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ON THE CAHN-HILLIARD EQUATION WITH NONLINEAR PRINCIPAL PART*

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Abstract We study the Cahn-Hilliard equation with nonlinear principal part

$$\frac{\partial u}{\partial t}+D[m(u)(kD^3u-DA(u))]=0$$

The existence of classical solutions is established by means of the method based on Campanato spaces and the energy estimates. The corresponding uniqueness is also proved.
Key Words Cahn-Hilliard equation; existence; uniqueness; Campanato space.
Classification 35K.

1. Introduction

The Cahn-Hilliard equation, namely

$$\frac{\partial u}{\partial t}+D[m(u)(kD^3u-DA(u))]=0 \quad \text{in } Q_T=(0,T)\times(0,1) \tag{1.1}$$

is based on a continuum model for phase transition in binary system such as alloy, glasses and polymer-mixtures, see [1], [2]. Here $u(t,x)$ is the concentration of one of the phase of the system, $m(u)$ the mobility, k a positive constant,

$$J=m(u)(kD^3u-DA(u))$$

the net flux, $D=\frac{\partial}{\partial x}$. Based on physical consideration, the equation (1.1) is supplemented with the zero net flux boundary value condition

$$J\Big|_{x=0,1}=0 \tag{1.2}$$

the natural boundary value condition

$$Du\Big|_{x=0,1}=0 \tag{1.3}$$

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and the initial value condition

$$u(0,x) = u_0(x) \tag{1.4}$$

Using the energy method, Elliott and Zheng Songmu [3] have successfully treated the problem (1.1)–(1.4) for (1.1) with linearized principal part, namely for (1.1) in which $m(u)$ is a positive constant. And for $A(s) = -s + \gamma_1 s^2 + \gamma_2 s^3$ with $\gamma_2 > 0$, the global existence of classical solutions of the problem (1.1)–(1.4) is established.

Since, in many situations, the mobility $m(u)$ depends on the concentration u in general, the investigation for (1.1) with nonlinear principal part seems to be a natural continuation of the pioneering work [3]. In this paper, we discuss the solvability of classical solutions of the problem (1.1)–(1.4) under the following much more general assumptions

$$m(s) > 0, \quad H(s) \equiv \int_0^s A(\sigma)d\sigma \geq -\mu, \quad \mu > 0$$

in which the non-uniform parabolicity for (1.1) is allowed and $A(s)$ is permitted to be some polynomial of odd order like $-s + \gamma_1 s^2 + \gamma_2 s^3$ with $\gamma_2 > 0$. The discussion for the degenerate case of the equation (1.1) will be subsequently presented in our next paper, where certain structure conditions are proposed ensuring the existence of "Physical Solutions", namely the solutions with the property that $0 \leq u \leq 1$. An interesting work for such kind of equation of degenerate type can be found in the recent work by Bernis and Friedman [7].

The main difficulties for treating the problem (1.1)–(1.4) are caused by the nonlinearity of the principal part and the lack of maximum principle. Due to the nonlinearity of the principal part, there are more difficulties in establishing the global existence of classical solutions. The method we use is based on the Schauder type priori estimates, which are relatively less used for such kind of parabolic equations of fourth order. Here the Schauder type estimates will be obtained by means of a modified Campanato space. We note that the Campanato spaces have been widely used to the discussion of partial regularity of solutions of parabolic systems of second order. Because of the lack of maximum principle, the actually used Campanato space is a modified version. In fact, after such modification, the terms related to supremum norm will not appear in deriving the key estimate (3.9). A detailed description and the associated properties of such space will be given in Section 2. Subsequently we prove the existence of classical solutions of the problem (1.1)–(1.4). The uniqueness is also discussed in this section.

2. A Modified Campanato Space and the Hölder Norm Priori Estimates

Let $Q_T = (0,T) \times (0,1)$, $y_0 = (t_0, x_0) \in \overline{Q}_T$. For any fixed $R > 0$, define

$$B_R = B_R(x_0) = (x_0 - R, x_0 + R)$$
$$I_R = I_R(t_0) = (t_0 - R^4, t_0 + R^4)$$
$$Q_R = Q_R(y_0) = I_R(t_0) \times B_R(x_0)$$
$$S_R = Q_R \cap Q_T$$
$$E_R = E_R(x_0) = B_R(x_0) \cap (0,1)$$
$$J_R = J_R(t_0) = I_R(t_0) \cap (0, +\infty)$$
$$d(y_1, y_2) = |x_1 - x_2| + |t_1 - t_2|^{\frac{1}{4}}$$

Let u be a function defined on Q_T and set

$$u_R = u_{y_0,R} = \frac{1}{S_R} \iint_{S_R} u\,dt\,dx$$
$$\hat{u}_R = \hat{u}_{y_0,R} = \begin{cases} u_R, & \text{if } Q_R \cap \partial_p Q_T = \emptyset \\ 0, & \text{if } Q_R \cap \partial_p Q_T \neq \emptyset \end{cases}$$

where $\partial_p Q_T$ is the parabolic boundary of Q_T, $|S_R| = \text{mes } S_R$.

Definition *Let* $\lambda > 0$,

$$C_*(\overline{Q}_T) = \{u \in C(\overline{Q}_T);\quad u = 0 \ on\ \partial_p Q_T\}$$

For any $u \in C_*(\overline{Q}_T)$, *denote*

$$M^2[u] = \sup_{y_0 \in \overline{Q}_T} \sup_{0<R\le R_0} \frac{1}{R^\lambda} \iint_{S_R(y_0)} |u(t,x) - \hat{u}_{y_0,R}|^2 dt\,dx$$

where $R_0 = \text{diam } Q_T$. *By the space* $\mathcal{L}_0^{2,\lambda}(Q_T)$ *we mean the subset of* $C_*(\overline{Q}_T)$, *each element of which satisfies* $M[u] < +\infty$. *For* $u \in \mathcal{L}_0^{2,\lambda}(Q_T)$, *we define its norm as*

$$\|u\|_{\mathcal{L}_0^{2,\lambda}(Q_T)} = \sup_{Q_T} |u(t,x)| + M[u]$$

For a bounded n-dimensional domain Ω, Campanato defined a similar space $\mathcal{L}^{2,\lambda}(\Omega)$, the set of all functions in $L^2(\Omega)$ satisfying

$$[u]^2_{\mathcal{L}^{2,\lambda}(\Omega)} \equiv \sup_{\substack{z_0 \in \Omega \\ \rho > 0}} \frac{1}{\rho^\lambda} \int_{\Omega(z_0,\rho)} |u(z) - u_{z_0,\rho}|^2 dz < +\infty$$

where

$$u_{z_0,\rho} = \frac{1}{|\Omega(z_0,\rho)|} \int_{\Omega(z_0,\rho)} u(z)dz$$
$$\Omega(z_0,\rho) = \{z \in \Omega; |z - z_0| < \rho\}$$

Taking $n=2$, $\Omega = Q_T$, we obtain a space $\mathcal{L}^{2,\lambda}(Q_T)$ which has some similarity with $\mathcal{L}_0^{2,\lambda}(Q_T)$. For a function u defined on Q_T, if the boundary value of the function is not considered, it is really possible to apply $\mathcal{L}^{2,\lambda}(Q_T)$ to describe its interior behaviour. But when prescribed the boundary value, we have to use the modified Campanato space $\mathcal{L}_0^{2,\lambda}(Q_T)$ to describe its global behaviour, in particular the behaviour near the parabolic boundary. This advantage and the following property of the space $\mathcal{L}_0^{2,\lambda}(Q_T)$ will be useful in estimating the Hölder norm of solutions of some fourth order parabolic problems, in particular, the Hölder norm of the derivative Du of solutions of the problem (1.1)-(1.4).

