J.Partial Diff. Eqs., 7(1994), 35–48
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EXISTENCE OF A WEAK SOLUTION FOR THE PHASE CHANGE PROBLEM WITH JOULE'S HEATING

Yuan Guangwei
(Dept. of Math., Peking University, Beijing 100871)
(Received Dec.11, 1991; revised May 12, 1992)

Abstract A phase change problem with Joule's heating describes the processes of electric heating in a conducting material. It is modeled as a coupled system of nonlinear partial differential equations with quadratic growth in the gradient. We establish the existence of a weak solution for the problem in two dimensions.
Key Words phase change; system of nonlinear partial differential equations; quadratic growth.
Classification 35K

1. Introduction

In this paper we consider a model that describes the combined effects of heat and electrical current flows in a metal. When an electrical current flows across the matal, Joule heating is generated by the resistance of the metal to the electrical current, which brings about the increase of the temperature. A phase change will take place once the melting temperature is crossed and the latent heat is absorbed.

Let $u = u(x,t)$ denote the temperature, u_* the melting temperature, $h = h(x,t)$ be the enthalpy density, $\varphi = \varphi(x,t)$ the electrical potential and $\sigma = \sigma(u)$ be the temperature dependent electrical conductivity. The mathematical model for the evolution under consideration is the following nonlinear system:

Find a triplet $\{h, u, \varphi\}$ such that

$$\frac{\partial h}{\partial t} - \Delta u = \sigma(u)|\nabla\varphi|^2 \tag{1.1}$$

$$\nabla(\sigma(u)\nabla\varphi) = 0 \tag{1.2}$$

$$h \subset u + \lambda H(u - u_*) \tag{1.3}$$

and the initial and boundary conditions, where

$$H(s) = \begin{cases} -1 & \text{if } s < 0 \\ [-1,1] & \text{if } s = 0 \\ 1 & \text{if } s > 0 \end{cases} \tag{1.4}$$

When $h \equiv u (i.e. \lambda = 0)$ in (1.1)–(1.3), Cimatti [1] proved the existence of weak solutions in two space dimensions and Chipot and Cimatti [2] proved the uniqueness for the problem in one and two space dimensions. For the physical background and the known results for the problem (1.1)–(1.3) we refer to [3] for more details and the references therein. In [3] by using regularization and time discretization the existence of the solutions $\{u_n, \varphi_n\}$ for the discretized approximated problems is proved, and then the strong convergence of $\{u_n\}$ and $\{\varphi_n\}$ in L^2 is proved. But we find that the proof of the latter step includes a mistake and the method breaks down. Here we shall give a new proof of the existence for the problem in two space dimensions.

The plan of the paper is as follows. In Section 2 the definition of the weak solution and the main result are stated. In Section 3 an approximating problem is solved by using Schauder fixed-point theorem. Further a priori estimates on the approximating solutions are obtained is Section 4. Since the right term of (1.1) involves the quadratic growth in the gradient of φ, we will use Meyers' estimate [4] to obtain the higher integratility of $|\nabla\varphi|$ and then prove the local equicontinuity of $\{u_n\}$ by using the modified method of the De Giorgi estimates (see [5]). In Section 5 it will be concluded that there exists a sequence of approximating solutions converging to the weak solution of the problem under consideration.

2. The Definition of the Weak Solutions and the Main Result

Let Ω be a smooth bounded domain of $\mathcal{R}^2$, which is occupied by a conducting material. Denote $\Omega_T = \Omega \times (0,T)$. We shall adopt the notation and symbol in [7] and make the following assumptions.

$$\sigma(s) \in C^1(\mathcal{R}^1),\ 0 < \sigma_* \le \sigma(s) \le \sigma^* < +\infty\ \forall s \in \mathcal{R}^1 \tag{2.1}$$

$$u_0(x) \in C(\bar{\Omega}),\ u_0(x) = 0 \text{ on } \partial\Omega,\ u_0(x) \ne u_* \text{ a.e. in } \Omega,\ u_* > 0 \tag{2.2}$$

$$\varphi_0 \in C^{1+\alpha,0}(\bar{\Omega}_T)\ (0 < \alpha < 1) \tag{2.3}$$

The enthalpy formulation of the problem is as follows:

Problem (P): Determine a triplet $\{h, u, \varphi\}$ such that

$$h \in \alpha(u) \quad \text{in } \Omega_T \tag{2.4}$$

$$\frac{\partial h}{\partial t} - \Delta u = \sigma(u)|\nabla\varphi|^2 \quad \text{in } \Omega_T \tag{2.5}$$

$$u = 0 \quad \text{on } \partial\Omega \times [0,T] \tag{2.6}$$

$$u = u_0(x) \quad \text{on } \Omega \times \{0\} \tag{2.7}$$

$$\nabla(\sigma(u)\nabla\varphi) = 0 \quad \text{in } \Omega_T \tag{2.8}$$

$$\varphi = \varphi_0 \quad \text{on } \partial\Omega \times [0,T] \tag{2.9}$$

Here $\alpha = \alpha(u)$ is the maximal monotone graph modelling the phase change process,

$$\alpha(s) = \begin{cases} s-1 & \text{if } s < u_* \\ [u_*-1, u_*+1] & \text{if } s = u_* \\ s+1 & \text{if } s > u_* \end{cases} \tag{2.10}$$

and we assume the latent heat $\lambda = 1$ and the melting temperature u_* =constant.

