phi_j^n|^2 \to 0, \quad n \to +\infty$$

where $u_j^n = G(u_{j-1}^{n-1}, u_{j+1}^{n-1})$, $\phi_j^n = G(\phi_{j-1}^{n-1}, \phi_{j+1}^{n-1})$, by the scheme (1.1).

Proof From Lemma 2.1, we can find a travelling wave ϕ_j^n of scheme (2.1), such that

$$\sum_{j+n=odd} (u_j^n - \phi_j^n) = 0, \quad \forall n \in \mathbf{Z}_+ \tag{2.3}$$

and

$$\sum_{j+n=odd} |u_j^0 - \phi_j^0| \le 2b_2$$

From condition (ii), we have

$$|u_j^n - u_r(u_l)| = O(|j|^{-\mu})$$

Define

$$v_j^n = \sum_{k \le j, k+n=odd} (u_j^n - \phi_j^n) 2r$$

then $v_j^n \to 0$ and $|v_j^n| = O(|j|^{-\mu+1})$, so all the following summations with j are reasonable. From (1.2), we have

$$\frac{v_j^{n+1} - \frac{1}{2}(v_{j-1}^n + v_{j+1}^n)}{h} + f(u_{j+1}^n) - f(\phi_{j+1}^n) = 0 \tag{2.4}$$

Multiplying (2.4) by $\frac{1}{2}(v_{j-1}^n + v_{j+1}^n)$ and summing over j for $j+n = odd$ we get

$$\sum_{j+n=odd} \frac{v_{j-1}^n + v_{j+1}^n}{2} \cdot \frac{v_j^{n+1} - \frac{1}{2}(v_{j-1}^n + v_{j+1}^n)}{h}$$

$$= \frac{1}{2h}\sum_j \Big[(v_j^{n+1})(v_{j-1}^n + v_{j+1}^n) - \frac{1}{2}(v_{j-1}^n + v_{j+1}^n)^2\Big]$$

$$= \frac{1}{2h}\sum_j \Big[(v_j^{n+1})^2 - (v_j^{n+1})^2 + v_j^{n+1}(v_{j-1}^n + v_{j+1}^n)$$

$$-\frac{1}{2}(v_{j-1}^n + v_{j+1}^n)^2\Big]$$

$$= \frac{1}{2h}\sum_j \Big[(v_j^{n+1})^2 - (v_j^{n+1} - \frac{1}{2}(v_{j-1}^n + v_{j+1}^n))^2 - \Big(\frac{v_{j-1}^n + v_{j+1}^n}{2}\Big)^2\Big]$$

We can see that

$$\sum_{j+n=odd} \Big[(v_j^{n+1})^2 - \Big(\frac{v_{j-1}^n + v_{j+1}^n}{2}\Big)^2\Big]$$

$$= \sum_{j+n=odd} (v_j^{n+1})^2 - \sum_{j+n=odd} \frac{1}{4}[2(v_{j+1}^n)^2 + 2(v_{j-1}^n)^2 - (v_{j+1}^n - v_{j-1}^n)^2]$$

$$= \sum_{j+n=odd} (v_j^{n+1})^2 - \sum_{j+n=odd} \frac{1}{2}(v_{j+1}^n)^2 - \sum_{j+n=odd} \frac{1}{2}(v_{j-1}^n)^2$$

$$+ \sum_{j+n=odd} \frac{1}{4}(v_{j+1}^n - v_{j-1}^n)^2$$

$$= \sum_{j+n=odd} [(v_j^{n+1})^2 - (v_{j+1}^n)^2 + \frac{1}{4}(v_{j+1}^n - v_{j-1}^n)^2] \tag{2.5}$$

From (2.4), we obtain

$$\Big[v_j^{n+1} - \frac{1}{2}(v_{j-1}^n + v_{j+1}^n)\Big]^2 = h^2[f(u_{j+1}^n) - f(\phi_{j+1}^n)]^2$$

so,

$$\frac{1}{2h}\sum_{j+n=odd} \Big[v_j^{n+1} - \frac{1}{2}(v_{j-1}^n + v_{j+1}^n)\Big]^2 = \frac{1}{2h}\sum_{j+n=odd} h^2[f(u_{j+1}^n) - f(\phi_{j+1}^n)]^2$$

$$= \frac{1}{2h}\sum_{j+n=odd} h^2[f'(\xi_j^n)]^2(u_{j+1}^n - \phi_{j+1}^n)^2 = \frac{1}{2h}\sum_{j+n=odd} h^2[f'(\xi_j^n)]^2\Big(\frac{v_{j+1}^n - v_{j-1}^n}{2r}\Big)^2$$

$$= \frac{1}{2h}\sum_{j+n=odd} \frac{h^2}{4r^2}[f'(\xi_j^n)]^2(v_{j+1}^n - v_{j-1}^n)^2$$

From CFL condition $\left|\frac{h}{r}f'\right| \le c_0 < 1$, we have

$$\frac{1}{2h}\sum_{j+n=odd}\left[v_j^{n+1}-\frac{1}{2}(v_{j-1}^n+v_{j+1}^n)\right]^2 \le \frac{1}{2h}\sum_{j+n=odd}\frac{c_0^2}{4}[f'(\xi_j^n)]^2(v_{j+1}^n-v_{j-1}^n)^2 \tag{2.6}$$

(2.5) and (2.6) yield

$$\begin{aligned}
&\sum_{j+n=odd}\frac{v_{j-1}^n+v_{j+1}^n}{2}\cdot\frac{v_j^{n+1}-\frac{1}{2}(v_{j-1}^n+v_{j+1}^n)}{h}\\
&=\frac{1}{2h}\sum_{j+n=odd}\left[(v_j^{n+1})^2-\left(\frac{v_{j-1}^n+v_{j+1}^n}{2}\right)^2\right]\\
&\quad-\frac{1}{2h}\sum_{j+n=odd}\left[v_j^{n+1}-\frac{1}{2}(v_{j-1}^n+v_{j+1}^n)\right]^2\\
&\ge\frac{1}{2h}\sum_{j+n=odd}\left[(v_j^{n+1})^2-(v_{j+1}^n)^2+\frac{1}{4}(v_{j+1}^n+v_{j-1}^n)^2\right]\\
&\quad-\frac{1}{2h}\sum_{j+n=odd}\frac{c_0^2}{4}(v_{j+1}^n-v_{j-1}^n)^2\\
&\ge\frac{1}{2h}\sum_{j+n=odd}\left[(v_j^{n+1})^2-(v_{j+1}^n)^2+\frac{1}{4}(1-c_0^2)(v_{j+1}^n-v_{j-1}^n)^2\right]
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&\sum_{j+n=odd}\frac{v_{j-1}^n+v_{j+1}^n}{2}\cdot[f(u_{j+1}^n)-f(\phi_{j+1}^n)]\\
&=\sum_{j+n=odd}\frac{v_{j-1}^n+v_{j+1}^n}{2}\cdot f'(\xi_{j+1}^n)(u_{j+1}^n-\phi_{j+1}^n)\\
&=\sum_{j+n=odd}\frac{v_{j-1}^n+v_{j+1}^n}{2}\cdot f'(\xi_{j+1}^n)\frac{v_{j+1}^n-v_{j-1}^n}{2r}\\
&=\sum_{j+n=odd}\frac{1}{4r}[(v_{j+1}^n)^2-(v_{j-1}^n)^2]f'(\xi_{j+1}^n)\\
&=\frac{1}{4r}\sum_{j+n=odd}f'(\phi_{j+1}^n)[(v_{j+1}^n)^2-(v_{j-1}^n)^2]\\
&\quad+\frac{1}{4r}\sum_{j+n=odd}f''(\hat{u})(\xi_{j+1}^n-\phi_{j+1}^n)[(v_{j+1}^n)^2-(v_{j-1}^n)^2]
\end{aligned}$$