Theorem 2.1 *Let $\lambda > 5$. Then there is an embedding*

$$\mathcal{L}_0^{2,\lambda}(Q_T) \subset C^{\frac{\alpha}{4},\alpha}(\overline{Q}_T)$$

where $\alpha = \dfrac{\lambda-5}{2}$. *Moreover, the embedding is continuous and*

$$\|u\|_{C^{\frac{\alpha}{4},\alpha}(\overline{Q}_T)} \le C(\lambda)\|u\|_{\mathcal{L}_0^{2,\lambda}(Q_T)}$$

where $C(\lambda)$ depends only on λ.

Proof The method we use follows the idea of Campanato [5]. For $0<\rho<R$ and $y_0 \in \overline{Q}_T$, we have

$$|\hat{u}_{y_0,R} - \hat{u}_{y_0,\rho}|^2 \le 2|u(t,x) - \hat{u}_{y_0,\rho}|^2 + 2|u(t,x) - \hat{u}_{y_0,R}|^2$$

Integrating the inequality with respect to (t,x) over $S_\rho = S_\rho(y_0)$,

$$\begin{aligned}
&|S_\rho||\hat{u}_{y_0,R} - \hat{u}_{y_0,\rho}|^2 \\
&\le 2\int\int_{S_\rho} |u(t,x) - \hat{u}_{y_0,R}|^2 dtdx + 2\int\int_{S_\rho} |u(t,x) - \hat{u}_{y_0,\rho}|^2 dtdx \\
&\le 4M^2[u]R^\lambda
\end{aligned}$$

Since $S_\rho| \ge \rho^5$, it follows that

$$|\hat{u}_{y_0,R} - \hat{u}_{y_0,\rho}|^2 \le 4M^2[u]\frac{R^\lambda}{\rho^5}$$

In particular, we have

$$|\hat{u}_{y_0,R} - \hat{u}_{y_0,\frac{R}{2}}|^2 \le 128M^2[u]R^{\lambda-5}$$

$$|\hat{u}_{y_0,\frac{R}{2}} - \hat{u}_{y_0,\frac{R}{4}}|^2 \le 128M^2[u]R^{\lambda-5}\left(\frac{1}{2}\right)^{\lambda-5}$$

$$\cdots \quad \cdots \quad \cdots$$

$$|\hat{u}_{y_0,\frac{R}{2^k}} - \hat{u}_{y_0,\frac{R}{2^{k+1}}}|^2 \le 128M^2[u]R^{\lambda-5}\left(\frac{1}{2^k}\right)^{\lambda-5}$$

Summing up these inequalities, we get

$$|\hat{u}_{y_0,R} - \hat{u}_{y_0,\frac{R}{2^{k+1}}}|^2 \le C(\lambda)M^2[u]R^{\lambda-5} \tag{2.1}$$

By the definition of $\hat{u}_{y_0,\rho}$ and the continuity of u, it is easily seen that

$$\lim_{\rho\to 0} \hat{u}_{y_0,\rho} = u(t_0, x_0)$$

and hence by letting k tend to infinity,

$$|\hat{u}_{y_0,R} - u(t_0,x_0)|^2 \le C(\lambda)M^2[u]R^{\lambda-5} \tag{2.2}$$

Let $y_1 = (t_1, x_1)$, $y_2 = (t_2, x_2) \in Q_T$ and set $R = d(y_1, y_2)$. If $d(y_1, \partial_p Q_T) \le 2R$, then

$$\hat{u}_{y_1,4R} = \hat{u}_{y_2,4R} = 0$$

and hence by (2.2),

$$\begin{aligned} &|u(t_1,x_1) - u(t_2,x_2)|^2 \\ &\le 2|u(t_1,x_1) - \hat{u}_{y_1,4R}|^2 + 2|u(t_2,x_2) - \hat{u}_{y_2,4R}|^2 \\ &\le C(\lambda)M^2[u]R^{\lambda-5} \end{aligned}$$

The same is true for the case where $d(y_2, \partial_p Q_T) \le 2R$.

If $d(y_1, \partial_p Q_T) > 2R$ and $d(y_2, \partial_p Q_T) > 2R$, then

$$\begin{aligned} |u(t_1,x_1) - (t_2,x_2)|^2 \le\ & 4|u(t_1,x_1) - u_{y_1,2R}|^2 \\ & + 4|u(t_2,x_2) - u_{y_2,2R}|^2 + 4|u_{y_1,2R} - u_{y_2,2R}|^2 \end{aligned}$$

From (2.2), we conclude that the first and second terms in the right hand-side of the above inequality can be estimated by $C(\lambda)M^2[u]R^{\lambda-5}$. For the third term, we notice that

$$|u_{y_1,2R} - u_{y_2,2R}|^2 \le 2|u(t,x) - u_{y_1,2R}|^2 + 2|u(t,x) - u_{y_2,2R}|^2$$

Integrating this inequality with respect to (t,x) over $Q_{y_1,2R} \cap Q_{y_2,2R}$ and noticing that

$$|Q_{y_1,2R} \cap Q_{y_2,2R}| \ge |Q_{y_1,2R}| \ge R^5$$

we have

$$\begin{aligned} &|u_{y_1,2R} - u_{y_2,2R}|^2 \\ &\le \frac{2}{R^5}\int\int_{Q_{y_1,2R}\cap Q_{y_2,2R}} [|u(t,x) - u_{y_1,2R}|^2 + |u(t,x) - u_{y_2,2R}|^2] dtdx \\ &\le \frac{2}{R^5}\int\int_{Q_{y_1,2R}} |u(t,x) - u_{y_1,2R}|^2 dtdx + \frac{2}{R^5}\int\int_{Q_{y_2,2R}} |u(t,x) - u_{y_2,2R}|^2 dtdx \\ &\le C(\lambda)M^2[u]R^{\lambda-5} \end{aligned}$$

Thus

$$|u(t_1,x_1)-u(t_2,x_2)|^2\leq C(\lambda)M^2[u]R^{\lambda-5}$$

The proof is completed.