Definition 2.1 We say that a triplet $\{h, u, \varphi\}$ is a weak solution of (2.4)–(2.9) if

$$\begin{aligned} & h \in L^\infty(\Omega_T),\ h \in \alpha(u),\ h(x,0) = \alpha(u_0(x)) \text{ in } \Omega \\ & u \in \overset{\circ}{V}{}_2^{1,0}(\Omega_T) \cap C(\Omega \times [0,T]),\ u(x,0) = u_0(x) \text{ in } \Omega \\ & \varphi \in C(\bar{\Omega}_T) \cap L^\infty(0,T; W^{1,p^*}(\Omega)) \cap C(0,T; H^1(\Omega)) \text{ for some } p^* > 2 \\ & \varphi = \varphi_0 \text{ on } \partial\Omega \times [0,T] \end{aligned} \tag{2.11}$$

and $\forall v \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ with $v = 0$ on $\Omega \times \{T\}$ there holds

$$\int_{\Omega_T} \Big\{ -h\frac{\partial v}{\partial t} + \nabla u \cdot \nabla v \Big\} dxdt = \int_{\Omega_T} \sigma(u)|\nabla\varphi|^2 v dxdt + \int_\Omega v(x,0)h(x,0)dx \tag{2.12}$$

and $\forall \psi \in H_0^1(\Omega), \forall t \in [0,T]$ there holds

$$\int_\Omega \sigma(u)\nabla\varphi \cdot \nabla\psi dx = 0 \tag{2.13}$$

Remark 2.2 Since for any $q > 1$, $\overset{\circ}{W}{}_2^{1,1}(\Omega_T) \hookrightarrow L^q(\Omega_T)$ holds in two space dimensions, the first term of the right in (2.12) makes sense.

In this paper the following existence theorem will be proved.

Theorem 2.3 *Under the assumptions (2.1)–(2.3), Problem (P) possesses at least one weak solution.*

3. An Approximating Problem

Set $\alpha_n(s) = s + H_n(s-u_*)$, $(n = 1, 2, \cdots)$ and $H_n(s)$ satisfies the following conditions:

$$H_n(s) \in C^1(\mathcal{R}^1),\ 0 \le H_n'(s) \le 4n \quad \forall s \in \mathcal{R}^1,\ \forall n \ge 1 \tag{3.1}$$

$$H_n'(s-u_*) \le \frac{4}{|s-u_*|}, \quad \forall s \in \mathcal{R}^\infty \setminus \{u_*\},\ \forall n \ge 1$$

$$H_n'(s-u_*) = 0 \quad \forall s \in \mathcal{R}^1 \setminus \Big[u_* - \frac{1}{n}, u_* + \frac{1}{n}\Big],\ \forall n \ge 1 \tag{3.2}$$

$H_n'(s-u_*)$ is increasing over $s \in (-\infty, u_*]$ and decreasing over $s \in [u_*, \infty)$ (3.3)

$\alpha_n(s) \to \alpha(s)$ in $C^1[a,b]$ for all $[a,b]$ such that $u_* \notin [a,b]$. (3.4)

Let $u_{0n}(x)(n=1,2,\cdots)$ satisfy

$$u_{0n}(x)\in C^1(\bar{\Omega}),\ u_{0n}(x)=0 \text{ on } \partial\Omega,\ \|u_{0n}(x)-u_0(x)\|_{C^0(\bar{\Omega})}\to 0(n\to\infty) \tag{3.5}$$

Denote $[a]_n=\min(a,n)$. For each n we consider the following problem (P_n): Find a pair $\{u_n,\varphi_n\}$ such that

$$\frac{\partial\alpha_n(u_n)}{\partial t}-\Delta u_n=\sigma(u_n)[|\nabla\varphi_n|^2]_n \text{ in } \Omega_T \tag{3.6}$$

$$\nabla(\sigma(u_n)\nabla\varphi_n)=0 \text{ in } \Omega,\ \forall t\in[0,T] \tag{3.7}$$

$$u_n(x,0)=u_{0n}(x) \text{ in } \Omega \tag{3.8}$$

$$u_n=0 \text{ on } \partial\Omega\times[0,T] \tag{3.9}$$

$$\varphi_n=\varphi_0 \text{ on } \partial\Omega,\ \forall t\in[0,T] \tag{3.10}$$

Lemma 3.1 *For $n=1,2,\cdots$, Problem (P_n) has a weak solution satisfying*

$$u_n\in W_p^{2,1}(\Omega_T)\cap \overset{\circ}{W}{}_p^{1,\frac12}(\Omega_T) \text{ for any } p>2$$
$$\varphi_n\in C(\bar{\Omega}_T)\cap L^\infty(0,T;C^{1+\alpha}(\bar{\Omega}))\cap C(0,T;H^1(\Omega)) \text{ and for some } p^*>2 \tag{3.11}$$
$$\|\nabla\varphi_n\|_{L_{p^*,\infty}(\Omega_T)}\equiv\sup_{0\le t\le T}\|\nabla\varphi_n(\cdot,t)\|_{L_{p^*}(\Omega)}\le C$$

where p^* and C are independent of n, and for any $\xi\in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ with $\xi=0$ on $\Omega\times\{T\}$

$$\int_{\Omega_T}\Big\{-\alpha_n(u_n)\frac{\partial\xi}{\partial t}+\nabla u_n\cdot\nabla\xi\Big\}dxdt$$
$$=\int_{\Omega_T}\sigma(u_n)[|\nabla\varphi_n|^2]_n\xi dxdt+\int_\Omega\alpha_n(u_{0n}(x))\xi(x,0)dx \tag{3.12}$$

and for all $\eta\in H_0^1(\Omega)$ and all $t\in[0,T]$

$$\int_\Omega\sigma(u_n)\nabla\varphi_{u_n}\cdot\nabla\eta dx=0 \tag{3.13}$$

Proof Introduce the Banach space $B=C^{\bar{\alpha},\frac{\bar{\alpha}}{2}}(\bar{\Omega}_T)$ and the closed convex subset

$$\mathcal{K}=\{v\in B;\|v\|_B\le C,\ v=0 \text{ on } \partial\Omega\times[0,T]\ v(x,0)=u_{0n}(x) \text{ in } \Omega\}$$

where $C>0$ and $\bar{\alpha}\in(0,1)$ are constants to be determined.

