J.Partial Diff. Eqs., 7(1994), 49–63
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THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF A DEGENERATE REACTION-DIFFUSION PROBLEM*

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(Received June 25; revised Nov.30, 1992)

Abstract Consider in this paper the existence, uniqueness and asymptotic behaviour of the solutions of the mixed problem for a class of degenerate quasilinear parabolic equations and properties of the corresponding stationary solutions.

Key Words Variational principle; stationary solution; equilibrium solution; asymptotic behaviour, region attractivity.

Classification 35K60, 35K65

1. Introduction

Let $D \subset R^n$ be a bounded domain with the boundary $\partial D \in C^1$. Consider the non-negative solutions of the problem

$$\begin{cases} w_t - L\Phi(w) = a(x)f(w), & (x,t) \in D \times (0,t) \\ w = \chi, & (x,t) \in \partial D \times [0,T) \\ w = w_0, & (x,t) \in \overline{D} \times \{0\} \end{cases} \tag{P}$$

where $L = \partial_i(a^{ij}\partial_j)$ with $a^{ij} = a^{ij}(x) = a^{ji}(x) \in C(\overline{D})$ is a uniformly elliptic operator, i.e., there are constants Λ, λ such that $\Lambda \geq \lambda > 0$, and for any $\xi \in R^n$ it holds that $\Lambda|\xi|^2 \geq a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$. Here and throughout the notation ∂_i is used for $\dfrac{\partial}{\partial x_i}$, and the summation convention over twice repeated indices is often used. Suppose that

(H1) $\Phi \in C^1[0,\infty)$ is a monotonically increasing function satisfying that $\Phi(0) = \Phi'(0) = 0$ and Φ^{-1} is Hölder continuous;

(H2) $f \in C^1[0,\infty)$ is an increasing function with $f(0) = 0$;

(H3) $a \in C^0(\overline{D}), w_0 \in L^\infty(D), \chi \in C^1(\partial D \times (0,T))$ and $\chi = w_0$ on $\partial D \times \{0\}$.

* Work supported by National Natural Science Foundation of China.

Problems of this type arise in population dynamics and in reaction-diffusion processes. The investigation of this paper was motivated by the special case of that (see [1] and [2])

$$u_t = \Delta\Phi(u) + f(\lambda, u), \lambda \in R; u_t = \Delta\Phi(u) = uf(x,t), (x,t) \in D \times (0,\infty)$$

On the former equation, the authors have proved a uniqueness and existence theorem for its mixed problem. Moreover, the asymptotic behaviour of the solution when $t \to \infty$ has been discussed for some values of λ. For the mixed problem of the later equation, the authors have given a good discription for the solution and the corresponding stationary solution. In this paper, based on above two works, we consider, for more general degenerate quasilinear parabolic equations, the properties of the solution of the problem (P) as well as the corresponding stationary solution.

Definition 1.1 *We call $w \in L^\infty(D)$ is an equilibrium solution of problem (P) if for any $\eta \in C^2(\overline{D})$ with zero value on ∂D, it holds that*

$$-\int_D \Phi(w)L\eta dx + \int_{\partial D} \Phi(\chi)\frac{\partial \eta}{\partial \nu}ds = \int_D a(x)f(w)\eta(x)dx \tag{1.1}$$

where $\dfrac{\partial \eta}{\partial \nu} = a^{ij}\dfrac{\partial \eta}{\partial x_i}\cos(n, x_j)$ is the oblique derivative of η. By an upper (lower) solution $\overline{w} \in L^\infty(D)(\underline{w} \in L^\infty(D))$ of (1.1) (i.e., by an equilibrium upper (lower) solution of (P)) we mean that $\overline{w}(\underline{w})$ satisfies (1.1) with the inequality sign $\geq (\leq)$, for any positive η as above.

Because of Hölder continuity of Φ^{-1}, we know from [2] that any solution of (1.1) is a classical solution, and $u = \Phi(w) \in C^{2,\alpha}(\overline{D})$ is a classical solution of the problem

$$\begin{cases} Lu + a(x)g(u) = 0, & in\ D \\ u = \varphi, & on\ \partial D \end{cases} \tag{P$_s$}$$

where $g = f \circ \Phi^{-1}, \varphi = \Phi[\chi(\cdot)]$. We call such u as a *stationary solution* of problem (P). u is called as an *upper (lower) solution* of (P$_s$) if it satisfies

$$Lu + a(x)g(u) \leq (\geq) 0\ in\ D, u \geq (\geq)\varphi\ on\ \partial D$$

Definition 1.2 *$w \in C([0,T]; L^1(D)) \cap L^\infty(Q_T)$ is called as a solution of (P) provided that for any $t \in [0,T]$ and non-negative $\zeta \in C^2(\overline{Q}_t)$ with $\zeta = 0$ on $\partial D \times (0,T)$ it holds that*

$$\begin{aligned} &\int_D w(x,t)\zeta(x,t)dx - \int_{Q_t}[w\zeta_t + \Phi(w)L\zeta]dxdt + \int_0^t dt\int_{\partial D}\Phi(\chi)\frac{\partial \zeta}{\partial \nu}ds \\ &= \int_D w_0\zeta(x,0)dx + \int_{Q_t} a(x)f(w)\zeta(x,t)dxdt \end{aligned} \tag{1.2}$$

where $Q_t = D \times (0,t)$ and $\dfrac{\partial}{\partial \nu}$ is as the one as in Definition 1.1. Instead of sign "=" in (1.2) by "$\geq (\leq)$", then w is called as an upper (lower) solution of (P).

2. Existence and Uniqueness

Lemma 2.1. *Let $\underline{w}, \overline{w}$ be respectively the lower and upper solution of* (P), *then for any $t \in [0,T]$ the following holds*

$$\int_D [\underline{w}(x,t) - \overline{w}(x,t)]_+ dx \le e^{kt} \int_D [\underline{w}(x,0) - \overline{w}(x,0)]_+ dx$$

where $r_+ = \max\{r, 0\}, k \ge 0$ is a constant.