Now, we consider the following linear problem

$$\frac{\partial u}{\partial t}+D^2(a(t,x)D^2u)=D^2f \quad \text{in } Q_T=(0,T)\times(0,1) \tag{2.3}$$

$$u(t,0)=u(t,1)=D^2u(t,0)=D^2u(t,1)=0 \tag{2.4}$$

$$u(0,x)=0 \tag{2.5}$$

Here we do not restrict the smoothness of the given functions $a(t,x)$ and $f(t,x)$, but simply assume that they are sufficiently smooth. Our main purpose is to find the relation between the Hölder norm of the solution u and $a(t,x)$, $f(t,x)$. The parabolicity assumption is

$$0<a_0\leq a(t,x)\leq A_0 \tag{2.6}$$

Let $y_0=(t_0,x_0)\in\overline{Q}_T$ be a fixed point and define

$$\varphi(u,\rho)=\int\int_{S_\rho}(|u-\hat{u}_\rho|^2+\rho^4|D^2u|^2)dtdx,\quad(\rho>0)$$

Let u be the solution of the problem (2.3)-(2.5). We split u on $S_R=S_R(y_0)$ as $u=u_1+u_2$, where u_1 is the solution of the problem

$$\frac{\partial u_1}{\partial t}+a(t_0,x_0)D^4u_1=0 \quad \text{in } S_R \tag{2.7}$$

$$u_1\Big|_{\partial_pS_R}=u\Big|_{\partial_pS_R},\quad B(x,D)u_1\Big|_{\partial E_R}=B(x,D)u\Big|_{\partial E_R} \tag{2.8}$$

and u_2 solves the problem

$$\frac{\partial u_2}{\partial t}+a(t_0,x_0)D^4u_2=D^2[a(t_0,x_0)-a(t,x))D^2u]+D^2f \quad \text{in } S_R \tag{2.9}$$

$$u_2\Big|_{\partial_pS_R}=0,\quad B(x,D)u_2\Big|_{\partial E_R}=0 \tag{2.10}$$

where $B(x,D)=D^2$ for $x=0,1$ and $B(x,D)=D$ for $x\neq 0,1$. By classical linear theory, the above decomposition is uniquely determined by u. The solution u_1, u_2 are sufficiently smooth in

$$S_R^*=\{(t,x)\in\overline{S}_R;t>\inf J_R\}$$

and satisfy u_i, $Du_i\in C(\overline{S}_R)$, $D^4u_i\in L^2(S_R)$ $(i=1,2)$ (see Section 4).

Some essential estimates on u_1 and u_2 are based on the following lemmas.

Lemma 2.2 *For the solution u_2 of the problem (2.9), (2.10), we have*

$$\sup_{J_R}\int_{E_R} u_2^2(t,x)dx+\int\int_{S_R}(D^2u_2)^2dtdx \le CR^{2\sigma}\int\int_{S_R}(D^2u)^2dtdx+C\sup|f|^2R^5 \tag{2.11}$$

where C depends on a_0, A_0 and $\|a\|_\sigma$, $\|a\|_\sigma$ denotes the norm of a in the space $C^{\frac{\sigma}{4},\sigma}(\overline{Q}_T)$.

Proof Denote by

$$Q_t=(0,t)\times(0,1),\quad S_R^t=S_R\cap Q_t,\quad J_R^t=J_R\cap(0,t)$$

Multiply the equation (2.9) by u_2 and integrate the resulting relation over S_R^t. Integrating by parts, we have

$$\frac{1}{2}\int_{E_R}u_2^2(t,x)dx+a(t_0,x_0)\int\int_{S_R^t}(D^2u_2)^2dsdx$$
$$=\int\int_{S_R^t}[a(t_0,x_0)-a(t,x)]D^2uD^2u_2dsdx+\int\int_{S_R^t}fD^2u_2dsdx$$
$$-\int_{J_R^t}f(s,\beta_R)Du_2(s,\beta_R)ds+\int_{J_R^t}f(s,\alpha_R)Du_2(s,\alpha_R)ds$$

where α_R and β_R are the left and right endpoints of the interval E_R. Noticing that

$$\sup_{x\in E_R}|Du_2(s,x)|^2\le R\int_{E_R}(D^2u_2(s,x))^2dx+\frac{C}{R^3}\int_{E_R}u_2^2(s,x)dx$$

we have

$$\Big|\int_{J_R^t}f(s,\beta_R)\,Du_2(s,\beta_R)ds\Big|+\Big|\int_{J_R^t}f(s,\alpha_R)Du_2(s,\alpha_R)ds\Big|$$
$$\le\varepsilon\int\int_{S_R^t}(D^2u_2)^2dsdx+\frac{\varepsilon}{R^4}\int\int_{S_R^t}u_2^2dsdx$$
$$+C_\varepsilon R\int_{J_R^t}(|f(s,\beta_R|^2+|f(s,\alpha_R|^2)ds$$
$$\le\varepsilon\int\int_{S_R^t}(D^2u_2)^2dsdx+\varepsilon\sup_{J_R}\int_{E_R}u_2^2(s,x)dx+C_\varepsilon R^5\sup|f|^2$$

This and the facts that

$$\Big|\int\int_{S_R^t}[a(t_0,x_0)-a(t,x)]D^2uD^2u_2dsdx\Big|$$
$$\le\varepsilon\int\int_{S_R^t}(D^2u_2)^2dsdx+C_\varepsilon\|a\|_\sigma^2R^{2\sigma}\int\int_{S_R^t}(D^2u)^2dsdx$$

$$\Big|\int\int_{S_R^t}fD^2u_2dsdx\Big|$$
$$\le\varepsilon\int\int_{S_R^t}(D^2u_2)^2dsdx+C_\varepsilon R^5\sup|f|^2$$

yield the desired estimate (2.11) and the proof is completed.

Lemma 2.3 *For any* (t,x_1), (t,x_2), (t_1,x), $(t_2,x)\in S_\rho$,

$$|u_1(t,x_1)-u_1(t,x_2)|^2\le CM(u_1,\rho)|x_1-x_2| \tag{2.12}$$

$$|u_1(t_1,x)-u_1(t_2,x)|^2\le CM(u_1,\rho)|t_1-t_2|^{\frac14} \tag{2.13}$$

where

$$M(u_1,\rho)=\sup_{J_\rho}\int_{E_\rho}(Du_1(t,x))^2dx+\int\int_{S_\rho}(D^3u_1)^2dtdx$$

and the constant C *depends only on* a_0 *and* A_0.