where $\hat{u}$ is a mean value. It is easy to see that

$$|\xi_{j+1}^n-\phi_{j+1}^n|\le|u_{j+1}^n-\phi_{j+1}^n|$$

therefore

$$\begin{aligned}\frac{1}{4r}\sum_{j+n=odd} f'(\phi^n_{j+1})[(v^n_{j+1})^2-(v^n_{j-1})^2] \\ &= \frac{1}{4r}\sum_{j+n=odd} f'(\phi^n_{j+1})(v^n_{j+1})^2 - \frac{1}{4r}\sum_{j+n=odd} f'(\phi^n_{j+1})(v^n_{j-1})^2 \\ &= \frac{1}{4r}\sum_{j+n=odd}[f'(\phi^n_{j-1})-f'(\phi^n_{j+1})](v^n_{j-1})^2\end{aligned}$$

Since ϕ^n_j is decreasing, $f'' > 0$, we can get

$$\frac{1}{4r}\sum_{j+n=odd} f'(\phi^n_{j+1})[(v^n_{j+1})^2-(v^n_{j-1})^2] \geq 0$$

Besides,

$$\begin{aligned}&\left|\frac{1}{4r}\sum_{j+n=odd} f''(\hat{u})(\xi^n_{j+1}-\phi^n_{j+1})[(v^n_{j+1})^2-(v^n_{j-1})^2]\right| \\ &\leq \frac{C}{4r}\sum|u^n_{j+1}-\phi^n_{j+1}|\,|(v^n_{j+1})^2-(v^n_{j-1})^2| \\ &= \frac{C}{4r}\sum\left|\frac{v^n_{j+1}-v^n_{j-1}}{2r}\right||(v^n_{j+1})^2-(v^n_{j-1})^2| \\ &= \frac{C}{8r^2}\sum|v^n_{j+1}-v^n_{j-1}|^2|v^n_{j+1}+v^n_{j-1}|\end{aligned}$$

so

$$\begin{aligned}\sum_{j+n=odd}&\frac{v^n_{j-1}+v^n_{j+1}}{2}\cdot[f(u^n_{j+1})-f(\phi^n_{j+1})] \\ &\geq \frac{C}{8r^2}\sum_{j+n=odd}|v^n_{j+1}-v^n_{j-1}|^2|v^n_{j+1}+v^n_{j-1}| \qquad (2.8)\end{aligned}$$

Finally, from (2.7), (2.8), we get

$$\begin{aligned}0 &= \sum_{j+n=odd}\frac{v^n_{j-1}+v^n_{j+1}}{2}\cdot\frac{v^{n+1}_j-\frac{1}{2}(v^n_{j-1}+v^n_{j+1})}{h} \\ &+ \sum_{j+n=odd}\frac{v^n_{j-1}+v^n_{j+1}}{2}\cdot[f(u^n_{j+1})-f(\phi^n_{j+1})] \\ &\geq \frac{1}{2h}\sum\Big[\Big((v^{n+1}_j)^2-(v^n_{j+1})^2+\frac{1}{4}(1-c_0^2)(v^n_{j+1}-v^n_{j-1})^2\Big) \\ &\quad -\frac{2hC}{8r^2}|v^n_{j+1}-v^n_{j-1}|^2|v^n_{j+1}+v^n_{j-1}|\Big]\end{aligned}$$

$$= \frac{1}{2h}\sum((v_j^{n+1})^2-(v_j^n)^2)$$
$$+\sum\Big(\frac{1}{8h}(1-c_0^2)-\frac{C}{8r^2}|v_{j+1}^n+v_{j-1}^n|\Big)(v_{j+1}^n-v_{j-1}^n)^2$$

We choose $b_2 = \dfrac{r^2(1-c_0^2)}{8hC}$, then if $\sum\limits_{j=odd}|u_j^0-\phi_j^0|\le b_2$, we have

$$\sum_{j+n=odd}|u_j^n-\phi_j^n|\le b_2 \quad \forall n\in \mathbf{Z}_+$$

Then

$$|v_j^n|\le b_2$$

Then

$$0\ge \frac{1}{2h}\sum_{j+n=odd}((v_j^{n+1})^2-(v_j^n)^2)+\sum_{j+n=odd}\frac{1-c_0^2}{16h}|v_{j+1}^n-v_{j-1}^n|^2$$

i.e.,

$$\frac{1}{2h}\sum_{j+n=odd}((v_j^{n+1})^2-(v_j^n)^2)+M_0\sum_{j+n=odd}|v_{j+1}^n-v_{j-1}^n|^2\le 0$$

i.e.,

$$\frac{1}{2h}\sum_{j+n=odd}(v_j^{n+1})^2-\frac{1}{2h}\sum_{j+n=odd}(v_j^n)^2\le -M_0\sum_{j+n=odd}|v_{j+1}^n-v_{j-1}^n|^2 \tag{2.9}$$

where $M_0>0$. Summing (2.9) from $n=0$ to $n=N$, we obtain

$$\frac{1}{2}\sum_{j+n=odd}(v_j^{N+1})^2rh+M_0\sum_{n=0}^{N}\sum_{j+n=odd}|v_{j+1}^n-v_{j-1}^n|^2rh\le\frac{1}{2}\sum_{j+n=odd}(v_j^0)^2rh \tag{2.10}$$

(2.10) shows that the series

$$\sum_{n=0}^{+\infty}\sum_{j+n=odd}|v_{j+1}^n-v_{j-1}^n|^2<+\infty$$

thus,

$$\lim_{n\to+\infty}\sum_{j+n=odd}|u_j^n-\phi_j^n|^2=0$$

which completes the proof of Theorem 2.1.