Proof For any non-negative $\varphi \in C^2(\overline{Q}_t)$ with $\varphi = 0$ on $\partial D \times (0,t)$, by Definition 1.2 it holds that

$$\begin{aligned}\int_D [\underline{w}(x,t) - \overline{w}(x,t)]\varphi(x,t)dx - \int_{Q_t} [(\underline{w} - \overline{w})\varphi_t + (\Phi(\underline{w}) - \Phi(\overline{w}))L\varphi]dxdt \\ \le \int_D [\underline{w}(x,0) - \overline{w}(x,0)]\varphi(x,0)dx + \int_{Q_t} a(x)[f(\underline{w}) - f(\overline{w})]\varphi(x,t)dxdt\end{aligned} \tag{2.1}$$

where $t \in [0,T]$. Let

$$\alpha = \begin{cases} \dfrac{\Phi(\underline{w}) - \Phi(\overline{w})}{\underline{w} - \overline{w}}, & \underline{w} \ne \overline{w} \\ 0, & \underline{w} = \overline{w} \end{cases}$$

then (2.1) becomes

$$\begin{aligned}\int_D [\underline{w}(x,t) - \overline{w}(x,t)]\varphi(x,t)dx - \int_{Q_t} (\underline{w} - \overline{w})(\varphi_t + \alpha L\varphi)dxdt \\ \le \int_D [\underline{w}(x,0) - \overline{w}(x,0)]\varphi(x,0)dx + \int_{Q_t} a(x)[f(\underline{w}) - f(\overline{w})]\varphi dxdt\end{aligned} \tag{2.2}$$

By (H1) and the boundedness of $\underline{w}$ and $\overline{w}$, we know $\alpha \in L^\infty(Q_t)$. Taking a series of smooth functions α_n satisfying that

$$\frac{1}{n} \le \alpha_n \le \|\alpha\|_{L^\infty(Q_t)} + \frac{1}{n},\ (\alpha_n - \alpha)/\sqrt{\alpha_n} \to 0 \ in\ L^2(Q_t)\ as\ n \to \infty$$

consider the following mixed problem

$$\begin{cases} \varphi_{nt} + \alpha_n L\varphi_n = \mu\varphi_n & in\ D \times (0,T) \\ \varphi_n = 0 & on\ \partial D \times (0,T) \\ \varphi_n(x,T) = \beta_n(x) & in\ D \times T \end{cases}$$

where $\beta_n \in C_0^\infty, 0 \le \beta_n \le 1; \mu \ge 0$ is a constant to be determined below. Based on Chapter 4 of [3], the above problem has a classical solution $\varphi_n \in C^{2,1}(\overline{Q}_t)$. Moreover, we have $0 \le \varphi_n \le e^{-\mu t}$. Multiply $L\varphi_n$ on both sides of the equation and integrate by parts, it yields

$$\Big[-\frac{1}{2}\int_D a^{ij}\frac{\partial\varphi_n}{\partial x_i}\frac{\partial\varphi_n}{\partial x_j}\Big]\Big|_0^T + \int_{Q_T} \alpha_n (L\varphi_n)^2 dxdt = -\mu \int_{Q_T} a^{ij}\frac{\partial\varphi_n}{\partial x_i}\frac{\partial\varphi_n}{\partial x_j} dxdt$$

By the uniform ellipticity of a^{ij} we find

$$\int_{Q_T}\alpha_n(L\varphi_n)^2dxdt+\lambda\mu\int_{Q_T}|D_x\varphi_n|^2dxdt\le\frac{\Lambda}{2}\int_D|D\beta_n|^2dx-c_1$$

then $\int_{Q_T}\alpha_n(L\varphi_n)^2dxdt<c$, which and (2.2) with $t=T,\varphi=\varphi_n$ yield

$$\begin{aligned}&\int_D[\underline{w}(x,T)-\overline{w}(x,T)]\beta_n(x)dx-\int_{Q_T}(\underline{w}-\overline{w})(\alpha-\alpha_n)L\varphi_ndxdt\\&\le\int_D[\underline{w}(x,0)-\overline{w}(x,0)]\varphi_n(x,0)dx+\int_{Q_T}[a(x)f(\underline{w})-a(x)f(\overline{w})+\mu(\underline{w}-\overline{w})]\varphi_ndxdt\\&\le\int_D[\underline{w}(x,0)-\overline{w}(x,0)]_+dx+\int_{Q_T}[a(x)f(\underline{w})-a(x)f(\overline{w})+\mu(\underline{w}-\overline{w})]_+e^{\mu s}dxds\end{aligned}\tag{2.3}$$

Choosing β_n such that $\beta_n\to\beta=\text{sign}(\underline{w}(x,T)-\overline{w}(x,T))$, and noticing that

$$\begin{aligned}\|(\alpha-\alpha_n)L\varphi_n\|_{L^1}&\le\left\|\frac{\alpha-\alpha_n}{\sqrt{\alpha_n}}\right\|_{L^2}\|\sqrt{\alpha_n}L\varphi_n\|_{L^2}\\&\le C^{\frac12}\left\|\frac{\alpha-\alpha_n}{\sqrt{\alpha_n}}\right\|_{L^2}\to0,\quad when\ n\to\infty\end{aligned}$$

then letting $n\to\infty$ in (2.3) we obtain

$$\begin{aligned}&e^{-\mu T}\int_D[\underline{w}(x,T)-\overline{w}(x,T)]_+dx\\&\le\int_D[\underline{w}(x,0)-\overline{w}(x,0)]_+dx+\int_0^T\int_De^{-\mu s}[af(\underline{w})-af(\overline{w})+\mu(\underline{w}-\overline{w})]_+dxds\end{aligned}\tag{2.4}$$