Let $u\in\mathcal{K}$ and $t\in[0,T]$. Denote by $\varphi_u=\varphi_u(\cdot,t)$, the unique solution to the problem

$$(\varphi_u-\varphi_0)(\cdot,t)\in H_0^1(\Omega),\ \int_\Omega\sigma(u)\nabla\varphi_u\cdot\nabla\eta dx=0\quad\forall\eta\in H_0^1(\Omega) \tag{3.14}$$

By the standard elliptic theory we have

$$\|\varphi_u(\cdot,t)\|_{C^\alpha(\bar\Omega)} \le C_1 \tag{3.15}$$

$$\|\varphi_u(\cdot,t)\|_{W^{1,p^*}(\Omega)} \le C_2 \text{ for some } p^* > 2 \tag{3.16}$$

Here $C_i (i = 1, 2)$ depends only on $\sigma_*, \sigma^*, \varphi_0$ and the smoothness of $\partial\Omega$. (3.16) is the known Meyers' estimates [4]. Also $\varphi_u(\cdot,t) \in C^{1+\alpha}(\bar\Omega)$ for any $t \in [0,T]$ (see [8, Chapt. 8]).

Next solve the following problem

$$\begin{cases} \alpha_n'(u)\dfrac{\partial v}{\partial t} - \Delta v = \sigma(u)[|\nabla\varphi_u|^2]_n & \text{in } \Omega_T \\ v = 0 \text{ on } \partial\Omega \times [0,T],\ v = u_{0n} \text{ on } \Omega \times \{0\} \end{cases} \tag{3.17}$$

From the theory of linear parabolic equation there exists a unique solution $v \in W_p^{2,1}(\Omega_T) \cap \overset{\circ}{W}{}_p^{1,\frac{1}{2}}(\Omega_T)$ for any $p > 2$ to problem (3.17) and by the Krylov's estimates we have $\|v\|_{C^{\alpha,\frac{\alpha}{2}}(\bar\Omega_T)} \le \bar C$, so that $\|v\|_{C^{\frac{\alpha}{2},\frac{\alpha}{4}}(\bar\Omega_T)} \le \bar{\bar C}$. Here $\bar C$ and $\alpha \in (0,1)$ are constants independent of u. Now we can choose $\bar\alpha = \dfrac{\alpha}{2}$ in the definition of Banach space B and the constant C in the definition of the subset $\mathcal{K}$ can be taken as $\bar{\bar C}$. Define a mapping $\Lambda : \mathcal{K} \to \mathcal{K}$ as follows: $v = \Lambda u$. Obviously the image $\Lambda\mathcal{K}$ is precompact. We need only to show that Λ is continuous. Let $u_i \in \mathcal{K}(i = 1, 2, \cdots)$ converge to u in $C^{\bar\alpha,\frac{\bar\alpha}{2}}(\bar\Omega_T)$. Denote $v_i = \Lambda u_i (i = 1, 2, \cdots)$ and $v = \Lambda u$. Since $\|v_i\|_{W_p^{2,1}(\Omega_T)} \le C$ and $\|v_i\|_{C^{\alpha,\frac{\alpha}{2}}(\bar\Omega_T)} \le C$, where C is independent of i, a subsequence out of $\{v_i\}$ can be selected (and relabelled with i) such that

$$v_i \to \tilde v \text{ in } W_p^{2,1}(\Omega_T), \quad v_i \to \tilde v \quad \text{in } C^{\bar\alpha,\frac{\bar\alpha}{2}}(\bar\Omega_T)$$

If we can prove that there exists a subsequence of $\{\nabla\varphi_{u_i}\}$ such that

$$\nabla\varphi_{u_i} \to \nabla\varphi_u \ a.e. \text{ in } \Omega_T (i \to \infty)$$

then $\tilde v$ satisfies that

$$\begin{cases} \alpha_n'(u)\dfrac{\partial \tilde v}{\partial t} - \Delta \tilde v = \sigma(u)[|\nabla\varphi_u|^2]_n & \text{in } \Omega_T \\ \tilde v = 0 \quad \text{on } \partial\Omega \times [0,T],\ \tilde v = u_{0n} & \text{on } \Omega \times \{0\} \end{cases}$$

We must have $\tilde v \equiv v = \Lambda u$, and hence the sequence $\{v_i\}$ itself converges to v in $C^{\bar\alpha,\frac{\bar\alpha}{2}}(\bar\Omega_T)$. Then Schauder's theorem applies, and clearly, any fixed point u of Λ yields a solution $\{u_n, \varphi_n\}$ to the problem (P_n) by setting $u_n = u$ and $\varphi_n = \varphi_u$.

To complete the proof, two simple propositions will be shown.

Proposition (A) Let $u \in \mathcal{K}$. For each $t_0 \in [0,T]$, we have

$$\varphi_u(\cdot,t) \to \varphi_u(\cdot,t_0) \text{ in } C^0(\bar{\Omega})(\text{as } t \to t_0) \tag{3.18}$$

$$\nabla\varphi_u(\cdot,t) \to \nabla\varphi_u(\cdot,t) \text{ in } L^2(\Omega)(\text{as } t \to t_0) \tag{3.19}$$

Proposition (B) Let $\{u_i\}$ converge to u in $C^{\bar{\alpha},\frac{\bar{\alpha}}{2}}(\bar{\Omega}_T)(i \to \infty)$, then

$$\varphi_{u_i} \to \varphi_u \qquad \text{in } L^2(\Omega_T) \tag{3.20}$$

$$\nabla\varphi_{u_i} \to \nabla\varphi_u \quad \text{in } L^2(\Omega_T) \tag{3.21}$$

Proof of (A) Denote $\varphi = \varphi_u$ for simplicity. Let $\{t_n\} \subset [0,T], t_n \to t_0(n \to \infty)$. From (3.15) and (3.16) it follows that there exists a subsequence $\{t_{n_k}\}$ and a function $\tilde{\varphi}(x) \in H^1(\Omega)$ such that

$$\varphi(x,t_{n_k}) \to \tilde{\varphi}(x) \text{ in } C^0(\bar{\Omega}),\ \nabla\varphi(x,t_{n_k}) \to \nabla\tilde{\varphi}(x) \text{ weakly in } L^2(\Omega)$$