Proof The estimate (2.12) is obvious. For (2.13), we only consider the case where $\Delta t=t_2-t_1>0$, $x,x+2(\Delta t)^{\frac14}\in E_\rho$. Integrating the equation (2.7) over the region $(t_1,t_2)\times(y,y+(\Delta t)^{\frac14})$, we get

$$\begin{aligned}0=&\int_y^{y+(\Delta t)^{\frac14}}[u_1(t_2,z)-u_1(t_1,z)]dz\\&+a(t_0,x_0)\int_{t_1}^{t_2}[D^3u_1(s,y+(\Delta t)^{\frac14})-D^3u_1(s,y)]ds\end{aligned}$$

or equivalently

$$\begin{aligned}0=&(\Delta t)^{\frac14}\int_0^1[u_1(t_2,y+\theta(\Delta t)^{\frac14})-u_1(t_1,y+\theta(\Delta t)^{\frac14})]d\theta\\&+a(t_0,x_0)\int_{t_1}^{t_2}[D^3u_1(s,y+(\Delta t)^{\frac14})-D^3u_1(s,y)]ds\end{aligned}$$

Integrating this equality with respect to y over $(x,x+(\Delta t)^{\frac14})$ and using the mean value theorem, we have

$$\begin{aligned}&(\Delta t)^{\frac12}[u_1(t_2,x^*)-u_1(t_1,x^*)]\\&=\int_{t_1}^{t_2}\int_x^{x+(\Delta t)^{\frac14}}[D^3u_1(s,y+(\Delta t)^{\frac14})-D^3u_1(s,y)]dsdy\end{aligned}$$

and hence

$$|u_1(t_2,x^*)-u_1(t_1,x^*)|^2\le C|t_1-t_2|^{\frac14}\int\int_{S_\rho}(D^3u_1)^2dtdx$$

where $x^*=y^*+\theta^*(\Delta t)^{\frac14}$, $y^*\in(x,x+(\Delta t)^{\frac14})$, $\theta^*\in(0,1)$. This and (2.12) yield (2.13) and hence complete the proof of the lemma.

Lemma 2.4 (Caccioppoli type inequality)

$$\sup_{J_{\frac{R}{4}}}\int_{E_{\frac{R}{4}}}(u_1(t,x)-\lambda)^2dx+\int\int_{S_{\frac{R}{4}}}(D^2u_1)^2dtdx$$
$$\leq\frac{C}{R^4}\int\int_{S_{\frac{R}{2}}}(u_1-\lambda)^2dtdx \tag{2.14}$$

$$\sup_{J_{\frac{R}{4}}}\int_{E_{\frac{R}{4}}}(Du_1(t,x))^2dx+\int\int_{S_{\frac{R}{4}}}(D^3u_1)^2dtdx$$
$$\leq\frac{C}{R^4}\int\int_{S_{\frac{R}{2}}}(Du_1)^2dtdx\leq\frac{C}{R^6}\int\int_{S_R}(u_1-\lambda)^2dtdx \tag{2.15}$$

where C depends only on a_0, A_0,

$$\lambda=\begin{cases}\text{arbitrary constant}, & \text{if } Q_R\cap\partial_pQ_T=\emptyset\\ 0, & \text{if } Q_R\cap\partial_pQ_T\neq\emptyset\end{cases}$$

Proof We discuss in the following two cases.

1°. The case $t_0-R^4<0$. In such a case, $\lambda=0$. Choose a C^∞ function $\chi(x)$ satisfying the following requirements. If $0,1\notin E_R$, then supp $\chi\subset\left(x_0-\frac{R}{2},x_0+\frac{R}{2}\right)$, $\chi(x)=1$ in $\left(x_0-\frac{R}{4},x_0+\frac{R}{4}\right)$, $0\leq\chi(x)\leq 1$,

$$|\chi'(x)|\leq\frac{C}{R},\ |\chi''(x)|\leq\frac{C}{R^2},\ |\chi'''(x)|\leq\frac{C}{R_3},\ |\chi^{(4)}(x)|\leq\frac{C}{R^4}$$

In $0\in E_R$, then the value of $\chi(x)$ for $x\leq x_0$, is changed into 1. If $1\in E_R$, then the value of $\chi(x)$ for $x\geq x_0$ is changed into 1. Multiplying the equation (2.7) by χ^4u_1 and integrating the resulting relation over S_R^t, we have

$$\int\int_{S_R^t}\frac{\partial u_1}{\partial t}\chi^4u_1dsdx+a(t_0,x_0)\int\int_{S_R^t}D^4u_1\chi^4u_1dsdx=0$$

From the boundary value conditions (2.4) and (2.8), it is easily seen that

$$\chi^4u_1\big|_{\partial E_R}=0,\quad D^2u_1D(\chi^4u_1)\big|_{\partial E_R}=0$$

Thus

$$
\begin{aligned}
0 &= \frac{1}{2}\int_{E_R} \chi^4 u_1^2(t,x)dx + \int\int_{S_R^t} a(t_0,x_0)D^2u_1D^2(\chi^4u_1)dsdx \\
&= \frac{1}{2}\int_{E_R} \chi^4 u_1^2(t,x)dx + \int\int_{S_R^t} a(t_0,x_0)\chi^4(D^2u_1)^2dsdx \\
&\quad +8\int\int_{S_R^t} a(t_0,x_0)\chi^3\chi' Du_1D^2u_1dsdx \\
&\quad +\int\int_{S_R^t} a(t_0,x_0)(24\chi^2\chi'^2+8\chi^3\chi'')u_1D^2u_1dsdx
\end{aligned}
$$

By Cauchy's inequality, we have

$$
\begin{aligned}
&\left|\int\int_{S_R^t} a(t_0,x_0)(24\chi^2\chi'^2+8\chi^3\chi'')u_1D^2u_1dsdx\right| \\
&\le \frac{1}{4}a(t_0,x_0)\int\int_{S_R^t}\chi^4(D^2u_1)^2dsdx + \frac{C}{R^4}\int\int_{S_{\frac{R}{2}}} u_1^2dsdx
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
&\left|8\int\int_{S_R^t} a(t_0,x_0)\chi^3\chi' Du_1D^2u_1dsdx\right| \\
&\le \frac{1}{4}a(t_0,x_0)\int\int_{S_R^t}\chi^4(D^2u_1)^2dsdx + C\int\int_{S_R^t}\chi^2\chi'^2(Du_1)^2dsdx
\end{aligned}
$$

Noticing that

$$
\begin{aligned}
\int\int_{S_R^t}\chi^2\chi'^2(Du_1)^2dsdx &= -\int\int_{S_R^t} u_1D(\chi^2\chi'^2Du_1)dsdx \\
&= -\int_{S_R^t}\chi^2\chi'^2u_1D^2u_1dsdx + \int\int_{S_R^t} u_1^2D^2(\chi^2\chi'^2)dsdx \\
&\le \frac{1}{4}a(t_0,x_0)\int\int_{S_R^t}\chi^4(D^2u_1)^2dsdx + \frac{C}{R^4}\int\int_{S_{\frac{R}{2}}} u_1^2dsdx
\end{aligned}
$$

we obtain

$$
\sup_{J_R}\int_{E_R}\chi^4u_1^2(t,x)dx + \int\int_{S_R}\chi^4(D^2u_1)^2dtdx \le \frac{C}{R^4}\int\int_{S_{\frac{R}{2}}} u_1^2dsdx
$$

from which (2.14) follows.