Obviously, replace T by any $t\in[0,T]$ the above inequality still holds. Denote

$$\underline{M}=\operatorname*{ess\,sup}_{Q_t}|\underline{w}|,\ \overline{M}=\operatorname*{ess\,sup}_{Q_t}|\overline{w}|,\ M=\max\{\underline{M},\overline{M}\}$$

$$K=\max_{\overline{D}}|a(x)|\max_{\xi\in[-M,M]}|f'(\xi)|,\quad\mu=K$$

then

$$[a(x)f(\underline{w})-a(x)f(\overline{w})]+\mu(\underline{w}-\overline{w})]_+\le2K(\underline{w}-\overline{w})_+$$

and by (2.4)

$$h(t)\le h(0)+2K\int_0^th(s)ds$$

where $h(t)=e^{-Kt}\int_D[\underline{w}(x,t)-\overline{w}(x,t)]_+dx$. By virtue of Gronwall lemma, we obtain $h(t)\le h(0)e^{2Kt}$, i.e.,

$$\int_D[\underline{w}(x,t)-\overline{w}(x,t)]_+dx\le e^{kt}\int_D[\underline{w}(x,0)-\overline{w}(x,0)]_+dx$$

where $k = 3K$.

Theorem 2.1 *Suppose that* (H1) *to* (H3) *hold. Moreover*

$$\lim_{s\to\infty}\frac{g(s)}{s}=0,\quad \lim_{s\to 0^+}\frac{g(s)}{s}=+\infty$$

Then, for any $T\geq 0$ *the problem* (P) *has a unique solution.*

Proof From the definition, it is obvious that the upper-lower solutions of (1.1) are also the ones of (1.2). By virtue of Lemma 3.3, there exist the upper solution ψ and the lower solution φ of (1.1), i.e., the equilibrium upper and lower solutions of problem (p), satisfying that $\psi,\varphi\in L^\infty(D), 0\leq\varphi\leq\psi$. By (H2) and (H3), $|a(x)f(s)|\leq M$ when $\varphi\leq s\leq\psi$. Therefore, on the base of the theorem in [4] (see Remark 1 following the theorem in [4]), the existence of the solution of (P) follows. If (P) has two solutions w_1, w_2, by Lemma 2.1 we have

$$\int_D [w_1(x,t)-w_2(x,t)]_+dx\leq e^{Kt}\int_D [w_1(x,0)-w_2(x,0)]_+dx=0,$$

then $w_1\leq w_2$. Similarly, the inequality $w_2\leq w_1$ holds too. Hence, $w_1=w_2$.

Using Lemma 2.1 and conditions (H1) to (H3), similar to the proof of the theorem in [1] we have

Theorem 2.2 *Let* $\underline{w}_0$ *and* $\overline{w}_0$ *be respectively the lower and upper solution of* (1.1) *satisfying* $0\leq\underline{w}_0\leq\overline{w}_0$ *a.e. in* D*, then,*

(a) *If* $\underline{w}_0\leq w_0\leq\overline{w}_0$ *a.e. in* D*, then for any* $t\geq 0$ *we have* $\underline{w}_0\leq w(\cdot,t;w_0)\leq\overline{w}_0$ *a.e. in* D*, where* $w(\cdot,t;w_0)$ *is a solution of* (P) *with the initial value of* w_0*;*

(b) *the mapping* $t\to w(x,t;\underline{w}_0)$ *is non-decreasing for a.e.* $x\in D$*; the mapping* $t\to w(x,t;\overline{w}_0)$ *is non-increasing for a.e.* $x\in D$*;*

(c) $w(\cdot,t;\underline{w}_0)$ *and* $w(\cdot,t;\overline{w}_0)$ *converge (in* C^0 *when* $n=1$*, and in* $L^p(D)$ *with* $p\geq 1$ *when* $n\geq 2$*) respectively to the equilibrium solutions* w *and* W *at* $t\to\infty$*, where* w *and* W *are respectively the minimal and the maximal equilibrium solution of* (P).

3. Stationary Case

In this section, we consider problem (P_s). Because $g=f\circ\Phi^{-1}$, (H1) and (H2) the following holds:

(A1) $g\in C^\alpha([0,\infty))\cap C^1((0,\infty)), \alpha\in(0,1)$, and $g(s)>0$ when $s>0, g(0)=0$;

(A2) $g'(s)>0$ when $s>0$.

Moreover, we suppose

(A3) $h(s)=\int_0^s\frac{1}{g(\sigma)}d\sigma\leq+\infty, h(0)=0$;

(A4) $g(s)$ is strictly concave in $(0,\infty)$.

At first we consider the unique solvability of problem (P_s). When $a(x)\geq 0$, by the discussion in [5], (P_s) has at most one solution, and so does it when $a(x)\leq 0$. If $a(x)$ changes its sign in D, we deal with it as follows.

Define $D^+ = \{x \in D | a(x) > 0\}, D^- = \{x \in D | a(x) < 0\}, D^0 = \text{int}(x \in D | a(x) = 0\}$. In view of the continuity of $a(x)$, D^+ consists at most of a countable number of connected components denoted by $D_k^+, k \in M = \{1, 2, \cdots, r\}, r \leq \infty$. Then, we have

Lemma 3.1 *Let u be a non-negative solution of (P_s), then*

(a) *Either $u \equiv 0$ or $u > 0$ in D_k^+;*

(b) *If $u > 0$ in D_k^+, then $u > 0$ in $\overline{D_k^+} \cap D$.*

This lemma gives rise to the following classification of the positive solutions of (P_s).