So $\tilde{\varphi}(x) = \varphi_0(x,t_0)$ on $\partial\Omega$ and for any $\eta \in H_0^1(\Omega)$.

$$\begin{aligned}
&\left|\int_\Omega \sigma(u(x,t_0))\nabla\tilde{\varphi}(x)\cdot\nabla\eta(x)dx\right| \\
\le &\left|\int_\Omega \{\sigma(u(x,t_0)) - \sigma(u(x,t_{n_k}))\}\nabla\varphi(x,t_{n_k})\cdot\nabla\eta(x)dx\right| \\
&+\left|\int_\Omega \sigma(u(x,t_0))\nabla(\tilde{\varphi}(x) - \varphi(x,t_{n_k}))\cdot\nabla\eta(x)dx\right| \\
&\to 0(\text{as } k \to \infty)
\end{aligned}$$

We conclude that $\tilde{\varphi}(x) = \varphi(x,t_0)$ in $\bar{\Omega}$ and (3.18) follows. And for all $\eta \in H_0^1(\Omega)$ we have

$$\begin{aligned}
&\int_\Omega \sigma(u(x,t_n))\nabla(\varphi(x,t_n) - \varphi(x,t_0))\cdot\nabla\eta(x)dx \\
&+\int_\Omega [\sigma(u(x,t_n)) - \sigma(u(x,t_0))]\nabla\varphi(x,t_0)\cdot\nabla\eta(x)dx = 0
\end{aligned}$$

Choose $\eta(x) = \varphi(x,t_n) - \varphi(x,t_0) - (\varphi_0(x,t_n) - \varphi_0(x,t_0)) \in H_0^1(\Omega)$ to obtain

$$\begin{aligned}
&\int_\Omega |\nabla(\varphi(x,t_n) - \varphi(x,t_0))|^2 \\
\le &C\int_\Omega |\sigma(u(x,t_n)) - \sigma(u(x,t_0))|^2|\nabla\varphi(x,t_0)|^2 \\
&+C\int_\Omega |\nabla\varphi(x,t_0)|^2|\nabla(\varphi_0(x,t_n) - \varphi_0(x,t_0))|^2
\end{aligned}$$

Here $C = C(\sigma_*,\sigma^*)$. Therefore (3.19) is proved.

Remark 3.2 From the argument above we see that in order to prove (A) it suffices to assume that $u \in C^0(\bar{\Omega}_T), u = u_{0n}(x)$ on $\Omega \times \{0\}$ and $u = 0$ on $\partial\Omega \times [0,T]$. Also using (A) we deduce that $\varphi_u(x,t) \in C^0(\bar{\Omega}_T)$. Then $\nabla_x \varphi_u(x,t)$ is a measurable function on Ω_T, and from (3.16) it follows that $\|\nabla\varphi_u\|_{L_{p^*,\infty}(\Omega_T)} \le C_2$.

Proof of (B) For any $\eta \in H_0^1(\Omega)$ and any $t \in [0,T]$ we have

$$\int_\Omega \sigma(u_i)\nabla\varphi_{u_i}\cdot\nabla\eta dx = 0, \quad \int_\Omega \sigma(u)\nabla\varphi_u\cdot\nabla\eta dx = 0$$

so

$$\int_\Omega \{\sigma(u_i)\nabla(\varphi_{u_i}-\varphi_u)\cdot\nabla\eta + (\sigma(u_i)-\sigma(u))\nabla\varphi_u\cdot\nabla\eta\}dx = 0$$

Let $\eta(x) = \varphi_{u_i}(x,t) - \varphi_u(x,t) \in H_0^1(\Omega)$ and obtain

$$\begin{aligned}
&\sigma_*\int_\Omega |\nabla(\varphi_{u_i}-\varphi_u)(x,t)|^2 dx \\
&\le \frac{\sigma_*}{2}\int_\Omega |\nabla(\varphi_{u_i}-\varphi_u)(x,t)|^2 dx \\
&\quad + \frac{1}{2\sigma_*}\int_\Omega |\sigma(u_i(x,t)) - \sigma(u(x,t))|^2 |\nabla\varphi_u(x,t)|^2 dx
\end{aligned}$$

Therefore

$$\int_{\Omega_T} |\nabla(\varphi_{u_i}-\varphi_u)|^2 dxdt \le \frac{1}{\sigma_*^2}\int_{\Omega_T} |\sigma(u_i)-\sigma(u)|^2|\nabla\varphi_u|^2 dxdt \to 0 \ (i\to\infty)$$

and also $\|\varphi_{u_i}-\varphi_u\|^2_{L^2(\Omega_T)} \le C\|\nabla(\varphi_{u_i}-\varphi_u)\|^2_{L^2(\Omega_T)}$. (B) follows.

The proof of Lemma 3.1 is completed.

4. Estimates on the Solution of (P_n)

From the results of Section 3 we have that

$$u_n \in W_p^{2,1}(\Omega_T) \cap \overset{\circ}{W}{}_p^{1,\frac{1}{2}}(\Omega_T) \text{ for any } p > 2$$

$$\nabla\varphi_n \in L_{p^*,\infty}(\Omega_T), \text{ for some } p^* > 2, \text{ and } \|\nabla\varphi_n\|_{L_{p^*,\infty}(\Omega_T)} \le C$$

Here C is a constant independent of n. And

$$\begin{cases} \dfrac{\partial\alpha_n(u_n)}{\partial t} - \Delta u_n = \sigma(u_n)[|\nabla\varphi_n|^2]_n & \text{in } \Omega_T \\ u_n = u_{0n}(x) \quad \text{on } \Omega\times\{0\}, u_n = 0 & \text{on } \partial\Omega\times[0,T] \end{cases} \tag{4.1}$$

In this section we shall establish the following three uniform estimates:

$\{u_n\}$ is bounded uniformly in Ω_T (4.2)

$\|u_n\|_{V_2^1(\Omega)} \le C(C$ independent of $n)$ (4.3)

$\{u_n\}$ is equicontinuous in $\Omega'\times[0,T]$ for any compacts of Ω (4.4)

Proof of (4.2) Set $A_k^+(t) = \{x \in \Omega; u(x,t) > k\}, A_k^-(t) = \{x \in \Omega; u(x,t) < k\}$ and

$$\eta = \pm(u_n - k)^{\pm}$$

where $k \geq \sup_n \max_{\bar{\Omega}} u_{0n}^+(x)$ for $(u_n - k)^+$ and $k \leq \inf_n \min_{\bar{\Omega}}(-u_{0n}^-(x))$ for $-(u_n-k)^-$. So $\eta \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$.