It can be seen that through the same argument process, we can prove that there exists another travelling wave ϕ_2 of scheme (2.1) such that

$$\sum_{j=even}(u_j^0-(\phi_2)_j^0)=0$$

and

$$\sum_{j+n=even}|u_j^n-(\phi_2)_j^n|^2\to 0,\quad n\to+\infty$$

We define

$$\phi_j^0 = \begin{cases} (\phi_1)_j^0, & j = \text{ odd} \\ (\phi_2)_j^0, & j = \text{even} \end{cases}$$

then

$$\sum_{j\in\mathbf{Z}} (u_j^0 - \phi_j^0) = 0$$

$$\sum_{j\in\mathbf{Z}} |u_j^0 - \phi_j^0| \le \sum_{j=odd} |u_j^0 - (\phi_1)_j^0| + \sum_{j=even} |u_j^0 - (\phi_2)_j^0|$$

and under the conditions of Theorem 2.1, we have

$$\sum_{j\in\mathbf{Z}} |u_j^n - \phi_j^n|^2 \to 0, \quad n \to +\infty \tag{2.11}$$

where $u_j^n = G(u_{j-1}^{n-1}, u_{j+1}^{n-1})$, $\phi_j^n = G(\phi_{j-1}^{n-1}, \phi_{j+1}^{n-1})$ by the L-F scheme (1.2).

We notice that ϕ_j^0 may not be monotone about j, but it decays to u_r or u_l at the exponential rate as $|j| \to +\infty$. We might as well call it "a travelling wave of the Lax-Friedrichs scheme".

3. l^1 Asymptotic Behavior, Large Perturbation

In this section, we shall make use of the l^2 results in Section 2 and obtain l^1 asymptotic behavior of scheme (1.2). At the same time, we shall relax the requirement that the initial value must be a small perturbation of a travelling wave.

In this section, we require that η is a rational number,

$$\eta = \frac{p}{q} \quad p, q \in \mathbf{Z}, q > 0$$

In the following process, we shall often use the operator T introduced by Jennings (see [7]). Its definition is:

$$(Tu)_{j-\eta} = G(u_{j-r}, u_{j-r+1}, \cdots, u_{j+t}), \quad j \in \mathcal{L}_\eta \tag{3.1}$$

It is easy to see, if u_j^n is

$$u_j^n = G(u_{j-r}^{n-1}, \cdots, u_{j+t}^{n-1}), \quad j \in \mathcal{L}_\eta$$

then

$$u_j^n = (T^n u^0)_{j-n\eta} \tag{3.2}$$

When ϕ is a travelling wave of a strictly monotone, we must have

$$\phi_j^n = \phi_{j-n\eta}^0 \tag{3.3}$$

this is because travelling waves are the fixed points of T.

Jennings investigated the l^1 compactness of $\{(T^n u^0)_x\}_{n=1}^{+\infty}$ in [7]. Denote

$$E = \{u_x | x \in \mathcal{L}_\eta, \sum_x |u_x - v_x| < +\infty, v_x \text{ is a certain travelling wave}\} \tag{3.4}$$

If v is a travelling wave, then for any μ, $0 < \mu < |u_l - u_r|$, there is a sufficiently large $x_0 \in \mathcal{L}_\eta$ so that

$$u_r \le v_x \le u_l - \mu, \text{ for } x \ge x_0, \ u_r + \mu \le v_x \le u_l, \text{ for } x \le -x_0$$

Denote by $E(v_x, \mu, x_0)$ a subset of E as the following

$$E(v_x, \mu, x_0) = \Big\{u_x : \sum_x |u_x - v_x| < \mu; v_x^l, v_x^r \text{are two travelling waves, such that}$$
$$u_r \le u_x < v_x^r, \ \forall x > x_0; v_x^l < u_x < u_l, \ \forall x < -x_0\Big\} \tag{3.5}$$

Then we have

Lemma 3.1 (Jennings) *The following statements are true*

(a) $$TE(v_x, \mu, x_0) \subset E(v_x, \mu, x_0);$$

(b) *Every $E(v_x, \mu, x_0)$ is compact in the l^1 topology.*

Our main result in this section is the following theorem.

Theorem 3.1 *For Lax-Friedrichs scheme* (1.2), *if* $\lambda \max |f'| \le c_0 < 1$, $\eta = \dfrac{p}{q}$, *and the initial value u_j^0 satisfies*

(i) $$\sum_j |u_j^0 - w_j^0| < +\infty$$

where w_j^0 is a certain travelling wave of the scheme (2.1),

(ii) $$|u_j^0 - u_l(u_r)| = O(|j|^{-\mu}), |j| \to +\infty, \ \mu > \frac{3}{2}$$

then there are two travelling waves ϕ_1, ϕ_2 of scheme (2.1), *such that, if we define*

$$\phi_j^0 = \begin{cases} (\phi_1)_j^0 & j = odd \\ (\phi_2)_j^0 & j = even \end{cases} \tag{3.6}$$

then,

$$\sum_{j \in \mathbf{Z}} |u_j^n - \phi_j^n| \to 0 \quad n \to +\infty$$

where $u_j^n = G(u_{j-1}^{n-1}, u_{j+1}^{n-1})$, $\phi_j^n = G(\phi_{j-1}^{n-1}, \phi_{j+1}^{n-1})$ by the L-F scheme.

Proof Let T be the operator defined by scheme (2.1) in the manner of (3.1). According to the relation between the Lax-Friedrichs scheme and scheme (2.1), the solution of the L-F scheme on the 2nq level u_j^{2nq} can be expressed by

$$u_j^{2nq} = (T^{nq} u^0)_{j-np} \tag{3.7}$$

If ϕ is defined by (3.6), we also have

$$\phi_j^{2nq} = \phi_{j-np}^0 \tag{3.8}$$

We notice that if p is even, then j and $j-p$ in (3.7), (3.8) are odd (or even) simultaneously.

Now, assume the initial value $u^0 \in E(v_x, \mu, x_0)$, where v_x is a travelling wave of scheme (2.1), $E(v_x, \mu, x_0)$ defined by (3.5), then $\{(T^{nq}u)_j : j \in \boldsymbol{Z}\}$ is compact in $l^1(\boldsymbol{Z})$. So there is a subsequence $\{n_k\}$ and a $\bar{u}_j$ such that

$$\sum_{j\in\boldsymbol{Z}} |(T^{n_k q}u)_j - \bar{u}_j| \to 0, \quad k \to +\infty \tag{3.9}$$

On the other hand, if u_j^0 satisfies the hypothesis in Theorem 2.1 then by (2.11) we have

$$\sum_{j\in\boldsymbol{Z}} |u_j^{2n_k q} - \phi_j^{2n_k q}|^2 \to 0, \quad n \to +\infty \tag{3.10}$$

Using (3.7), (3.8), we can rewrite (3.10) into

$$\sum_{j\in\boldsymbol{Z}} |(T^{n_k q}u^0)_j - \phi_j^0|^2 \to 0, \quad n \to +\infty \tag{3.11}$$

Combine (3.9) with (3.11), we can see $\bar{u}_j = \phi_j^0$, $j \in \boldsymbol{Z}$.