Definition 3.1 (a) *For any subset $I \subseteq M$, let S_I be the class of solutions of (P_s) which are positive in $D_I^+ \equiv \cup_{k \in I} D_k^+$;* (b) *$N_I$ denotes the set $\{u \in S_I : u \equiv 0$ on $D^+ - D_I^+\}$.*

Then, we have

Lemma 3.2 *Let $\varepsilon > 0$ be a fixed number and u be a non-negative solution of (P_s). The function $U = h(u + \varepsilon)$(h being defined in (A3)) satisfies the following equation*

$$LU = -g'(u + \varepsilon) a^{ij} \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} - a(x) \frac{g(u)}{g(u + \varepsilon)}$$

Because the proof of above two lemmas is standard and trivial, we omit it here. Now, we have a uniqueness result as follows:

Theorem 3.1 *Suppose* (A1) *to* (A4) *hold. Then, for any finite $I \subset M, N_I$ has at most one element.*

Proof Suppose that there exist two different solutions $u_1, u_2 \in N_I$. Let $D' = \{x \in D | u_1(x) > u_2(x)\}$, then by the definition of the solution, we know $u_1 = u_2$ on ∂D and hence $u_1 = u_2$ on $\partial D'$. Denote $U_i = h(u_i), i = 1, 2$, because $h' > 0$ we have

$$U_1 > U_2 \text{ in } D' \text{ and } U_1 = U_2 \text{ on } \partial D' \tag{3.1}$$

Hence, there exists a point $x_0 \in D'$ at the place where the difference of $\delta(x) = U_1(x) - U_2(x)$ attains its maximum, i.e., $\delta(x_0) = \max_{x \in D'} \delta(x)$. Let us distinguish two cases:

(a) There exists some $x_0 \in D'$ such that $\delta(x_0) = \max_{x \in D'} \delta(x)$ and $U_2(x_0) > 0$. Denote by V the maximal connected component of the set $D_1' \equiv \{x \in D' | U_2(x) > 0\}$ containing x_0, then $\delta(x) \in C^2(V)$. Taking $\varepsilon = 0$ in Lemma 3.2 yields

$$LU_k = -g'(u_k) a^{ij} \frac{\partial U_k}{\partial x_i} \frac{\partial U_k}{\partial x_j} - a(x), \quad k = 1, 2$$

Then

$$L\delta = LU_1 - LU_2 = -g'(u_1) a^{ij} \frac{\partial U_1}{\partial x_i} \frac{\partial U_1}{\partial x_j} + g'(u_2) a^{ij} \frac{\partial U_2}{\partial x_i} \frac{\partial U_2}{\partial x_j}$$

By (A4),

$$g'(u_1) < g'(u_2) \text{ in } D' \tag{3.3}$$

which together with (3.2) implies

$$L\delta + g'(u_2) \sigma_i \frac{\partial \delta}{\partial x_i} \geq 0 \text{ in } V \tag{3.4}$$

where $\sigma_i = \sum_{j=1}^{n} \frac{\partial(U_1+U_2)}{\partial x_j} a^{ij}$.

Since $\delta(x)$ assumes its maximum at an interior point of V, the maximum principle entails that $\delta =$ const. in V. It then follows that

$$0 = D\delta = \frac{1}{g(u_1)} Du_1 - \frac{1}{g(u_2)} Du_2 \text{ in } V \tag{3.5}$$

i.e.,

$$\frac{\partial U_1}{\partial x_i} = \frac{\partial U_2}{\partial x_i}, \quad i = 1, 2, \cdots, n$$

The ellipticity of L together with (3.3) gives

$$\begin{aligned} 0 = L\delta &= [g'(u_2) - g'(u_1)] a^{ij} \frac{\partial U_1}{\partial x_i}\frac{\partial U_1}{\partial x_j} \\ &= \frac{g'(u_2) - g'(u_1)}{g^2(u_1)} a^{ij} \frac{\partial u_1}{\partial x_i}\frac{\partial u_1}{\partial x_j} \\ &\geq \frac{g'(u_2) - g'(u_1)}{g^2(u_1)} \lambda |Du_1|^2 \geq 0 \end{aligned} \tag{3.6}$$

Therefore, $|Du_1| = 0$ in V. By (3.5), $|Du_2| = 0$ in V. Hence,

$$u_1 = c_1, \; u_2 = c_2 \text{ in } V$$

where c_1 and c_2 are two constants with $c_1 > c_2$. On the other hand, $V \subseteq D_1' \subseteq D'$ and hence either $U_2 = 0$ or $U_2 = U_1$ on ∂V. This implies that either $u_2 = c_2 = 0$ or $c_1 = c_2$. Both cases lead to a contradiction of $U_2 > 0$ and $c_1 > c_2$ respectively.

(b) $U_2(x_0) = 0$ for all x_0 where δ achieves its maximum in D'. Denote by C_0 the maximal connected component of the set $C \equiv \{x \in D' : \delta(x) = \delta(x_0)\}$ which contains x_0. Then, $U_2 \equiv 0$ in C_0 and

$$\delta = \delta(x_0) > 0 \tag{3.7}$$

Therefore, $U_1 > 0$ in C_0. On the other hand, $u_1 \in N_I$ implies $U_1 \equiv 0$ on $\overline{D^+ - D_I^+}$. Hence, by (3.7) we find

$$C_0 \cap (\overline{D^+ - D_I^+}) = \emptyset \tag{3.8}$$

By virtue of Lemma 3.1(b), $U_2 > 0$ on $\overline{D_k^+}$. If $\partial D_k^+ \cap \partial D$ is non-empty for some $k \in I$, we have $u_1 = u_2$ and hence $\delta \equiv 0$ in such a intersection. Therefore,

$$C_0 \cap \overline{D_k^+} = \emptyset, \quad \forall k \in I \tag{3.9}$$

By the hypothesis, I is a finite set, and then (3.8) and (3.9) implies

$$C_0 \cap \overline{D_k^+} = \emptyset \tag{3.10}$$

C_0 and $\overline{D^+}$ are therefore at a positive distance from each other. Then, there exists a connected neighbourhood $O \subset C_0$ such that $\overline{O} \cap \overline{D_+} = \emptyset, \overline{O} \cap (C - C_0) = \emptyset$ and $\delta(x) > 0$ in $\overline{O}$. Therefore, the monotonicity of h implies that

$$\min_{x \in \overline{O}} [u_1(x) - u_2(x)] > 0 \tag{3.11}$$

Thus, there exists a constant $b > 0$ such that $\delta(x) \le b < \delta(x_0)$ for any $x \in \partial O$. For any $\epsilon > 0$ we define

$$U_{2\epsilon} \equiv h(u_2 + \epsilon), \quad \delta_\epsilon \equiv U_1 - U_{2\epsilon}$$