Since $w = Du_1$ satisfies

$$
\frac{\partial w}{\partial t} + a(t_0,x_0)D^4w = 0 \quad \text{in } S_R
$$

and $Dw(t,0) = D^3w(t,0) = 0$ if $0 \in \partial E_R$, $Dw(t,1) = D^3w(t,1) = 0$ if $1 \in \partial E_R$, we can prove the estimate (2.15) by the way similar to the proof of (2.14).

2°. The case $t_0 - R^4 \geq 0$. Choose another function $\eta(t) \in C^\infty$ such that $\eta(t) = 1$ in $\left(t_0 - \left(\frac{R}{4}\right)^4, +\infty\right)$, $\eta(t) = 0$ in $\left(-\infty, t_0 - \left(\frac{R}{2}\right)^4\right)$, $0 \leq \eta(t) \leq 1$, $|\eta'(t)| \leq \frac{C}{R^4}$ for all $t \in \boldsymbol{R}$.

With λ stated in the lemma, we multiply (2.7) by $\chi^4\eta(u_1 - \lambda)$ and integrate the resulting relation over S_R^t. Then we derive an equality similar to (2.16) in which u_1 is replaced by $u_1 - \lambda$ and a term

$$-\int_{t_0-R^4}^{t}\int_{E_R}\chi^4\eta'(u_1-\lambda)^2 dsdx$$

is added. Then following the argument as in Case 1°, we can complete the proof of the lemma.

Lemma 2.5 *For any $0 < \rho < R$,*

$$\varphi(u_1, \rho) \leq C\left(\frac{\rho}{R}\right)^6 \varphi(u_1, R) \tag{2.17}$$

where C depends only on a_0, A_0 and $\|a\|_\sigma$.

Proof It sufficies to show (2.17) for $\rho \leq \frac{R}{4}$. From Lemma 2.3 and Lemma 2.4, we have

$$\iint_{S_\rho} |u_1 - \hat{u}_{1\rho}|^2 dtdx \leq CM\left(u_1, \frac{R}{4}\right)\rho^6 \leq C\left(\frac{\rho}{R}\right)^6 \iint_{S_R}(u_1-\lambda)^2 dtdx$$

Taking $\lambda = \hat{u}_{1R}$, we obtain

$$\iint_{S_\rho} |u_1 - \hat{u}_{1\rho}|^2 dtdx \leq C\left(\frac{\rho}{R}\right)^6 \iint_{S_R}(u_1 - \hat{u}_{1R})^2 dtds \tag{2.18}$$

On the other hand, by (2.15),

$$\begin{aligned}
&\iint_{S_\rho} \rho^4 (D^2u_1)^2 dtdx \\
&\leq C_1 \iint_{S_\rho} \rho^6 (D^3u_1)^2 dtdx + C_2 \iint_{S_\rho} \rho^2 (Du_1)^2 dtdx \\
&\leq C_1\rho^6 \iint_{S_{\frac{R}{4}}} (D^3u_1)^2 dtdx + C_2\rho^6 \sup_{J_{\frac{R}{4}}} \int_{E_{\frac{R}{4}}} (Du_1(t,x))^2 dx \\
&\leq C\left(\frac{\rho}{R}\right)^6 \iint_{S_{\frac{R}{2}}} R^2 (Du_1)^2 dtdx \\
&\leq C\left(\frac{\rho}{R}\right)^6 \left[\iint_{S_R} R^4 (D^2u_1)^2 dtdx + \iint_{S_R}(u_1 - \hat{u}_{1R})^2 dtdx\right] \\
&= C\left(\frac{\rho}{R}\right)^6 \varphi(u_1, R)
\end{aligned}$$

which together with (2.18) imply (2.17). The proof is completed.

To estimate the Hölder norm of u, we also need the following technical lemma, whose proof can be found in [6].

Lemma 2.6 *Let $\varphi(\rho)$ be a nonnegative and nondecreasing function satisfying*

$$\varphi(\rho) \le A\Big[\Big(\frac{\rho}{R}\Big)^{\alpha} + \varepsilon\Big]\varphi(R) + BR^{\beta}$$

for all $0 < \rho \le R \le R_0$ with A, B, α, β positive constants with $\beta < \alpha$. Then there exists a constant $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta)$ such that for all $0 < \rho \le R \le R_0$ and $0 < \varepsilon < \varepsilon_0$,

$$\varphi(\rho) \le C\left[\Big(\frac{\rho}{R}\Big)^{\beta}\varphi(R) + BR^{\beta}\right]$$

where C is a constant depending only on α, β and A.

Now we state the main result in this section.

Theorem 2.7 *Let $a\,(t,x)$ and $f\,(t,x)$ be appropriately smooth function, u be the smooth solution of the problem (2.3)-(2.5). Then for any $\alpha \in \Big(0, \dfrac{1}{2}\Big)$, there exists a constant C depending only on a_0, A_0, α, T, $\|a\|_{\sigma}$, $\displaystyle\iint_{Q_T} u^2 dtdx$, $\displaystyle\iint_{Q_T}(D^2u)^2 dtdx$ such that*

$$|u(t_1,x_1) - u(t_2,x_2)| \le C(1 + \sup|f|)(|x_1 - x_2|^{\alpha} + |t_1 - t_2|^{\frac{\alpha}{4}}) \tag{2.19}$$

Proof For any fixed point $(t_0, x_0) \in \overline{Q}_T$, consider the function $\varphi(u,\rho)$, which is clearly nondecreasing with respect to ρ. By Lemma 2.5,

$$\begin{aligned}\varphi(u,\rho) &\le \varphi(u_1,\rho) + \varphi(u_2,\rho)\\ &\le C\Big(\frac{\rho}{R}\Big)^6 \varphi(u_1,R) + \varphi(u_2,R)\\ &\le C\Big(\frac{\rho}{R}\Big)^6 \varphi(u,R) + C\varphi(u_2,R)\end{aligned}$$

holds for any $0 < \rho < R$. By Lemma 2.2,

$$\begin{aligned}\varphi(u_2,R) &= \iint_{S_R}[(u_2 - \hat{u}_{2R})^2 + R^4(D^2u_2)^2]dtdx\\ &\le 4\iint_{S_R} u_2^2 dtdx + R^4\iint_{S_R}(D^2u_2)^2 dtdx\\ &\le 4R^4 \sup_{J_R}\int_{E_R} u_2^2(t,x)dx + R^4\iint_{S_R}(D^2u_2)^2 dtdx\\ &\le CR^{4+2\sigma}\iint_{S_R}(D^2u)^2 dtdx + C\sup|f|^2 R^9\\ &\le CR^{2\sigma}\varphi(u,R) + C\sup|f|^2 R^9\end{aligned}$$