In fact, by the process above, we have proved that

$$\sum_{j\in\boldsymbol{Z}} |(T^{nq}u^0)_j - \phi_j^0| \to 0, \quad n \to +\infty$$

Using (3.7), (3.8) again, we have

$$\sum_{j\in\boldsymbol{Z}} |u_j^{2nq} - \phi_j^{2nq}| \to 0, \quad n \to +\infty$$

Similarly, $\{(T^{nq}u^1)_j\}$ ($u_j^1 = G(u_{j-1}^0, u_{j+1}^0)$) is also compact in $l^1(\boldsymbol{Z})$, and we also have

$$\sum_{j\in\boldsymbol{Z}} |u_j^{2nq+1} - \phi_j^{2nq+1}|^2 \to 0$$

so we can also conclude that

$$\sum_{j\in\boldsymbol{Z}} |(T^{nq}u^1)_j - \phi_j^1| \to 0, \quad n \to +\infty$$

In fact, for $i = 0, 1, 2, \cdots, 2q-1$, we can obtain

$$\sum_{j\in\boldsymbol{Z}} |(T^{nq}u^i)_j - \phi_j^i| \to 0, \quad n \to +\infty$$

which is

$$\sum_{j\in \mathbf{Z}} |u_j^{2nq+i} - \phi_j^{2nq+i}| \to 0,\ n\to +\infty,\quad 0\le i\le 2q-1$$

This means

$$\sum_{j\in \mathbf{Z}} |u_j^n - \phi_j^n| \to 0,\quad n\to +\infty$$

At this moment, the initial value not only needs to satisfy the conditions of Theorem 2.1 but also belongs to $\bigcup\limits_{v_x,\mu,x_0} E(v_x,\mu,x_0)$. Now, let us remove the later requirement. In fact, it can be proved that

$$\overline{\bigcup_{v_x,\mu,x_0} E(v_x,\mu,x_0)}^{l^1} = \Big\{u_x : \sum_x |u_x - v_x| \le |u_l - u_r|, \text{ for some travelling wave } v_x\Big\} \tag{3.12}$$

Let us prove (3.12). Assume there is a travelling wave v_x such that

$$\sum_x |u_x - v_x| \le |u_l - u_r|$$

First, if

$$\sum_x |u_x - v_x| < \mu < |u_l - u_r|$$

then define

$$u_x^{(m)} = \begin{cases} u_x, & |x|\le m \\ v_x, & |x| > m \end{cases}$$

It can be seen that

(i) $\displaystyle\sum_x |u_x^{(m)} - v_x| = \sum_{|x|\le m} |u_x - v_x| < \mu$

(ii) $x > m,\ u_x^{(m)} = v_x \le v_x^r;\ x < -m,\ u_x^{(m)} = v_x \ge v_x^l$

that is, every $u_x^{(m)}$, $m = m_0, m_0+1, \cdots$, belongs to a certain $E(v_x,\mu,x_0)$, moreover,

$$\sum_x |u_x^{(m)} - u_x| = \sum_{|x|>m} |u_x - v_x| \to 0,\quad m\to +\infty$$

This means

$$u_x \in \overline{\bigcup_{v_x,\mu,x_0} E(v_x,\mu,x_0)}^{l_1}$$

Secondly, when u_x satisfies $\sum\limits_x |u_x - v_x| = |u_l - u_r|$, we might as well assume $u_0 - v_0 > 0$, set $u_0^{(m)} = u_0 - \dfrac{1}{m}$, and define

$$u_x^{(m)} = \begin{cases} u_0^{(m)}, & x = 0 \\ u_x, & x\ne 0 \end{cases}$$

then

(i) $$\sum_x |u_x^{(m)} - v_x| = \sum_{x\neq 0} |u_x - v_x| + |u_0^{(m)} - v_0|$$

$$= \sum_{x\in\mathcal{L}_\eta} |u_x - v_x| - \frac{1}{m} \quad (m \gg 1)$$

$$= |u_l - u_r| - \frac{1}{m} < |u_l - u_r|$$

(ii) $$\sum_{x\in\mathcal{L}_\eta} |u_x^{(m)} - u_x| = |u_0^{(m)} - u_0|$$

$$= \frac{1}{m} \to 0, \quad m \to +\infty$$

For every fixed m, there is a sequence

$$\{u_x^{(m,l)}\}_{l=1}^{+\infty} \subset \bigcup_{v_x,\mu,x_0} E(v_x,\mu,x_0)$$

and

$$\sum_{x\in\mathcal{L}_\eta} |u_x^{(m,l)} - u_x^{(m)}| \to 0, \quad l \to +\infty$$

Given an arbitrary $\varepsilon > 0$, there exists a sufficiently large m, such that

$$\sum_{x\in\mathcal{L}_\eta} |u_x - u_x^{(m)}| < \frac{\varepsilon}{2}$$

For this m, there is a l such that

$$\sum_{x\in\mathcal{L}_\eta} |u_x^{(m,l)} - u_x^{(m)}| < \frac{\varepsilon}{2}$$

Hence, there are m and l such that

$$\sum_{x\in\mathcal{L}_\eta} |u_x^{(m,l)} - u_x| \leq \sum_{x\in\mathcal{L}_\eta} |u_x^{(m)} - u_x| + \sum_{x\in\mathcal{L}_\eta} |u_x^{(m,l)} - u_x^{(m)}| < \varepsilon$$

which implies: $u_x \in \overline{\bigcup_{v_x,\mu,x_0} E(v_x,\mu,x_0)}^{l^1}$.

We turn now to show that if the initial value u_0 satisfies the conditions of Theorem 2.1, then

$$\sum_{j\in Z} |u_j^n - \phi_j^n| \to 0, \quad n \to +\infty$$

We notice that if the constant b_2 in Theorem 2.1 is smaller than $\frac{1}{2}|u_l - u_r|$, then u^0 belongs to $\overline{\bigcup_{v_x,\mu,x_0} E(v_x,\mu,x_0)}^{l^1}$. For this u_j^0, choose a sequence $\{(u_m)_j\}_{m=1}^{+\infty}$ such that every member of it not only satisfies Theorem 3.1 but also belongs to $\bigcup_{v_x,\mu,x_0} E(v_x,\mu,x_0)$ and

$$\sum_{j\in Z} |(u_m)_j^0 - u_j^0| \to 0, \quad m \to +\infty$$

From the conclusions above, for every m, there is a travelling wave ϕ_m such that