Clearly $\delta_\epsilon \le \delta$ in D. By (3.11), there exists some $\epsilon > 0$ such that

$$u_1 > u_2 + \epsilon \text{ and } \delta_\epsilon(x_0) > b \text{ on } \overline{O} \tag{3.12}$$

It then follows that $\delta_\epsilon(x) \le \delta(x) \le b < \delta_\epsilon(x_0)$ for any $x \in \partial O$. Hence, δ_ϵ attains its maximum at the same interior point in O and is not constant on $\overline{O}$. On the other hand, $U_1 > U_{2\epsilon} > 0$ in O. Then, by Lemma 3.2 we have

$$LU_1 = -g'(u_1) a^{ij} \frac{\partial U_1}{\partial x_i} \frac{\partial U_1}{\partial x_j} - a(x)$$

$$LU_{2\epsilon} = -g'(u_2 + \epsilon) a^{ij} \frac{\partial U_{2\epsilon}}{\partial x_i} \frac{\partial U_{2\epsilon}}{\partial x_j} - a(x) \frac{g(u_2)}{g(u_2 + \epsilon)}$$

in view of (3.12) and (A5), the difference of the above two identities gives

$$L\delta_\epsilon \ge -g'(u_2 + \epsilon) \sigma_i \frac{\partial \delta_\epsilon}{\partial x_i} - a(x) \Big(1 - \frac{g(u_2)}{g(u_2 + \epsilon)}\Big) \tag{3.13}$$

where $\sigma_i = \sum_{j=1}^{n} a^{ij} \frac{\partial (U_1 + U_2)}{\partial x_j}$. Since $\overline{O} \cap \overline{D^+} = \emptyset$, we have $a(x) \le 0$ in O. The monotonicity of g with (3.13) implies

$$L\delta_\epsilon + g'(u_2 + \epsilon) \sigma_i \frac{\partial \delta_\epsilon}{\partial x_i} \ge 0 \text{ in } O$$

By the maximum principle, δ_ϵ can not achieve its maximum in O unless it is constant. This is a contradiction, whence the result follows.

For obtaining the existence of the solution in S_I, we suppose that

(A5) $\frac{g(s)}{s} \to 0$ as $s \to \infty$, $\frac{g(s)}{s} \to +\infty$ as $s \to 0^+$.

The next lemma can be proved similar to its counterpart in [4] so we only write it down without the proof.

Lemma 3.3 *Assume* (A5) *holds, then*

(a) *For any $m > 0$ there exists a stationary upper solution $\overline{u}$ of problem* (P) *such that $\overline{u} \ge m$ on $\overline{D}$;*

(b) *For any* $I \subset M$ *and any open set* $\mathcal{U}$ *such that* $\overline{\mathcal{U}} \subset D_I^+, \overline{\mathcal{U}} \cap D_k^+ \neq \emptyset, \forall k \in I$, *there exists a family of lower solutions* $\underline{u}_\rho, \rho \in (0, \rho_0]$, *such that* $\mathrm{supp}\underline{u}_\rho \subset \mathcal{U}$, $\mathrm{supp}\underline{u}_\rho \cap D_k^+ \neq \emptyset, \forall k \in I$ *and* $\|\underline{u}_\rho\|_\infty \to 0$ *as* $\rho \to 0$.

Now, we give rise the main result in this section:

Theorem 3.2 *Assume* (A1) *and* (A5) *hold. Then for any* $I \subset M$ *we have*

(a) $S_I \neq \emptyset$;

(b) *there exist a minimal solution* u_I *and a maximal solution* U_I *in* S_I *such that* $u_I \leq u \leq U_I$ *for any other solution* $u \in S_I$;

(c) *the maximal solution* U_I *coincides with the maximal solution in* S_M;

(d) *if* $u_I \neq U_I$, *then* $u_I \in N_{I'}$, *where* $I' = \cap\{\tilde{I} | \tilde{I} \supseteq I, N_{\tilde{I}} \neq \emptyset\}$.

Proof (a) By Lemma 3.3 and the upper-lower solution principle, the result follows immediately.

(b) Denote $U_I(x) = \max_{u \in S_I} u(x)$, then, by virtue of Lemma 3.3(a) there exists an upper solution $\overline{u}$ of (P_s) such that $\overline{u} \geq U_I$. By Theorem 2.2(c), $w(\cdot, t; \bar{u})$ converges to the maximal equilibrium solution W. Let $U \equiv \Phi(W)$, then, $U \in S_I$ and $u \leq U_I$. On the other hand, for any $u \in S_I$, we have $\Phi^{-1}(u) \leq \Phi^{-1}(U)$. Because of the monotonicity of Φ^{-1}, $u \leq U$, and hence $U_I \leq U$. Therefore, $U_I \equiv U$, i.e., U_I is the maximal element in S_I. Similarly, we can show that $u_I(x) = \min_{u \in S_I} u(x)$ is a minimal element in S_I.

(c) Suppose that $U_I \neq U_M$. Then, by Theorem 3.1, there exists some $k \in M - I$ such that $U_I = 0$ in D_k^+. Since $U_M \in S_I$, then by (b), $U_M \leq U_I$ and thus $U_M \equiv 0$ in D_k^+ contradicting the definition of U_M.

(d) For $u' \in N_{\tilde{I}}$, since $I \subseteq \tilde{I}$ we have $u' \in S_I$ and thus $u_I \leq u'$. Hence, $u_I \equiv 0$ in $D_M^+ - D_{I'}^+$, i.e., $u_I \in N_{I'}$.