Thus

$$\varphi(u,\rho) \le C\left[\left(\frac{\rho}{R}\right)^6 + R^{2\sigma}\right]\varphi(u,R) + C\sup|f|^2 R^9$$

For ε_0 in Lemma 2.6, we choose $R_0 > 0$ such that $R^{2\sigma} < \varepsilon_0$ whenever $R \le R_0$. By Lemma 2.6, we have

$$\varphi(u,\rho) \le C\left[\left(\frac{\rho}{R_0}\right)^\lambda \varphi(u,R_0) + \sup|f|^2\rho^\lambda\right]$$

for some $5 < \lambda < 6$, and hence

$$M^2[u] \le C\left[\frac{1}{R_0^\lambda}\varphi(u,R_0) + \sup|f|^2\right]$$

Using Lemma 2.1, we immediately obtain (2.19) and complete the proof of the theorem.

3. The Existence and Uniqueness for Classical Solutions

We first consider the special problem where no lower order term appears.

$$\frac{\partial u}{\partial t} + D[m(u)D^3u] = 0, \quad \text{in } Q_T = (0,T)\times(0,1) \tag{3.1}$$

$$Du(t,0) = Du(t,1) = D^3u(t,0) = D^3u(t,1) = 0 \tag{3.2}$$

$$u(0,x) = u_0(x) \tag{3.3}$$

We have the following

Theorem 3.1 *Let $m(s) \in C^{1+\alpha}(R)$, $u_0 \in C^{4+\alpha}(\bar{I})$, $D^iu_0(0) = D^iu_0(1) = 0$ $(i = 0,1,2,3,4)$, $m(s) > 0$. Then the problem (3.1)–(3.3) admits a classical solution in the space $C^{1+\frac{\alpha}{4},4+\alpha}(\overline{Q_T})$, the norm of which is determined by the known quantities.*

Proof It suffices to restrict the consideration to $m(s) \in C^\infty(R)$, $u_0 \in C_0^\infty(I)$. We apply the Leray-Schauder principle of fixed point to solve the problem (3.1)–(3.3). To do this, we need some priori estimates on the smooth solution u of the problem (3.1)–(3.3).

Multiplying both sides of the equation (3.1) by D^2u and integrating the resulting relation over $Q_t = (0,t)\times(0,1)$, we have

$$\int\int_{Q_t} \frac{\partial u}{\partial t} D^2u\,ds\,dx + \int\int_{Q_t} D(m(u)D^3u)D^2u\,ds\,dx = 0$$

Integrating by parts and using the boundary condition (3.2), we have

$$\frac{1}{2}\int_0^1 (Du(t,x))^2dx + \int\int_{Q_t} m(u)(D^3u)^2ds\,dx = \frac{1}{2}\int_0^1 (Du_0)^2dx$$

It follows that

$$\sup_{0\le t\le T}\int_0^1 (Du(t,x))^2dx \le C \tag{3.4}$$

$$\iint_{Q_T} m(u)(D^3u)^2dtdx \le C \tag{3.5}$$

The integration of (3.1) over the interval (0, 1) yields

$$\int_0^1 u(t,x)dx = \int_0^1 u_0(x)dx$$

By the mean value theorem, there exists $x_t^* \in (0,1)$ such that

$$u(t,x_t^*) = \int_0^1 u_0(x)dx$$

and hence for any $(t,x)\in Q_T$,

$$\begin{aligned}|u(t,x)| &\le |u(t,x)-u(t,x_t^*)| + |u(t,x_t^*)|\\ &\le \left|\int_{x_t^*}^x Du(t,y)dy\right| + \int_0^1 |u_0(x)|dx\end{aligned}$$

Taking this into account and using (3.4), it follows that

$$\sup_{Q_T}|u(t,x)| \le C \tag{3.6}$$

Further we may obtain the Hölder estimate for u, namely

$$|u(t,x_1)-u(t,x_2)| \le C|x_1-x_2|^{\frac{1}{2}} \tag{3.7}$$

$$|u(t_1,x)-u(t_2,x)| \le C|t_1-t_2|^{\frac{1}{8}} \tag{3.8}$$

The estimate (3.7) follows from (3.4). For (3.8), we only consider the case where $0\le x\le \frac{1}{2}$, $\Delta t = t_1 - t_2 > 0$, $(\Delta t)^{\frac{1}{4}} \le \frac{1}{4}$. Integrating the equation (3.1) over $(t_1,t_2)\times(y, y+(\Delta t)^{\frac{1}{4}})$, we have

$$\begin{aligned}&\int_y^{y+(\Delta t)^{\frac{1}{4}}}[u(t_2,z)-u(t_1,z)]dz\\ &\quad= -\int_{t_1}^{t_2}[m(u(s,y+(\Delta t)^{\frac{1}{4}}))D^3u(s,y+(\Delta t)^{\frac{1}{4}}) - m(u(s,y))D^3u(s,y)]ds\end{aligned}$$

or equivalently

$$\begin{aligned}&(\Delta t)^{\frac{1}{4}}\int_0^1[u(t_2,y+\theta(\Delta t)^{\frac{1}{4}}) - u(t_1,y+\theta(\Delta t)^{\frac{1}{4}})]d\theta\\ &\quad= -\int_{t_1}^{t_2}[m(u(s,y+(\Delta t)^{\frac{1}{4}}))D^3u(s,y+(\Delta t)^{\frac{1}{4}}) - m(u(s,y))D^3u(s,y)]ds\end{aligned}$$

Integrating the above equality with respect to y over $(x, x+(\Delta t)^{\frac{1}{4}})$ and using the mean value theorem, we get

$$|u(t_2, x^*) - u(t_1, x^*)| \le C(\Delta t)^{\frac{1}{8}}.$$

where $x^* = y^* + \theta^*(\Delta t)^{\frac{1}{4}}$, $y^* \in (x, x+(\Delta t)^{\frac{1}{4}})$, $\theta^* \in (0,1)$. By virtue of this and (3.7), the estimate (3.8) follows immediately.

The key estimate is the Hölder estimate for Du, which can be obtained by the result in Section 2,

$$|Du(t_1, x_1) - Du(t_2, x_2)| \le C(|x_1 - x_2|^{\frac{1}{4}} + |t_1 - t_2|^{\frac{1}{16}}) \tag{3.9}$$

where C depends only on the known quantities. In fact, $w = Du - Du_0$ satisfies

$$\frac{\partial w}{\partial t} + D^2(m(u)D^2 w) = -D^2(m(u)D^3 u_0) \equiv D^2 f \quad \text{in } Q_T$$

$$w(t,0) = w(t,1) = D^2 w(t,0) = D^2 w(t,1) = 0$$

$$w(0,x) = 0$$

and hence by (3.5)–(3.8) and Theorem 2.7, the estimate (3.9) follows.