$$\sum_{j\in Z} |(u_m)_j^n - (\phi_m)_j^n| \to 0, \quad n \to +\infty$$

It follows that

$$\sum_{j\in Z} |(T^{nq}u_m)_j - (\phi_m)_j^0| \to 0, \quad n \to +\infty$$

Let $m_1, m_2 \in N$ and $\forall\varepsilon > 0$, then

$$\begin{aligned}
&\sum_{j\in Z} |(\phi_{m_2})_j^0 - (\phi_{m_1})_j^0| \\
&\le \sum_j |(T^{nq}u_{m_1})_j - (\phi_{m_1})_j^0| + \sum_j |(T^{nq}u_{m_2})_j - (\phi_{m_2})_j^0| \\
&\quad + \sum_j |(T^{nq}u_{m_2})_j - (T^{nq}u_{m_1})_j| \\
&\le \sum_j |(T^{nq}u_{m_1})_j - (\phi_{m_1})_j^0| + \sum_j |(T^{nq}u_{m_2})_j - (\phi_{m_2})_j^0| \\
&\quad + \sum_j |(u_{m_2})_j - (u_{m_1})_j|
\end{aligned}$$

We choose n sufficiently large, then

$$\sum_j |(T^{nq}u_{m_i})_j - (\phi_{m_i})_j^0| < \varepsilon, \quad i = 1, 2$$

therefore

$$\sum_{j\in Z} |(\phi_{m_2})_j^0 - (\phi_{m_1})_j^0| \le \sum_j |(u_{m_2})_j - (u_{m_1})_j| + 2\varepsilon$$

Since ε is an arbitrary small quantity, and $\{(u_m)_j\}_{m=1}^{+\infty}$ is a convergent sequence, $\{\phi_m\}$ must converge to a ϕ, i.e.

$$\sum_{j\in Z} |(\phi_m)_j - \phi_j| \to 0, \quad m \to +\infty$$

Now,

$$\begin{aligned}
\sum_j |u_j^n - \phi_j^n| &\le \sum_j |u_j^n - (u_m)_j^n| + \sum_j |u_j^n - (\phi_m)_j^n| + \sum_j |\phi_j^n - (\phi_m)_j^n| \\
&\le \sum_j |u_j^0 - (u_m)_j^0| + \sum_j |(u_m)_j^n - (\phi_m)_j^n| + \sum_j |\phi_j^0 - (\phi_m)_j^0|
\end{aligned}$$

We first choose m such that

$$\sum_j |u_j^0 - (u_m)_j^0| < \varepsilon, \quad \sum_j |\phi_j^0 - (\phi_m)_j^0| < \varepsilon$$

Fix the m, choose n sufficiently large, such that

$$\sum_j |(u_m)_j^n - (\phi_m)_j^n| < \varepsilon$$

Thereofe, it follows that

$$\sum_j |u_j^n - \phi_j^n| \to 0, \quad n \to +\infty$$

Finally, we can remove the restriction: $\sum_j |u_j^0 - \phi_j^0| \le b_2$.

Suppose u_j^0 satisfies the condition of Theorem 2.1, except the condition (i) is relaxed to

$$\sum_j |u_j^0 - w_j^0| < +\infty$$

Let ϕ_j^n be the travelling wave defined by (3.6), and assume

$$\sum_j |u_j^0 - \phi_j^0| = M$$

We choose a positive integer N, such that $\frac{M}{N} < b_2$, define

$$(u_i)_j^0 = \phi_j^0 + \frac{i}{N}(u_j^0 - \phi_j^0), \quad i = 1, 2, \cdots, N$$

Let us first exam $(u_1)_j^0$, since

$$\sum_j |(u_1)_j^0 - \phi_j^0| \le \frac{1}{N} \sum_j |(u_1)_j^0 - \phi_j^0| = \frac{M}{N} < b_2$$

and

$$\sum_j ((u_1)_j^0 - \phi_j^0) = 0$$

we get

$$\sum_j |(u_1)_j^n - \phi_j^n| \to 0, \quad n \to +\infty$$

By induction, supposing

$$\sum_j |(u_i)_j^n - \phi_j^n| \to 0$$

we study $(u_{i+1})_j^n$,

$$\begin{aligned}\sum_j |(u_{i+1})_j^n - \phi_j^n| &\le \sum_j |(u_{i+1})_j^n - (u_i)_j^n| + \sum_j |\phi_j^n - (u_i)_j^n| \\ &\le \sum_j |(u_{i+1})_j^0 - (u_i)_j^0| + \sum_j |\phi_j^n - (u_i)_j^n| \\ &= \frac{M}{N} + \sum_j |\phi_j^n - (u_i)_j^n|\end{aligned}$$

From the inductive hypothesis, we can find a sufficiently large n_0, such that

$$\sum_j |(u_{i+1})_j^{n_0} - \phi_j^{n_0}| < b_2$$

thus,

$$\sum_j |(u_{i+1})_j^n - \phi_j^n| \to 0, \quad n \to +\infty$$

So, for $(u_N)_j^0 = u_j^0$, we also have

$$\sum_j |u_j^n - \phi_j^n| \to 0, \quad n \to +\infty$$

which completes the proof.

4. Asymptotic Behavior for Systems

In this section, we investigate the asymptotic stability of the L-F scheme (1.2) approximating general systems of conservation laws (1.1). For simplicity, we focus our attention on the single-shock case, because the discussion for the multiple-shock case is similar.

Let (1.1) be a $m \times m$ system. We assume that it is strictly hyperbolic in the sense that at each state $u \in R^m$ the Jacobian $\nabla f(u)$ has m real and distinct eigenvalues,

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_m(u)$$

with corresponding left and right eigenvectors $l_i(u)$ and $r_i(u)$. We can normalize them into $l_i(u) r_k(u) = \delta_{ik}$, and let

$$L(u) = (l_1(u)^T, \cdots, l_m(u)^T)^T, \quad R(u) = (r_1(u), \cdots, r_m(u))$$

$$\Lambda(u) = \text{diag}\,(\lambda_1(u), \lambda_2(u), \cdots, \lambda_m(u))$$

Assume that the k-characteristic field is genuinely nonlinear, i.e., $\nabla\lambda_k \cdot r_k \neq 0$. Given two constant vectors u_- and u_+, such that, (1.1) has a single-shock solution which satisfies the Rankine-Hugoniot relation and Lax's geometric entropy condition,

$$\lambda_k(u_+) < s < \lambda_k(u_-)$$

From the results in [11], on the expanded grid the L-F scheme has a corresponding discrete shock solution. Now, suppose u_j^n to be a scheme solution satisfying

$$\lim_{j\to-\infty} u_j^n = u_-, \quad \lim_{j\to+\infty} u_j^n = u_+$$

Our main result in this section is as follows:

Theorem 4.1 *Let ϕ_j^n and ψ_j^n be two discrete shock profiles satisfying*

$$\sum_{j=odd}(u_j^0-\phi_j^0)=0, \quad \sum_{j=even}(u_j^0-\psi_j^0)=0 \tag{4.1}$$

and there are three constants c_1, c_2, c_3, such that

$$\varepsilon = |u_+ - u_-| \le c_1 \tag{4.2}$$

$$\sum_{j=odd}(1+j^2)|u_j^0-\phi_j^0|^2 + \sum_{j=even}(1+j^2)|u_j^0-\psi_j^0|^2 \le c_2 \tag{4.3}$$

$$\lambda \sup_i |\lambda_i(u)| \le c_3 \tag{4.4}$$

then

$$\lim_{n\to\infty}\Big(\sum_{j+n=odd}|u_j^n-\phi_j^n|^2 + \sum_{j+n=even}|u_j^n-\psi_j^n|^2\Big)=0 \tag{4.5}$$

Furthermore, if

$$\sum_{j=odd}(1+j^2)^{3/2}|u_j^0-\phi_j^0|^2 + \sum_{j=even}(1+j^2)^{3/2}|u_j^0-\psi_j^0|^2 < +\infty \tag{4.6}$$

then we have

$$\sup_{0\le n<\infty}\Big(\sum_{j+n=odd}|u_j^n-\phi_j^n| + \sum_{j+n=even}|u_j^n-\psi_j^n|\Big) < +\infty \tag{4.7}$$

Proof Most of the proofs for this theorem are the same as those of Theorem 1.1 in [10], we only give the different part here. And we only discuss the case for odd grid nodes, the discussion for the even nodes is a simple rewriting. Set

$$\bar v_j^n = \sum_{k\le j, k+n=odd}(u_k^n-\phi_k^n)$$

then from (1.2), v_j^n satisfies

$$\bar v_j^{n+1} - \frac{\bar v_{j+1}^n + \bar v_{j-1}^n}{2} + \frac{\lambda}{2}(f(u_{j+1}^n) - f(\phi_{j+1}^n)) = 0 \tag{4.8}$$

Let

$$l_j^n = L(\phi_j^n), r_j^n = R(\phi_j^n), \Lambda_j^n = \Lambda(\phi_j^n), v_j^n = l_j^n \bar v_j^n \tag{4.9}$$

then we have

$$\bar{v}_j^n = r_j^n v_j^n \tag{4.10}$$

and

$$u_{j+1}^n - \phi_{j+1}^n = \bar{v}_{j+1}^n - \bar{v}_{j-1}^n \tag{4.11}$$

Left-multiplying (4.8) by $\frac{1}{2}(l_{j+1}^n + l_{j-1}^n)$, we get

$$\begin{aligned} v_j^{n+1} - \frac{v_{j+1}^n + v_{j-1}^n}{2} + \frac{\lambda}{2}\frac{l_{j+1}^n + l_{j-1}^n}{2}\tilde{f}'(u_{j+1}^n - \phi_{j+1}^n) &= \Big(l_j^{n+1} - \frac{1}{2}l_{j+1}^n - \frac{1}{2}l_{j-1}^n\Big)\bar{v}_j^{n+1} \\ &\quad - \frac{l_{j-1}^n - l_{j+1}^n}{4}\bar{v}_{j+1}^n - \frac{l_{j-1}^n - l_{j+1}^n}{4}\bar{v}_{j-1}^n \end{aligned} \tag{4.12}$$

where $\tilde{f}'$ denotes that each element of the Jacobian $\nabla f(u)$ takes a mean value (need not at the same point). By Taylor's formula and (4.11), the left hand of (4.12) can be written into

$$\begin{aligned} &\frac{\lambda}{2}\,\frac{l_{j+1}^n + l_{j-1}^n}{2}\tilde{f}'(u_{j+1}^n - \phi_{j+1}^n) \\ &\quad = \frac{\lambda}{2}\frac{(l_{j+1}^n + l_{j-1}^n)}{2}\tilde{f}'(\bar{v}_{j+1}^n - \bar{v}_{j-1}^n) \\ &\quad = \frac{\lambda}{2}\frac{\Lambda_{j+1}^n + \Lambda_{j-1}^n}{2}(v_{j+1}^n - v_{j-1}^n) + \frac{\lambda}{4}\Lambda_{j-1}^n(l_{j-1}^n - l_{j+1}^n)\bar{v}_{j+1}^n \\ &\qquad - \frac{\lambda}{4}\Lambda_{j+1}^n(l_{j+1}^n - l_{j-1}^n)\bar{v}_{j-1}^n + O(1)(|\bar{v}_{j+1}^n - \bar{v}_{j-1}^n|^2) \end{aligned} \tag{4.13}$$

From (4.8) and (4.10)

$$\bar{v}_j^{n+1} = \frac{1}{2}(r_{j+1}^n v_{j+1}^n + r_{j-1}^n v_{j-1}^n) - \frac{\lambda}{2}\tilde{f}'(r_{j+1}^n v_{j+1}^n - r_{j-1}^n v_{j-1}^n)$$

Hence,

$$\begin{aligned} &\Big(l_j^{n+1} - \frac{1}{2}l_{j+1}^n - \frac{1}{2}l_{j-1}^n\Big)\bar{v}_j^{n+1} + \frac{\lambda}{4}\Lambda_{j+1}^n(l_{j+1}^n - l_{j-1}^n)\bar{v}_{j-1}^n \\ &\quad + \frac{\lambda}{4}\Lambda_{j-1}^n(l_{j+1}^n - l_{j-1}^n)\bar{v}_{j+1}^n \\ &= \Big(l_j^{n+1} - \frac{1}{2}l_{j+1}^n - \frac{1}{2}l_{j-1}^n + \frac{s\lambda}{2}(l_{j+1}^n - l_{j-1}^n)\Big)\cdot\frac{1}{2}(r_{j+1}^n v_{j+1}^n + r_{j-1}^n v_{j-1}^n) \\ &\quad - \Big(l_j^{n+1} - \frac{1}{2}l_{j+1}^n - \frac{1}{2}l_{j-1}^n\Big)\frac{\lambda}{2}\tilde{f}'(r_{j+1}^n v_{j+1}^n - r_{j-1}^n v_{j-1}^n) \\ &\quad + \frac{\lambda}{4}(\Lambda_{j+1}^n - s)(l_{j+1}^n - l_{j-1}^n)r_{j-1}^n v_{j-1}^n + \frac{\lambda}{4}(\Lambda_{j-1}^n - s)(l_{j+1}^n - l_{j-1}^n)r_{j+1}^n v_{j+1}^n \end{aligned} \tag{4.14}$$