Multiply (4.1) by the function $\eta = \pm(u_n - k)^{\pm}$, which vanishes on the parabolic boundary of Ω_T, and integrate over $\Omega_t \equiv \Omega \times (0,t)$ to obtain

$$\pm\int_{\Omega_t} \frac{\partial \alpha_n(u_n)}{\partial t}(u_n - k)^{\pm} + \int_{\Omega_t} |\nabla(u_n - k)^{\pm}|^2 = \pm\int_{\Omega_t} \sigma(u_n)[|\nabla\varphi_n|^2]_n (u_n - k)^{\pm}$$

We treat the term involving $\dfrac{\partial \alpha_n(u_n)}{\partial t}$ as follows:

$$\begin{aligned}
&\pm\int_{\Omega_t} \frac{\partial \alpha_n(u_n)}{\partial t}(u_n - k)^{+} \\
&= \int_{\Omega_t} \frac{\partial}{\partial t}\Big\{\pm\int_{\alpha_n(k)}^{\alpha_n(u_n)} [\alpha_n^{-1}(s) - k]^{\pm} ds\Big\} \\
&= \pm\int_{\Omega}\Big\{\int_{\alpha_n(k)}^{\alpha_n(u_n(x,t))} [\alpha_n^{-1}(s) - k]^{\pm} ds\Big\}dx \\
&= \pm\int_{\Omega}\Big\{\int_{k}^{u_n(x,t)} (\tilde{s} - k)^{\pm}\alpha_n'(\tilde{s})d\tilde{s}\Big\}dx \geq \frac{1}{2}\int_{\Omega}(u_n(x,t) - k)^{\pm 2}dx
\end{aligned}$$

Therefore we have

$$\frac{1}{2}\int_{\Omega}(u_n(x,t) - k)^{\pm 2}dx + \int_0^t\int_{A_k^{\pm}(t)} |\nabla u_n|^2 \leq \pm\int_{\Omega_t} \sigma(u_n)[|\nabla\varphi_n|^2]_n(u_n - k)^{\pm} \tag{4.5}$$

The right hand side of (4.5) is non-positive when we choose the lower sign. Hence

$$(u_n - k)^- = 0 \text{ in } \Omega \text{ i.e. } u_n(x,t) \geq \inf_n \min_{\bar{\Omega}}(-u_0^-(x)) \text{ in } \Omega_T$$

When we choose the upper sign we get

$$\begin{aligned}
&\int_{\Omega}(u_n(x,t) - k)^{+2}dx + \int_0^t\int_{A_k^{+}(t)} |\nabla u_n|^2 \\
&\leq C\int_{\Omega_t} |\nabla\varphi_n|^2 (u_n - k)^+ \\
&\leq C\|\nabla\varphi_n\|^2_{L_{p^*,\infty}(\Omega_T)}\|(u_n - k)^+\|_{L_{\frac{qq_1}{q-1},q_1}(\Omega_t)}\Big(\int_0^t |A_k^+|^{\frac{q-1}{q}} ds\Big)^{\frac{q_1-1}{q_1}}
\end{aligned}$$

Here $q = \frac{p^*}{2} > 1, q_1 = 2\left(2 - \frac{1}{q}\right)$.

From the embedded theorem

$$V_2^{1,0}(\Omega_t) \hookrightarrow L_{\frac{qq_1}{q-1}, q_1}(\Omega_t)$$

(see [7]), it follows that

$$\|(u_n - k)^+\|^2_{V_2^1(\Omega_t)}$$
$$\leq C\|\nabla\phi_n\|^2_{p^*,\infty,\Omega_T}\|(u_n - k)^+\|_{V_2^1(\Omega_t)}\Big(\int_0^t |A_k^+|^{\frac{q-1}{q}} ds\Big)^{\frac{q_1-1}{q_1}}$$

Then

$$\|(u_n - k)^+\|^2_{V_2^1(\Omega_t)} \leq \tilde{C}\Big(\int_0^t |A_k^+|^{\frac{q-1}{q}} ds\Big)^{\frac{q_1-1}{q_1}}$$

Here $\tilde{C}$ is independent of n. Therefore $\|u_n\|_{L^\infty(\Omega_T)} \leq M$. (see [7] Thm 6.1; pp.102) and (4.2) is proved.

From (4.1), for all $\phi \in \overset{\circ}{W}{}_p^{2,1}(\Omega_T)$ and all $[t_0, t] \subset [0, t]$ we have

$$\int_{t_0}^t \int_\Omega \Big\{\frac{\partial \alpha_n(u_n)}{\partial t}\phi + \nabla u_n \cdot \nabla\phi - \sigma(u_n)[|\nabla\varphi_n|^2]_n\phi\Big\}dxdt = 0 \tag{4.6}$$

Choosing $\phi = \alpha_n(u_n) - \alpha_n(0), t_0 = 0$ we obtain

$$\frac{1}{2}\int_\Omega (\alpha_n(u_n(x,t)) - \alpha_n(0))^2 dx - \frac{1}{2}\int_\Omega (\alpha_n(u_{0n}(x)) - \alpha_n(0))^2 dx$$
$$+\int_{\Omega_t} \alpha_n'(u_n)|\nabla u_n|^2$$
$$= \int_{\Omega_t} \sigma(u_n)[|\nabla\varphi_n|^2]_n(\alpha_n(u_n) - \alpha_n(0))$$

So (4.3) follows.