The maximal solution $U \equiv U_M$ can be constructed by means of a variational principle as follows. Consider the problem

$$\begin{cases} Lv + a(x)g(v+h) = 0 & \text{in } D \\ v = 0 & \text{on } \partial D \end{cases} \tag{3.17}$$

here $Lh = 0$ in D and $h = \varphi$ on ∂D. From §6 in [1] we know that (3.17) has a unique solution. Obviously, $u = v + h$ is a solution of (P_s). Let

$$J[w] = \int_D a^{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} dx - 2\int_D a(x) G(w+h) dx$$

$$G(s) = \int_0^s g(\tau) d\tau$$

We have

Theorem 3.3 *Under the same assumption as for Theorem 3.2, we have*

(a) *the problem*

$$J[w] = \inf, \quad w \in H_0^1(D) \tag{3.18}$$

has a solution w_0;

(b) *if in addition* $r < +\infty$ *and* (A2) *to* (A4) *hold, then* (3.17) *is uniquely solvable and* $U \equiv w_0 + h$ *is the maximal solution of* (P_s).

Proof (a) We first show that $J[w]$ is bounded from below. Let

$$\bar{a} = \begin{cases} \max_{\overline{D}} a(x), & \text{if } a(x) \geq 0 \text{ for some } x \in D \\ 0, & \text{if } a(x) \leq 0 \end{cases}$$

and $\bar{h} = \max_{\overline{D}} h(x)$. By virtue of the ellipticity of L, we have

$$J[w] \geq \lambda \int_D |Dw|^2 dx - 2\bar{a} \int_D G(w + \bar{h}) dx, \ \forall w \in H_0^1(D) \tag{3.19}$$

By the Sobolev imbedding theorem,

$$\int_D |Dw|^2 dx \geq c_0 \int_D w^2 dx, \ w \in H_0^1(D), \ c_0 = c_0(n, D) > 0 \tag{3.20}$$

By (A5), there exist constants c_1 and s_0 such that

$$G(s+h) \leq \frac{c_0 \lambda}{4\bar{a}} s^2 + c_1, \quad \forall s \geq s_0 \tag{3.21}$$

Combining (3.20) with (3.21) gives

$$J[w] \geq c_0 \lambda \int_D w^2 dx - \frac{1}{2} c_0 \lambda \int_D w^2 dx$$
$$-2\bar{a} \int_D c_1 dx \geq -2\bar{a} c_1 |D|, \text{ when } s = s_0$$

For $s < s_0$,

$$J[w] \geq \lambda c_0 \int_D w^2 dx - 2\bar{a} \int_D G(s_0 + \bar{h}) dx \geq -2\bar{a} G(s_0 + \bar{h})|D|$$

Denote $c_2 = \min\{-2\bar{a}c_1|D|, -2\bar{a}G(s_0 + \bar{h})|D|\}$. We then have

$$J[w] \geq c_2, \quad \forall w \in H_0^1(D) \tag{3.22}$$

where C_2 is independent of w. Therefore, there exists a minimal sequence $\{w_n\} \subset H_0^1$ such that

$$\lim_{n \to \infty} J[w] = \inf_{w \in H_0^1(D)} J[w] = d$$

Hence, for any $\epsilon > 0$, there is an integer $N_0(\epsilon)$ such that $d + \epsilon \geq J[w_0]$ for $n \geq n_0(\epsilon)$, i.e.,

$$d + \epsilon \geq \lambda \int_D |Dw_n|^2 dx - 2\bar{a} \int_D G(w_n + \bar{h}) dx \tag{3.23}$$

If $w_n < s_0$ for all $n \geq n_0(\epsilon)$, then the monotonicity of G and (3.23)implies that the integral $\int_D |Dw_n|^2 dx$ is uniformly bounded. If $w_n \geq s_0$ for some $n \geq n' \geq n_0(\epsilon)$, then by (3.20), (3.21) and (3.23) we have

$$d + \epsilon + \frac{\lambda}{2} \int_D |Dw_n|^2 dx + 2\bar{a}c_1|D| \geq \lambda \int_D |Dw_n|^2 dx$$

and hence,

$$\int_D |Dw_n|^2 dx \leq \frac{2(d + \epsilon + 2\bar{a}c_1|D|)}{\lambda} = c_3$$

Therefore,

$$\int_D w_n^2 dx \leq \frac{1}{c_0} \int_D |Dw_n|^2 dx \leq \frac{c_3}{c_0}$$

By virtue of the weak compactness of $H_0^1(D)$, there exists a subsequence of $\{w_n\}$, denoted still by $\{w_n\}$, converging weakly to some $w_0 \in H_0^1(D)$. The Sobolev imbedding theorem implies that w_n converges strongly to w_0 in $L^2(D)$, and hence there is again a subsequence of $\{w_n\}$, denoted still by $\{w_n\}$, converging to w_0 a.e. in D. Then, the lower semi-continuity of the integral $J[w_n]$ and the Lebesque's dominated convergence theorem yield

$$\lim_{n \to \infty} J[w_n] = J[w_0]$$

which establishes (a).

(b) If w_0 is a solution of (3.18), then by the variational principle w_0 is a solution of (3.17) and hence $U = w_0 + h$ solves (P_s).