Substituting (4.13), (4.14) into (4.12), one can get

$$v_j^{n+1}-\frac{v_{j+1}^n+v_{j-1}^n}{2}+\frac{\lambda}{2}\frac{(\Lambda_{j+1}^n+\Lambda_{j-1}^n)}{2}(v_{j+1}^n-v_{j-1}^n)$$

$$=-\frac{1}{4}(l_{j+1}^n-l_{j-1}^n)(r_{j+1}^n v_{j+1}^n-r_{j-1}^n v_{j-1}^n)$$

$$+\Big(l_j^{n+1}-\frac{1}{2}l_{j+1}^n-\frac{1}{2}l_{j-1}^n+\frac{s\lambda}{2}(l_{j+1}^n-l_{j-1}^n)\Big)\cdot\frac{1}{2}(r_{j+1}^n v_{j+1}^n+r_{j-1}^n v_{j-1}^n)$$

$$-\Big(l_j^{n+1}-\frac{1}{2}l_{j+1}^n-\frac{1}{2}l_{j-1}^n\Big)\frac{\lambda}{2}\bar{f}'(r_{j+1}^n v_{j+1}^n-r_{j-1}^n v_{j-1}^n)$$

$$+\frac{\lambda}{4}(\Lambda_{j+1}^n-s)(l_{j+1}^n-l_{j-1}^n)r_{j-1}^n v_{j-1}^n+\frac{\lambda}{4}(\Lambda_{j-1}^n-s)(l_{j+1}^n-l_{j-1}^n)r_{j+1}^n v_{j+1}^n$$

$$+O(|r_{j+1}^n v_{j+1}^n-r_{j-1}^n v_{j-1}^n|^2)$$

$$\equiv(A)+(B)+(C)+(D)+(E)+(F) \tag{4.15}$$

Taking scalar production with $(v_{j+1}^n+v_{j-1}^n)$, and summing up with respect to j, one can get

$$\sum_{j+n=odd}(v_{j+1}^n+v_{j-1}^n)\Big(v_j^{n+1}-\frac{v_{j+1}^n+v_{j-1}^n}{2}\Big)$$

$$\geq\sum_{j+n=odd}\Big(|v_j^{n+1}|^2-|v_{j+1}^n|^2+\frac{1}{4}(1-Cc_3)|v_{j+1}^n-v_{j-1}^n|^2\Big) \tag{4.16}$$

Since Λ is a diagonal matrix, the second term of the left hand of (4.15) equals

$$\frac{\lambda}{2}\sum_{j+n=odd}(v_{j+1}^n+v_{j-1}^n)^T\frac{\Lambda_{j+1}^n+\Lambda_{j-1}^n}{2}(v_{j+1}^n-v_{j-1}^n)$$

$$=\frac{\lambda}{2}\sum_{j+n=odd}\Big[(v_{j+1}^n)^T\frac{\Lambda_{j+1}^n+\Lambda_{j-1}^n}{2}v_{j+1}^n-(v_{j-1}^n)^T\frac{\Lambda_{j+1}^n+\Lambda_{j-1}^n}{2}v_{j-1}^n\Big]$$

$$=\frac{\lambda}{2}\sum_{j+n=odd}(v_{j-1}^n)^T\frac{\Lambda_{j-3}^n-\Lambda_{j+1}^n}{2}v_{j-1}^n$$

$$=\frac{\lambda}{2}\sum_{j+n=odd}\Big[(v_{j-1}^n)^T\frac{\lambda_{k,j-3}^n-\lambda_{k,j+1}^n}{2}v_{j-1}^n$$

$$+(v_{j-1}^n)^T\Big(\frac{\Lambda_{j-3}^n-\Lambda_{j+1}^n}{2}-\frac{\lambda_{k,j-3}^n-\lambda_{k,j+1}^n}{2}\Big)v_{j-1}^n\Big]$$

of which the first term has the following lower bound

$$\frac{\lambda}{2}\sum_{j+n=odd}\frac{\lambda_{k,j-1}^n-\lambda_{k,j+1}^n}{2}|v_{j-1}^n|^2$$

From (2.24c) in [10], the second term can be estimated as

$$\begin{aligned}
&\Big|\frac{\lambda}{2}\sum_{j+n=odd}(v_{j-1}^n)^T\Big(\frac{\Lambda_{j-3}^n-\Lambda_{j+1}^n}{2}-\frac{\lambda_{k,j-3}^n-\lambda_{k,j+1}^n}{2}\Big)v_{j-1}^n\Big| \\
&\qquad\le C\sum_{j+n=odd}|\phi_{j+1}^n-\phi_{j-1}^n|\Big[\sum_{i\neq k}|v_{i,j-1}^n|^2+|v_{j+1}^n-v_{j-1}^n|^2\Big] \\
&\qquad\le C\sum_{j+n=odd}(\lambda_{k,j-1}^n-\lambda_{k,j+1}^n)\Big(\sum_{i\neq k}|v_{i,j-1}^n|^2+|v_{j+1}^n-v_{j-1}^n|^2\Big)
\end{aligned}$$

Next, we estimate every term in the right hand of (4.15). From (2.24b), (2.24c) in [10], we get

$$\begin{aligned}
&\Big|\sum_{j+n=odd}(v_{j+1}^n+v_{j-1}^n)\cdot(A)\Big| \\
&\quad=\frac{1}{4}\Big|\sum_{j+n=odd}(v_{j+1}^n+v_{j-1}^n)^T(l_{j+1}^n-l_{j-1}^n)(r_{j+1}^n(v_{j+1}^n-v_{j-1}^n)+(r_{j+1}^n-r_{j-1}^n)v_{j-1}^n)\Big| \\
&\quad\le C\sum_{j+n=odd}(\lambda_{k,j-1}^n-\lambda_{k,j+1}^n)(|v_{j+1}^n-v_{j-1}^n|^2+(\lambda_{k,j-1}^n-\lambda_{k,j+1}^n)|v_{j-1}^n|^2) \\
&\quad\le C\varepsilon\sum_{j+n=odd}(|v_{j+1}^n-v_{j-1}^n|^2+(\lambda_{k,j-1}^n-\lambda_{k,j+1}^n)|v_{j-1}^n|^2)
\end{aligned}$$

In order to estimate (B), since ϕ_j^n satisfies (1.2), we have

$$\begin{aligned}
l_j^{n+1}&-\frac{1}{2}l_{j+1}^n-\frac{1}{2}l_{j-1}^n+\frac{s\lambda}{2}(l_{j+1}^n-l_{j-1}^n) \\
&=(1+O(\varepsilon))\nabla l(\phi_{j-1}^n)\Big(\phi_j^{n+1}-\frac{1}{2}\phi_{j+1}^n-\frac{1}{2}\phi_{j-1}^n+\frac{s\lambda}{2}(\phi_{j+1}^n-\phi_{j-1}^n)\Big) \\
&=(1+O(\varepsilon))\nabla l(\phi_{j-1}^n)\Big(\frac{\lambda}{2}(\tilde{f}'-s)(\phi_{j+1}^n-\phi_{j-1}^n)\Big) \\
&=O(\varepsilon)(\lambda_{k,j-1}^n-\lambda_{k,j+1}^n) \qquad (4.17)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Big|\sum_{j+n=odd}(v_{j+1}^n+v_{j-1}^n)\cdot(B)\Big| \\
&\qquad\le O(\varepsilon)\sum_{j+n=odd}(\lambda_{k,j-1}^n-\lambda_{k,j+1}^n)(|v_{j-1}^n|^2+|v_{j+1}^n-v_{j-1}^n|^2)
\end{aligned}$$