Proof of (4.4). For simplicity of notation we let $u_* = 0$ (by using the transformation $v_n = u_n - u_*$) and drop the subscript n. Following the notation of [5], let $\sigma_1, \sigma_2 \in (0,1)$ and $(x_0, t_0) \in \Omega_T$ be fixed. Set $B(R) = \{x \in \Omega; |x - x_0| < R\}$ and consider the cylinders $Q(R,\lambda) = B(R) \times [t_0, t_0 + \lambda], Q(R - \sigma_1 R, \lambda - \sigma_2\lambda) = B(R - \sigma_1 R) \times [t_0 + \sigma_2\lambda, t_0 + \lambda], \lambda > 0$.

Define cutoff functions in $Q(R,\lambda)$ as follows:

(a) $\xi \in C_0[Q(R,\lambda)]$ such that $\xi(x,t) = 0$ on $\partial B(R) \times [t_0, t_0 + \lambda], \xi(x, t_0) = 0$ in

$$B(R), \xi(x,t) = 1 \text{ in } Q(R - \sigma_1 R, \lambda - \sigma_2\lambda),\ 0 \leq \frac{\partial\xi}{\partial t} \leq \frac{c}{\sigma_2\lambda}$$

$$|\nabla_x \xi| \leq \frac{c}{\sigma_1 R},\ 0 \leq \xi \leq 1$$

(b) $\xi \in C_0(B(R))$ such that $\xi(x) = 1$ in $B(R - \sigma_1 R), |\nabla \xi| \le C(\sigma_1 R)^{-1}$

For any cylinder $Q(R, \lambda) \subset \Omega_T$, we choose the following test function in (4.6)

$$\phi = \pm(u_n - k)^{\pm}\xi^2$$

where $k \in \mathcal{R}^1$ satisfies $|k| \le M$. Obviously

$$I \equiv \int_{t_0}^{t}\int_{\Omega} \pm\frac{\partial \alpha(u)}{\partial t}(u-k)^{\pm}\xi^2(x,\tau)dxd\tau = \int_{t_0}^{t}\int_{\Omega}\xi^2(x,\tau)\frac{\partial \Lambda}{\partial t}$$

where $\Lambda = \pm\int_{\alpha(k)}^{\alpha(u)}[\alpha^{-1}(\xi) - k]^{\pm}d\xi$. We perform an integration by parts to obtain

$$\begin{aligned} I &= \int_{\Omega}\Lambda(x,t)\xi^2(x,t)dx - \int_{t_0}^{t}\int_{\Omega}\Lambda(x,t)\frac{\partial \xi^2}{\partial t}dxd\tau \\ &\ge \frac{1}{2}\int_{\Omega}(u(x,t)-k)^{\pm 2}\xi^2(x,t)dx - \int_{t_0}^{t}\int_{\Omega}\Lambda(x,\tau)\frac{\partial \xi^2(x,\tau)}{\partial t}dxd\tau \end{aligned} \tag{4.7}$$

Using assumption (2.1) we have

$$\begin{aligned} &\int_{t_0}^{t}\int_{\Omega}\{\nabla u\cdot\nabla\phi - \sigma(u)[|\nabla\varphi|^2]_n\phi\}dxd\tau \\ &\ge \int_{t_0}^{t}\int_{\Omega}\{|\nabla(u-k)^{\pm}|^2\xi^2(x,\tau) \pm 2(u-k)^{\pm}\xi\nabla u\cdot\nabla\xi \\ &\quad -\sigma(u)[|\nabla\varphi|^2]_n(\pm(u-k)^{\pm})\xi^2\}dxd\tau \\ &\ge \frac{1}{2}\int_{t_0}^{t}\int_{\Omega}|\nabla(u-k)^{\pm}|^2\xi^2(x,\tau) \\ &\quad -2\int_{t_0}^{t}\int_{\Omega}[(u-k)^{\pm}]^2|\nabla\xi|^2 - \int_{0}^{t}\int_{\Omega}2M\sigma^*|\nabla\varphi|^2\xi^2\chi[(u-k)^{\pm}>0] \end{aligned} \tag{4.8}$$

Here $\chi(\Sigma)$ denotes the characteristic function of the set Σ. We set

$$A_{k,R}^{\pm}(\tau) \equiv \{x \in B(R); (u-k)^{\pm}(x,\tau) > 0\}$$

By Hölder inequality

$$\begin{aligned} J &\equiv \int_{t_0}^{t}\int_{\Omega}2M\sigma^*|\nabla\varphi|^2\xi^2\chi[(u-k)^{\pm}>0] \\ &\le 2M\sigma^*\|\nabla\varphi\|_{p^*,\infty,\Omega_T}^2\int_{t_0}^{t}|A_{k,R}^{\pm}(\tau)|^{\frac{p^*-2}{p^*}}d\tau \end{aligned}$$

Denote $r = 2\left(2 - \frac{2}{p^*}\right), q = \frac{p^* r}{p^* - 2}$, we have

$$\frac{1}{2}\int_\Omega (u(x,t)-k)^{\pm 2}\xi^2(x,t)dx + \int_{t_0}^t \int_\Omega |\nabla(u-k)^\pm|^2\xi^2$$
$$\leq \gamma \int_{t_0}^{t_0+\lambda}\int_\Omega [(u-k)^\pm]^2(|\nabla\xi|^2 + \xi\xi_t)$$
$$+\gamma \int_{t_0}^t \int_\Omega \Lambda(x,\tau)\frac{\partial\xi^2(x,\tau)}{\partial t}dxd\tau + \gamma\int_{t_0}^t |A^\pm_{k,R}(\tau)|^{\frac{r}{q}}d\tau \tag{4.9}$$

$\forall t \in [t_0, t_0+\lambda]$, where γ is a constant (independent of n) depending only upon the data. A change of variable in the integral defining $\Lambda(x,t)$ gives

$$\Lambda(x,t) = \int_0^{\pm(u-k)^\pm} \eta\alpha'(k+\eta)d\eta \tag{4.10}$$

Therefore $\Lambda(x,t) \leq C(M)(u-k)^\pm$.