Now we change the equation (3.1) into the form

$$\frac{\partial u}{\partial t} + a(t,x)D^4 u + b(t,x)D^3 u = 0$$

where

$$a(t,x) = m(u(t,x)) \ge \inf m(u(t,x)) \equiv m_0 > 0$$

$$b(t,x) = m'(u(t,x))Du(t,x)$$

From the estimates (3.7), (3.8) and (3.9) we see that

$$|a(t_1, x_1) - a(t_2, x_2)| \le C(|x_1 - x_2|^{\frac{1}{2}} + |t_1 - t_2|^{\frac{1}{8}})$$

$$|b(t_1, x_1) - b(t_2, x_2)| \le C(|x_1 - x_2|^{\frac{1}{4}} + |t_1 - t_2|^{\frac{1}{16}})$$

and hence by the classical linear theory (cf. Theorem 1 in Section 4 of [4]),

$$\left|\frac{\partial u}{\partial t}(t_1, x_1) - \frac{\partial u}{\partial t}(t_2, x_2)\right| \le C(|x_1 - x_2|^{\beta} + |t_1 - t_2|^{\frac{\beta}{4}}) \tag{3.10}$$

$$|D^4 u(t_1, x_1) - D^4 u(t_2, x_2)| \le C(|x_1 - x_2|^{\beta} + |t_1 - t_2|^{\frac{\beta}{4}}) \tag{3.11}$$

where $\beta = \min\left(\frac{1}{4}, \alpha\right)$, C depends only on the known quantities.

Taking (3.10) and (3.11) into account, we may improve the estimates (3.7)–(3.9) as

$$|u(t_1, x_1) - u(t_2, x_2)| \le C(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{4}})$$

$$|Du(t_1, x_1) - Du(t_2, x_2)| \le C(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{4}})$$

and so the exponent β in the estimates (3.10) and (3.11) can be replaced by α.

Define the linear space

$$\mathbf{X} = \{u \in C^{\frac{1+\alpha}{4}, 1+\alpha}(\overline{Q_T});\ Du(t,0) = Du(t,1) = 0,\ u(0,x) = u_0(x)\}$$

and the associated operator $\mathbf{T}$ on $\mathbf{X}$,

$$\mathbf{T} : \mathbf{X} \to \mathbf{X}, \quad u \longmapsto w$$

where w is determined by the following linear problem

$$\frac{\partial w}{\partial t} + m(u(t,x))D^4 w + m'(u(t,x))Du(t,x)D^3 w = 0$$

$$Dw(t,0) = Dw(t,1) = D^3 w(t,0) = D^3 w(t,1) = 0$$

$$w(0,x) = u_0(x)$$

By classical linear theory (see Theorem 2 and Theorem 3 in Section 4 of [4]), the above problem admits a unique solution in the space $C^{\frac{4+\beta}{4}, 4+\beta}(\overline{Q_T})$. So, the operator $\mathbf{T}$ is well-defined and compact. Moreover, if $u = \sigma\, \mathbf{T}\, u$, for some $\sigma \in (0,1]$, then u satisfies (3.1) (3.2) and $u(0,x) = \sigma u_0(x)$. Thus from the discussions above, we see that the norm of u in the space $C^{\frac{4+\alpha}{4}, 4+\alpha}(\overline{Q_T})$ can be estimated by some constant C depending only on the known quantities. By Leray-Schauder principle of fixed point, the operator $\mathbf{T}$ has a fixed point u, which is the desired classical solution of the problem (3.1)–(3.3). The proof is completed.

Theorem 3.2 *The problem* (3.1)–(3.3) *has at most one solution in the space* $C^{\frac{4+\alpha}{4}, 4+\alpha}(\overline{Q_T})$.

Proof Suppose u_1 and u_2 are two solutions of the problem (3.1)–(3.3). Then for any smooth function $\varphi(t,x)$ with $D\varphi(t,0) = D\varphi(t,1) = D^3\varphi(t,0) = D^3\varphi(t,1) = \varphi(T,x) = 0$, there holds

$$\iint_{Q_T} (u_1 - u_2)\frac{\partial \varphi}{\partial t} dt dx + \iint_{Q_T} [m(u_1)D^3 u_1 - m(u_2)D^3 u_2] D\varphi dt dx = 0$$

or

$$\iint_{Q_T} (u_1 - u_2)\frac{\partial \varphi}{\partial t} dt dx - \iint_{Q_T} (u_1 - u_2) D^3(A(t,x)D\varphi) dt dx + \iint_{Q_T} (u_1 - u_2) B(t,x) D\varphi dt dx = 0$$

where

$$A(t,x) = m(u_1(t,x))$$

$$B(t,x) = \int_0^1 m'(\lambda u_1 + (1-\lambda)u_2) d\lambda \cdot D^3 u_2$$

For any fixed $f \in C_0^\infty(Q_T)$, consider the linear problem

$$\frac{\partial \varphi}{\partial t} - D^3(A(t,x)D\varphi) + B(t,x)D\varphi = f(t,x)$$
$$D\varphi(t,0) = D\varphi(t,1) = D^3\varphi(t,0) = D^3\varphi(t,1) = 0$$
$$\varphi(T,x) = 0$$

Since $A(t,x) \in C^{\frac{3+\alpha}{4},3+\alpha}(\overline{Q_T})$, $B(t,x) \in C^{\frac{\alpha}{4},\alpha}(\overline{Q_T})$ by classical linear theory, the above problem admits a unique solution $\varphi \in C^{\frac{4+\alpha}{4},4+\alpha}(\overline{Q_T})$. Thus

$$\int\int_{Q_T} (u_1 - u_2) f dt dx = 0$$

By the arbitrariness of f we see that $u_1(t,x) \equiv u_2(t,x)$ and complete the proof of the theorem.

Now, we consider the general problem (1.1)–(1.4) which is clearly equivalent to the problem (1.1), (3.2), (3.3).

Theorem 3.3 *Let $m(s) \in C^{1+\alpha}(R)$, $A(s) \in C^{2+\alpha}(R)$, $u_0 \in C^{4+\alpha}(\overline{I})$, $D^i u_0(0) = D^i u_0(1) = 0$ $(i = 0,1,2,3,4)$, $m(s) > 0$, and for some $\mu > 0$,*

$$H(s) \equiv \int_0^s A(\sigma) d\sigma \geq -\mu \tag{3.12}$$

Then the problem (1.1), (3.2), (3.3) admits a unique classical solution in the space $C^{1+\frac{\alpha}{4},4+\alpha}(\overline{Q_T})$, the norm of which is determined by the known quantities.