From (4.17),

$$l_j^{n+1}-\frac{1}{2}l_{j+1}^n-\frac{1}{2}l_{j-1}^n=O(1)(\lambda_{k,j-1}^n-\lambda_{k,j+1}^n)$$

Therefore,

$$\Big|\sum_{j+n=odd}(v^n_{j+1}+v^n_{j-1})\cdot(C)\Big|$$
$$\le\sum_{j+n=odd}\Big(\frac{\lambda}{16}(\lambda^n_{k,j-1}-\lambda^n_{k,j+1})|v^n_{j-1}|^2+O(\varepsilon)|v^n_{j+1}-v^n_{j-1}|^2\Big)$$

Similarly, from (2.26b), (2.26c) in [10] we have

$$\Big|\sum_{j+n=odd}(v^n_{j+1}+v^n_{j-1})\cdot((D)+(E))\Big|$$
$$\le O(\varepsilon)\sum_{j+n=odd}(\lambda^n_{k,j-1}-\lambda^n_{k,j+1})(|v^n_{j-1}|^2+|v^n_{j+1}-v^n_{j-1}|^2)$$

Finally, since

$$|(r^n_{j+1}v^n_{j+1}-r^n_{j-1}v^n_{j-1})|^2$$
$$\le 2(|r^n_{j+1}(v^n_{j+1}-v^n_{j-1})|^2+|(r^n_{j+1}-r^n_{j-1})v^n_{j-1}|^2)$$
$$\le C(|(v^n_{j+1}-v^n_{j-1})|^2+(\lambda^n_{k,j-1}-\lambda^n_{k,j+1})|v^n_{j-1}|^2)$$

therefore,

$$\Big|\sum_{j+n=odd}(v^n_{j+1}+v^n_{j-1})\cdot(F)\Big|$$
$$\le C\sup_{j,n}|v^n_j|\sum_{j+n=odd}(|v^n_{j+1}-v^n_{j-1}|^2+(\lambda^n_{k,j-1}-\lambda^n_{k,j+1})|v^n_{j-1}|^2)$$

By taking c_1, c_3 suitably small, and *a priori* assuming that $\sup_{j,n}|v^n_j|$ is sufficiently small, then we have

$$\sum_{j+n=odd}\Big(|v^n_{j+1}|^2-|v^n_{j-1}|^2+\frac{1}{8}|v^n_{j+1}-v^n_{j-1}|^2+\frac{\lambda}{8}(\lambda^n_{k,j-1}-\lambda^n_{k,j+1})|v^n_{k,j-1}|^2\Big)$$
$$\le C\sum_{j+n=odd}(\lambda^n_{k,j-1}-\lambda^n_{k,j+1})\Big|\sum_{i\neq k}\Big|v^n_{i,j-1}|^2 \qquad (4.18)$$

In order to use the results in [10], on the grid nodes which satisfy $j+n=even$, we temporarily set

$$u^n_j=u^n_{j-1},\quad \phi^n_j=\phi^n_{j-1}$$

then the distributions of the functions on odd and even nodes are identical. Hence, on even nodes, an estimation like (4.18) holds. If put them together we can get

$$\sum_j\Big(|v^n_{j+1}|^2-|v^n_{j-1}|^2+\frac{1}{8}|v^n_{j+1}-v^n_{j-1}|^2+\frac{\lambda}{8}(\lambda^n_{k,j}-\lambda^n_{k,j+1})|v^n_{k,j}|^2\Big)$$
$$\le C\sum_j(\lambda^n_{k,j}-\lambda^n_{k,j+1})\Big|\sum_{i\neq k}|v^n_{i,j}|^2$$

We can see that this is already identical to (3.20) in [10], and all the process followed (3.20) of [10] can be applied here without any change. So we complete the proof.

References

[1] Oleinik O., Discontinuous solutions of nonlinear differential equations, *Usp. Mat. Nauk.*, **27** (1957), 3–73.

[2] Diperna R., Convergence of approximation solutions of conservation laws, *Arch. Rat. Mech. Anal.*, **82** (1983), 27–70.

[3] Ding Xiaxi, Chen Guiqiang, Luo Peizhu, Convergence of the Lax-Friedrichs schemes for isentropic gas dynamics, *Acta Mathematica Scientia*, **5** (1985), 433–472.

[4] Crandall M.G. and Majda A., Monotone Difference approximations for scalar conservation laws, *Math. Comp.*, **34** (1980), 1–28.

[5] Kuznetzov N.N., On stable methods for solving nonlinear first order partial differential equations in the class of discontimuous functions, *Topics in Numerical Analysis*, Acad. Press Inc. London. **3** (1977), 183–197.

[6] Lax P.D. and Wendroff B., Systems of conservation laws, *Commu. Pure Appl. Math.*, **13** (1960), 217–237.

[7] Jennings G., Discrete Shocks, *Comm. Pure Appl. Math.*, **27** (1974), 25–37.

[8] Enqquist B. and Osher S.. One-sided difference approximations for nonlinear conservation laws, *Math. Comp.*, **36** (1981), 321–351.

[9] Chern I.L., Large-Time Behavior of Solutions of Lax-Friedrichs Finite Difference Equations for Hyperbolic Systems of Conservation Laws, *Math. Comp.*, **56** (1991), 107–118.

[10] Liu J.G., Xin Z.P., Nonlinear stability of discrete shocks for systems of conservation laws, preprint (1992).

[11] Majda A. and Ralston J., Discrete shock profiles for systems of conservation laws, *Comm. Pure Appl. Math.*, **32** (1979), 445–483.

[12] Smyrlis Y.S., Existence and Stability of Stationary Profiles of the LW Scheme, *Comm. Pure Appl. Math.*, **43** (1990), 508–545.

[13] Szepessy A., Nonlinear stability of shock waves of a finite element method for systems of conservation laws, preprint (1992).

[14] Yu S.H., Nonexistence of the local travelling profiles for the Lax-Wendroff scheme, preprint (1992).

[15] Zhou T., The Long-Time Asymptotic Behavior of the Monotone Schemes, Master's Degree Dissertation, Peking University. (1990).

[16] Liu T.P., Nonlinear stability of shock waves for viscous conservation laws, *Mem. Amer. Math. Soc.*, **328** (1985), 1–108.