By redefining the constant γ and recalling the construction $\xi(x,t)$, (4.9) implies

$$|(u-k)^\pm|^2_{V_2^{1,0}(Q(R-\sigma_1 R, \lambda-\sigma_2\lambda))}$$
$$\leq \gamma[(\sigma_1 R)^{-2} + (\sigma_2\lambda)^{-1}]\|(u-k)^\pm\|^2_{2,Q(R,\lambda)}$$
$$+\gamma\int_{t_0}^{t_0+\lambda}|A^\pm_{k,R}(\tau)|^{\frac{r}{2}}d\tau + \gamma(\sigma_2\lambda)^{-1}\int_{Q(R,\lambda)}(u-k)^\pm dxd\tau \tag{4.11}$$

Here γ is independent of n and inequality (4.11) is valid for all $k \in [-M, M]$, all cylinder $Q(R,\lambda) \subset \Omega_T$ and all $\sigma_1, \sigma_2 \in (0,1)$.

Now suppose that (4.9) is written for the function $(u_n - k)^+$ for $k > 0$, then by (3.2) $\Lambda(x,t)$ in (4.10) can be estimated as follows:

$$\Lambda(x,t) \leq \frac{1}{2}\sup_{s\geq k}\alpha'(s)(u-k)^{+2} \leq \frac{1}{2}\left(1+\frac{4}{k}\right)(u-k)^{+2}$$

Hence

$$|(u-k)^+|^2_{V_2^{1,0}(Q(R-\sigma_1 R, \lambda-\sigma_2\lambda))}$$
$$\leq \gamma\left(1+\frac{1}{k}\right)\left(\frac{1}{(\sigma_2 R)^2} + \frac{1}{\sigma_2\lambda}\right)\|(u-k)^+\|^2_{2,Q(R,\lambda)} + \gamma\int_{t_0}^{t_0+\lambda}|A^+_{k,R}(\tau)|^{\frac{r}{q}}d\tau \tag{4.12}$$

Here γ does not depend upon n and (4.12) is valid for all $k \in (0, M]$, all cylinder $Q(R,\lambda) \subset \Omega_T$ and all $\sigma_1, \sigma_2 \in (0,1)$.

Remark 4.1 An analogous inequality holds for the function $(u_n - k)^-, k < 0$, to which we refer as $(3.10)^-$.

All the subsequent arguments in this section will be carried over cylinders of the form $Q(R, \theta R^2) \equiv B(R) \times [t_0, t_0 + \theta R^2], \theta > 0$.

Let $k>0$ and $\mu \geq \sup_{Q(R,\theta R^2)}(\alpha(u)-k)^+, 0<\eta<\mu$. Set

$$\psi(x,t)=\tilde{\psi}(\alpha(u)) \equiv \ln^+\Big[\frac{\mu}{\mu-(\alpha(u)-k)^+)+\eta}\Big]$$

Then there exists a constant $C=C(\theta)$ such that for all $t\in[t_0,t_0+\theta R^2]$,

$$\int_{B(R-\sigma_1 R)}\psi^2(x,t)dx \leq \int_{B(R)}\psi^2(x,t_0)dx+\frac{C(\theta)}{\sigma_1^2}\Big(\ln\frac{\mu}{\eta}\Big)\Big(1+\frac{R^{2(p^*-2)/p^*}}{\eta^2}\Big)R^2 \tag{4.13}$$

Proof of (4.13) In (4.6) choose $\phi=(\psi^2)'\xi^2(x)$, here $\xi(x)$ is as in (b) and the prime denoting differentiation $w.r.t.\alpha(u)$. Denote $\psi'=\dfrac{d\tilde{\psi}(v)}{dv}$.

The first term gives

$$\begin{aligned}&\int_{t_0}^t\int_\Omega\frac{\partial\alpha(u)}{\partial t}(\psi^2)'\xi^2(x)dxd\tau\\ &=\int_\Omega\psi^2(x,\tau)\xi^2(x)dx|_{t_0}^t\\ &\geq\int_{B(R-\sigma_1 R)}\psi^2(x,t)dx-\int_{B(R)}\psi^2(x,t_0)dx\end{aligned}$$

In estimating the second term we first observe that $(\psi^2)''=2(1+\psi)(\psi')^2$. Hence

$$\begin{aligned}&\int_{t_0}^t\int_\Omega\{\nabla u\cdot 2(1+\psi)(\psi')^2\alpha'(u)\nabla u\cdot\xi^2(x)+2(\psi^2)'\xi(x)\nabla\xi(x)\cdot\nabla u\}\\ &\geq\frac{3}{2}\int_{t_0}^t\int_\Omega(1+\psi)(\psi')^2\alpha'(u)|\nabla u|^2\xi^2(x)-32\int_{t_0}^t\int_\Omega\psi|\nabla\xi|^2\end{aligned}$$

For the remaining term we have

$$\begin{aligned}&-2\int_{t_0}^t\int_\Omega\sigma(u)[|\nabla\varphi|^2]_n\psi\psi'\xi^2(x)dxd\tau\\ &\leq\frac{2\sigma^*}{\eta}\ln\frac{\mu}{\eta}\int_{t_0}^t\int_\Omega|\nabla\varphi|^2\xi^2(x)\end{aligned}$$