The uniqueness of U as well as w_0 follows by Theorem 3.1, as soon as we prove that U is the maximal solution of (P_s). Hence, it is necessary to prove that $U > 0$ in D^+. By contradiction, we suppose $U \equiv 0$ in D_k^+ for some $k \in M$, i.e., $w_0 = -h$. Let B be the ball in D_k^+ and $\xi > 0$ be the first eigenfunction of the following problem:

$$\begin{cases} L\xi + \mu, & \xi = 0 \quad \text{in} \quad B \\ L\xi + \mu, & \xi = 0 \quad \text{on} \quad \partial B \end{cases}$$

Define

$$\tilde{w} = \begin{cases} w_0 & \text{in} \quad D - B \\ -h + \epsilon\xi & \text{in} \quad B \end{cases}$$

Because $w_0 = -h$ in D_k^+, we have

$$J[\tilde{w}] = \int_D a^{ij}\frac{\partial \tilde{w}}{\partial x_i}\frac{\partial \tilde{w}}{\partial x_j}dx - 2\int_D a(x)G(\tilde{w}+h)dx$$
$$= \int_D a^{ij}\frac{\partial w_0}{\partial x_i}\frac{\partial w_0}{\partial x_j}dx - 2\int_D a(x)G(w_0+h)dx$$
$$+\epsilon^2\int_B a^{ij}\frac{\partial \xi}{\partial x_i}\frac{\partial \xi}{\partial x_j}dx - 2\int_B a(x)G(\epsilon\xi)dx$$
$$-2\epsilon\int_B a^{ij}\frac{\partial h}{\partial x_i}\frac{\partial \xi}{\partial x_j}dx + 2\int_B a(x)G(w_0+h)dx \tag{3.26}$$

Since $B \subset D_k^+$ it must hold that $w_0 + h = 0$ in B, i.e., $G(w_0+h) = 0$. Because

$$\int_B a^{ij}\frac{\partial h}{\partial x_i}\frac{\partial \xi}{\partial x_j}dx = -\int_B \xi Lh dx = 0$$

(3.26) becomes

$$J[\tilde{w}] = J[w_0] + \epsilon^2\int_B a^{ij}\frac{\partial \xi}{\partial x_i}\frac{\partial \xi}{\partial x_j}dx - 2\int_B a(x)G(\epsilon\xi)dx$$
$$= J[w_0] - \epsilon^2\int_B \xi L\xi dx - 2\int_B a(x)G(\epsilon\xi)dx$$
$$= J[w_0] + \epsilon^2\mu\int_B \xi^2 dx - 2\int_B a(x)G(\epsilon\xi)dx \tag{3.27}$$

Take $\underline{a}$ such that $a(x) \geq \underline{a} > 0$ in B, then by (A5) there is a constant $\epsilon > 0$ such that

$$g(\epsilon\xi) > \frac{2\mu}{\underline{a}}\epsilon\xi \text{ in } B, \text{ and hence } G(\epsilon\xi) \geq \frac{\mu}{\underline{a}}\epsilon^2\xi^2$$

Hence, (3.27) becomes

$$J[\tilde{w}] \leq J[w_0] + \epsilon^2\mu\int_B \xi^2 dx - \frac{2\mu}{\underline{a}}\epsilon^2\int_B a(x)\xi^2 dx$$
$$\leq J[w_0] - \epsilon^2\mu\int_B \xi^2 dx < J[w_0]$$

It contradicts the definition of w_0.

4. Parabolic Case

In this section we consider the asymptotic behaviour of the solution of problem (P). The results are based on [4], and some methods in [4] and [5] are used.

Definition *The interval* $[w_1, w_2] = \{w \in L_+^\infty(D) | w_1 \leq w \leq w_2\}$ *is called as* L^p*-attractive if there exists a set* $\chi \subseteq L_+^\infty(D)$ *such that*

(a) $[w_1, w_2] \subseteq \chi$;

(b) *for any* $w_0 \in \chi$, *there exists a solution* $w(\cdot,t;w_0)$ *of* (P) *such that*

$$\text{dist}\{w(\cdot,t;w_0),[w_1,w_2]\} \to 0 \text{ in } L^p(D) \text{ as } t \to \infty$$

Let us denote $w_I = \Phi^{-1}(u_I), W = \Phi^{-1}(U)$, where u_I and U are respectively the minimal and maximal solutions of (P_s) in S_I. Because Φ is monotone, then w_I and W are respectively the minimal and maximal equilibrium solutions of (P) in D_I^+. From Section 2 we know that there exists uniquely a solution $w(x,t;w_0)$ of (P). Let us denote by $C_k(k=1,2,\cdots,l)$ the connected components of the set $\{x \in D|w(x,t;w_0)>0\}$ which do not intersect Γ^+. Then, we have

Theorem 4.1 *Suppose* (A1) *and* (A5) *hold, then*

(a) *For all* $w_0 \neq 0$ *in* $C_k \cap (\cup_{i\in I} D_i^+), k=1,2,\cdots,l$, *the interval* $[w_I, W]$ *attracts every solution* $w(\cdot,t;w_0)$ *of* (P) *in the* L^p *sense* ($p \in [1,\infty)$ *when* $n \geq 2, p=\infty$ *when* $n=1$);

(b) *If* $w_0 \leq w_I$ *and* $w_0 \not\equiv 0$ *in* $C_k \cap (\cup_{i\in I} D_i^+), k=1,2,\cdots,l$, *then* $w(\cdot,t;w_0) \to w_I$ *in* $L^p(D)$ *when* $t \to \infty$.

Proof (a) By Lemma 3.3 there exist an upper solution $\bar{u}$ and a lower solution $\underline{u}_\rho$ of (P_s) such that $\text{supp}\underline{u}_\rho \subset C_k \cap (\cup_{i\in I} D_i^+)$ and $\underline{u}_\rho \leq w_0 \leq \overline{u}$. By Lemma 2.1, we know $w(x,t;w_0) \leq w(x,t;\overline{u})$. Similarly, $w(x,t;w_0) \geq w(x,t;\underline{u}_\rho)$. Then, the conclusion follows after using Theorem 2.2(c).

(b) Similar to the proof of (a).

Corollary 4.1 *Suppose* (A1) *to* (A5) *hold,* $r<\infty$. *Then, for any* $w_0 \not\equiv 0$ *in* $D_i^+(i \in M)$ *we have* $\lim_{t\to\infty} w(\cdot,t;w_0)=w$, *where* w *is the unique positive equilibrium solution of* (P) *in* D^+.