Proof Most of the estimates needed in the proof of the theorem can be established by using the arguments similar to that in Theorem 3.1 and hence is omitted here. Some differences arise in the derivation of the estimates (3.4) and (3.9). For the proof of (3.4), we set

$$F(t) = \int_0^1 \Big[\frac{k}{2}(Du)^2 + H(u) + \mu\Big] dx$$

Using the assumption (3.12), we get from the equation (1.1)

$$\begin{aligned} F'(t) &= \int_0^1 \Big[kDu\frac{\partial Du}{\partial t} + A(u)\frac{\partial u}{\partial t}\Big] dx \\ &= -\int_0^1 [kD^2u - A(u)]\frac{\partial u}{\partial t} dx \\ &= -\int_0^1 m(u)[kD^3u - DA(u)]^2 dx \leq 0 \end{aligned}$$

Thus $F(t) \leq F(0)$ and the estimate (3.4) follows. Set $w = Du - Du_0$. Then w satisfies the equation

$$\frac{\partial w}{\partial t} + D^2(km(u)D^2 w) = D^2 \hat{f}$$

where

$$\hat{f} = -km(u)D^3u_0 + m(u)A'(u)Du$$

By Theorem 2.7, we have

$$\begin{aligned}
&|Du(t_1,x_1) - Du(t_2,x_2)| \\
&\leq C(1+\sup|\hat{f}|)(|x_1-x_2|^{\frac{1}{4}} + |t_1-t_2|^{\frac{1}{16}}) \\
&\leq C(1+\sup|Du|)(|x_1-x_2|^{\frac{1}{4}} + |t_1-t_2|^{\frac{1}{16}})
\end{aligned}$$

Taking this into account and using the interpolation inequality, the estimate (3.9) follows at once.

The remaining part of the proof of the theorem follows from the arguments as in Theorem 3.1. The proof is completed.

4. Appendix

Consider the problem

$$\frac{\partial u}{\partial t} + \alpha D^4 u = f \quad \text{in } Q_T = (0,T)\times(0,1) \tag{4.1}$$

$$u(t,0) = u(t,1) = B_1(D)u(t,0) = B_2(D)u(t,1) = 0 \tag{4.2}$$

$$u(0,x) = 0 \tag{4.3}$$

where α is a positive constant, f a sufficiently smooth function defined on Q_T, $B_i(D) = D$ or D^2 $(i=1,2)$.

When $f(0,0)\neq 0$ or $f(1,0)\neq 0$, the problem (4.1)–(4.3) does not satisfy the compatibility conditions. We devote this section to a brief discussions on the smoothness of the solutions of the problem (4.1)–(4.3), which have been used in Section 2.

1°. The global smoothness

Choose a sequence $\{f_j\}\subset C_0^\infty(Q_T)$ converging to f in $L^2(Q_T)$ and consider the equation

$$\frac{\partial u}{\partial t} + \alpha D^4 u = f_j \tag{4.4}$$

The solution u_j of the problem (4.4), (4.2), (4.3) are clearly in $C^\infty(Q_T)$ and satisfy the following estimate

$$\begin{aligned}
&\int_0^1 u_j^2(t,x)dx + \int_0^1 (D^2u_j(t,x))^2dx + \iint_{Q_T}(D^4u_j)^2dtdx \\
&\leq C(T)\iint_{Q_T} f_j^2 dtdx \leq C
\end{aligned}$$

from which and the equation (4.4) itself, we may further prove the estimates on the Hölder norms of u_j and Du_j by using the usual arguments

$$|u_j(t_1,x_1)-u_j(t_2,x_2)|\le C(T)(|x_1-x_2|+|t_1-t_2|^{\frac{1}{4}})$$

$$|Du_j(t_1,x_1)-Du_j(t_2,x_2)|\le C(T)(|x_1-x_2|^{\frac{1}{2}}+|t_1-t_2|^{\frac{1}{8}})$$

It follows that the solution u of the problem (4.1)–(4.3) satisfies

$$u\in C^{\frac{1}{4},1}(\overline{Q_T}),\quad Du\in C^{\frac{1}{8},\frac{1}{2}}(\overline{Q_T})$$

$$u\in L^\infty(0,T;H_0^2(I)),\quad \frac{\partial u}{\partial t}\in L^2(Q_T),\ D^4u\in L^2(Q_T)$$

2°. The smoothness for $t>0$

For any $\varepsilon>0$, we choose a sequence $\eta_\varepsilon(t)\in C^\infty(R)$ such that $\eta_\varepsilon(t)=1$ for $t\ge 2\varepsilon$, $\eta_\varepsilon(t)=0$ for $t\le\varepsilon$ and $0\le\eta_\varepsilon(t)\le 1$ for any t. Set $w_\varepsilon=\eta_\varepsilon(t)u$. Then w_ε satisfies

$$\frac{\partial w_\varepsilon}{\partial t}+\alpha D^4 w_\varepsilon=\eta_\varepsilon(t)f+\eta'_\varepsilon(t)u\equiv f_\varepsilon$$

and (4.2), (4.3). Since $f_\varepsilon(0,0)=f_\varepsilon(1,0)=0$, by classical linear theory we see that

$$w_\varepsilon\in C^{1+\frac{\beta}{4},4+\beta}(\overline{Q_T})$$

for some $\beta\in(0,1)$. By the arbitrariness of ε, we see that u is classical in $(0,T)\times[0,1]$. Moreover, if f is sufficiently smooth, u is also sufficiently smooth for $t>0$.

3°. The behaviour of D^2u as $t\to 0$

Set

$$H(t)=\int_0^1(D^2u(t,x))^2dx$$

Then from the discussion in 1°, $H(t)$ is bounded in $(0,T)$ and

$$H'(t)=2\int_0^1 f(t,x)D^4u(t,x)dx-2\alpha\int_0^1(D^4u(t,x))^2dx$$

Thus

$$\int_0^T|H'(t)|dt\le C(T)\int\int_{Q_T}f^2dtdx$$

This shows that $H(t)$ is absolutely continuous in $(0,T)$ and hence the limit $H(0)\equiv\lim\limits_{t\to 0}H(t)$ exists.

We conclude that $H(0)=0$. Let $\alpha_\varepsilon(t)$ be the kernel of mollifier in one dimension and set

$$\varphi_\varepsilon(t)=\int_t^{+\infty}\alpha_\varepsilon(s-\varepsilon)ds$$

Multiplying both sides of the equation (4.4) by $\varphi_\varepsilon(t)D^4u_j$, integrating the resulting relation over Q_T and integrating by parts, we have

$$-\frac{1}{2}\int\int_{Q_T}\varphi_\varepsilon'(t)(D^2u_j)^2dtdx+\alpha\int\int_{Q_T}\varphi_\varepsilon(t)(D^4u_j)^2dtdx$$
$$=\int\int_{Q_T}\varphi_\varepsilon(t)f_jD^4u_jdtdx$$

Letting $j\to\infty$ and then $\varepsilon\to 0$, we immediately see that $H(0)=0$.

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