Combining these estimates above we deduce

$$\begin{aligned}&\int_{B(R-\sigma_1 R)}\psi^2(x,t)dx-\int_{B(R)}\psi^2(x,t_0)dx\\ &\leq\gamma\ln\frac{\mu}{\eta}\int_{t_0}^t\int_\Omega|\nabla\xi|^2+\frac{\gamma}{\eta}\ln\frac{\mu}{\eta}\int_{t_0}^t\|\nabla\varphi\|^2_{p^*,\Omega}(\tau)R^{2(p^*-2)/p^*}d\tau\\ &\leq\gamma\ln\frac{\mu}{\eta}\frac{\theta R^2}{(\sigma_1 R)^2}\cdot\pi R^2+\frac{\gamma}{\eta}\ln\frac{\mu}{\eta}\|\nabla\varphi\|^2_{p^*,\infty,\Omega_T}\theta R^2\cdot(\pi R^2)^{(p^*-2)/p^*}\\ &\leq\frac{C(\theta)}{\sigma_1^2}\Big(\ln\frac{\mu}{\eta}\Big)\Big(1+\frac{R^{2(p^*-2)/p^*}}{\eta}\Big)R^2\end{aligned}$$

and (4.13) is proved.

Remark 4.2 If $k < 0$ and $\bar{\mu} \geq \sup_{Q(R,\theta R^2)} (\alpha_n(u_n) - k)^-$, then an analogous inequality holds for

$$\bar{\psi}(x,t) = \ln^+ \frac{\bar{\mu}}{\bar{\mu} - (\alpha_n(u_n) - k)^- + \eta}, \quad 0 < \eta < \bar{\mu}$$

to which we refer as $(4.13)^-$.

Now (4.4) follows from inequalities (4.11), $(4.12)^\pm$ and $(4.13)^\pm$ via the arguments of [5.pp.95–115]. The continuity up to $t = 0$ also can be proved as in [5. Th.6.1]. (see pp.114–115).

5. The Limit as $n \to \infty$

From the results of Section 4 it follows that there exists a subsequence of $\{u_n\}$ (and relabelled with n) such that

$$\begin{aligned} &u_n \to u && \text{in } C(\Omega' \times [0,T]),\ \forall \Omega' \subset\subset \Omega \\ &\nabla u_n \rightharpoonup \nabla u && \text{weakly in } L^2(\Omega_T) \\ &\alpha_n(u_n) \rightharpoonup h \subset \alpha(u) && \text{weakly in } L^2(\Omega_T) \\ &\alpha_n(u_{0n}(x)) \rightharpoonup \alpha(u_0(x)) && \text{weakly in } L^2(\Omega) \end{aligned} \tag{5.1}$$

Arguing as the proof of (3.21) we have

$$\int_\Omega |\nabla(\varphi_n - \varphi_m)(x,t)|^2 dx \leq \frac{1}{\sigma_*^2} \|\sigma(u_n) - \sigma(u_m)\|^2_{p^*/(p^*-2),\Omega}(t) \|\nabla \varphi_m\|_{p^*,\Omega}(t)$$

Therefore

$$\begin{aligned} &\int_\Omega |\nabla(\varphi_n - \varphi_m)(x,t)|^2 dx \to 0 \qquad \forall t \in [0,T] \\ &\int_{\Omega_T} |\nabla(\varphi_n - \varphi_m)(x,t)|^2 dxdt \to 0 \\ &\int_{\Omega_T} |\varphi_n - \varphi_m|^2 dxdt \to 0 \qquad (\text{as } n,m \to \infty) \end{aligned} \tag{5.2}$$

Let $\varphi_n \to \tilde{\varphi}$ in $L^2(\Omega_T)$, $\nabla \varphi_n \to \nabla \tilde{\varphi}$ in $L^2(\Omega_T)$ and

$$\varphi_n(x,t) \to \varphi(x,t) \text{ in } H^1(\Omega) \text{ for each } t \in [0,T]$$

Then

$$\int_\Omega \sigma(u(x,t)) \nabla \varphi(x,t) \nabla \eta dx = 0 \quad \forall \eta \in H_0^1(\Omega),\ \forall t \in [0,T] \tag{5.3}$$

We can now proceed as the proof of Proposition (A) in §3 and Remark 3.2 to derive

$$\varphi \in C(\bar{\Omega}_T) \cap C(0,T; H^1(\Omega)) \cap L^\infty(0,T; W^{1,p^*}(\Omega)) \text{ (for some } p^* > 2)$$

It is easily seen that $\varphi(x,t) \equiv \tilde{\varphi}(x,t)$ in Ω_T.

For $\forall \xi \in C^1(\bar{\Omega}_T)$, with $\xi = 0$ on $\Omega \times \{T\} \cup \partial\Omega \times [0,T]$, we have

$$\int_{\Omega_T} \Big\{ -\alpha_n(u_n)\frac{\partial \xi}{\partial t} + \nabla u_n \cdot \nabla \xi \Big\} dxdt$$

$$= \int_{\Omega_T} \sigma(u_n)[|\nabla \varphi_n|^2]_n \xi dxdt + \int_\Omega \alpha_n(u_{0n}(x))\xi(x,0)dx$$

Let $n \to \infty$ in the subsequence chosen above to obtain

$$\int_{\Omega_T} \Big\{ -h\frac{\partial \xi}{\partial t} + \nabla u \cdot \nabla \xi \Big\} dxdt$$

$$= \int_{\Omega_T} \sigma(u)|\nabla \varphi|^2 \xi dxdt + \int_\Omega \alpha(u_0(x))\xi(x,0)dx$$

This equality also holds for all $\xi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ with $\xi = 0$ on $\Omega \times \{T\}$, and Theorem 2.3 is proved.

Remark 5.1 We may use the method of [6] to prove u is continuous on $\partial\Omega \times [0,T]$. Similarly for the case of Neumann boundary data and mixed boundary data the existence of weak solutions can be obtained.

Acknowledgement The author heartily thanks Prof. Lishang Jiang for his guidance and valuable suggestions and also thanks all other fellows in the seminar of Suzhou University for their help.

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