Proof Based on Theorem 3.1, w is the unique solution in D^+. The conclusion then follows after using Theorem 4.1(a).

In the following, we suppose the continuity of $w(\cdot,t;w_0)$ in Q_∞.

Theorem 4.2 *Assume* (A1) *to* (A5) *hold,* $r <= \infty$. *Then, for any initial condition* w_0 *the solution* $w(\cdot,t,w_0)$ *of (P) converges to an equilibrium solution as* $t \to \infty$.

Proof First, we suppose that for any $k \in M$, there is a point $x_k \in D_k^+$ and $t_k \geq 0$ such that $w(x_k,t_k;w_0)>0$. By virtue of the continuity of $w(\cdot,t;w_0)$ and Lemma 3.3, there exists a lower solution $\underline{u}_\rho$ with $\underline{u}_\rho(x_k)>0$ such that $\underline{u}_\rho \leq w(\cdot,t_k;w_0)$. By Theorem 2.2 we then have

$$w(x_k,t;w_0) \geq \underline{u}_\rho(x_k)>0, \quad t \geq \max\{t_1,\cdots,t_r\} \equiv t_0$$

Using Corollary 4.1 to $w(\cdot,t+t_0;w_0)=w(\cdot,t;w(\cdot,t_0;w_0))$ yields $\lim_{t\to\infty} w(\cdot,t;w_0)=W$, an equilibrium solution.

Secondly, we suppose the case of $w(\cdot,t;w_0) \equiv 0$ in $D_j^+, j=1,2,\cdots,q \leq r$. Denote

$$\tilde{a} = \begin{cases} 0, & in \quad D_j^+, j=1,2,\cdots,q \\ a, & elsewhere \end{cases}$$

Replacing a by $\tilde{a}$ in (P) and then denote this new problem by $(\tilde{P})$, it is obvious that $w(\cdot,t;w_0)$ is a solution of $(\tilde{P})$. Similar to the preceding discusions, we have $\lim_{t\to\infty} w(\cdot,t;w_0) = \tilde{W}$, where $\tilde{W}$ is the maximal equilibrium solution of $(\tilde{P})$. Since $w(\cdot,t;w_0) \equiv 0$ in $D_1^+ \cup \cdots \cup D_q^+$ for any $t \geq 0$, then $\tilde{W} \equiv 0$ in $D_1^+ \cup \cdots \cup D_q^+$. Hence, $\tilde{W} \in N_{\{q+1,\cdots,r\}}$, i.e., $\tilde{W}$ is the equilibrium solution of (P).

Corrollary 4.2 *Assume* (A1) *to* (A5). *If for any* $I \subseteq M, N_I$ *has at most an element, then Theorem 4.2 still holds for* $r = +\infty$.

Proof It is sufficient to prove that for any $k \in M$ there exist $x_k \in D_k^+$ and $t_k \geq 0$ such that $w(x_k,t_k;w_0) > 0$. Otherwise, similar to the proof of Theorem 4.2, instead of problem (P) we consider problem $(\tilde{P})$, the conclusion follows. The case of finite $\tilde{M}$ associate in problem $(\tilde{P})$ has been proved in Theorem 4.2.

Consider $\{\underline{w}_n\}$ where $\underline{w}_i = w_{\{1,2,\cdots,i\}}$ is the minimal solution in $S_{\{1,2,\cdots,i\}}, i = 1,2,\cdots$. By Theorem 3.2, such a sequence uniquly exists and is non-decreasing, uniformly bounded by W from above, where W is the maximal equilibrium solution of (P). Hence, $\lim_{n\to\infty} w_n = \overline{w} \leq W$.

Let us denote the Green function of operator L by $G(x,y)$ which equals to zero on ∂D. Then,

$$\underline{w}_n(x) = \int_D G(x,y)a(y)g(\underline{w}_n(y))dy - \int_{\partial D} \frac{\partial G(x,y)}{\partial \nu_y}\varphi(y)ds \tag{4.1}$$

where

$$\frac{\partial G}{\partial \nu_y} = a^{ij}\frac{\partial G(x,y)}{\partial y_i}\cos(n,y_i)$$

is the oblique derivative. Obviously, $a(x)$ is uniformly bounded in $\overline{D}$. Due to the monotonicity of g we know $g(\underline{w}_n) \leq g(w)$, and then

$$|\underline{w}_n(x) - \underline{w}_n(x')| \leq c\int_D |G(x,y) - G(x',y)|dy + \int_{\partial D}\left|\frac{\partial G(x,y)}{\partial \nu_y} - \frac{\partial G(x',y)}{\partial \nu_y}\right|\varphi(y)ds_y$$

By virtue of the property of $G(x,y)$, we know that the right-hand side of above inequality tends to zero as $x \to x'$. Hence, $\underline{w}_n(x)$ is equi-continuous. Then, by the Arzela-Ascoli theorem, there is a subsequence of $\{\underline{w}_n(x)\}$, denoted still by $\{\underline{w}_n(x)\}$, converging uniformly to a continuous function $\overline{w}(x)$. Let $n \to \infty$ in (4.1) we get

$$\overline{w}(x) = \int_D G(x,y)a(y)g(\overline{w}(y))dy - \int_D \frac{\partial G(x,y)}{\partial \nu_y}\varphi(y)ds$$

Taking the derivatives under the integral sign yields the regularity of $\overline{w}$. Therefore, $\overline{w}$ is an equilibrium solution of (P). Obviously, $\overline{w} > 0$ in D_k^+ for any $k \in M$, and hence $\overline{w} = W$, the maximal equilibrium solution of (P), by the uniqueness assumption.

By Theorem 4.1,

$$w(\cdot, t; w_0) \to [\underline{w}_n, W] \text{ as } t \to \infty, \ \forall n \in M$$

then, $\lim_{t\to\infty} w(\cdot, t; w_0) = W$.

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