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EXISTENCE AND REGULARITY OF SOLUTIONS OF A NONLINEAR NONUNIFORMLY ELLIPTIC SYSTEM ARISING FROM A THERMISTOR PROBLEM

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Abstract A thermistor is an electric circuit device made of ceramic material whose electric conductivity depends on the temperature. If the only heat source is the electric heating, the temperature and the electric potential satisfy a nonlinear elliptic system which is also degenerate if the electric conductivity is not uniformly bounded from above or away from zero. Under general boundary conditions, we establish existence and Hölder continuity of solutions of such a nonlinear nonuniformly elliptic system. When the electric conductivity linearly depends on the temperature, we provide a non-uniqueness and non-existence example.

Key Words Thermistor; elliptic system; nonlinear; nonuniformly elliptic; mixed boundary value problem.

Classification 35J70, 35J55.

1. Introduction

A thermistor, or a thermally-sensitive-resistor, is an electrical device made of semiconducting materials whose electrical resistivity changes up to 5 orders of magnitude as the temperature increases over a certain range. It has many applications such as current regulation, switching, thermal conductivity analysis, and control and alarm; see, for example, Hyde [12] and Llewellyn [13].

When acting as a (renewable) circuit breaker, a thermistor operates as follows: an increase in current provides more (electrical) heating, leading to a rise in temperature of the material which causes a rise in the resistivity, thereby reducing the current (to almost zero if the temperature increases beyond a critical limit). When the thermistor cools down, its resistivity decreases and the normal operation of the circuit resumes. In this paper, however, we shall only study the steady state problems.

Denote by Ω the domain in R^N occupied by the thermistor ($N = 2, 3$ are cases of physical interest) and by $u, \phi, \sigma(u)$, and $k(u)$ the temperature, electrical potential, electrical conductivity, and thermal conductivity, respectively. Then the steady state thermistor problem is to solve the elliptic system

$$\nabla(\sigma(u)\nabla\varphi)=0 \quad \text{in } \Omega \quad \text{(conservation of current)} \tag{1.1}$$

$$-\nabla(k(u)\nabla u)=\sigma(u)|\nabla\varphi|^2 \quad \text{in } \Omega \quad \text{(conservation of energy)} \tag{1.2}$$

subject to the boundary conditions

$$\varphi=\varphi_0 \quad \text{on } \Gamma_D^\varphi, \quad \partial_n\varphi=0 \quad \text{on } \Gamma_N^\varphi\equiv\partial\Omega\setminus\overline{\Gamma_D^\varphi} \tag{1.3}$$

$$u=u_0 \quad \text{on } \Gamma_D^u, \quad \partial_n u+h(x,u)=0 \quad \text{on } \Gamma_N^u\equiv\partial\Omega\setminus\overline{\Gamma_D^u} \tag{1.4}$$

where ∂_n is the outward normal derivative and $\Gamma_D^\varphi, \Gamma_D^u$ are smooth hypersurfaces. Typically, Γ_D^φ consists of two disjoint hypersurfaces Γ_D^1 and Γ_D^2, and

$$\varphi=V \quad \text{on } \Gamma_D^1, \quad \varphi=0 \quad \text{on } \Gamma_D^2$$

where V is the voltage difference applied on the thermistor.

There has been recent mathematical interest in this thermistor problem in both the case when σ is positive [4, 5, 6, 11, 15, and the references therein] and the case when σ is vanishes at large temperature [1, 2, 3, 10, 16].

The obstacles in this thermistor problem are the quadratic growth on the right-hand side of (1.2) and the degeneracy of (1.1) when $\sigma(u)$ is not uniformly bounded from above or away from zero.

In this paper we shall consider the case when $\sigma(u)$ is positive but is not necessarily uniformly bounded from above and away from zero as $u\to\infty$. Since the change of thermal conductivity is of secondary importance, we shall assume that $k=1$. In fact, the method given here can be applied to the general case of $k>0$ as well. We shall establish the existence and Hölder continuity of the solution of (1.1)–(1.4) under certain conditions on $\sigma(u)$ and $h(x,u)$.

In the case when $k=1$ and σ is uniformly bounded from above and away from zero, existence of weak solutions to (1.1)–(1.4) was recently established by Howison, Rodrigues, and Shillor [11]. The strategy they used to get around the quadratic growth is to write $\sigma|\nabla\varphi|^2$ as $\nabla(\sigma(\varphi-\varphi_0)\nabla\varphi)+\sigma\nabla\varphi_0\nabla\varphi$ which is a bounded functional on $H^1(\Omega)$. They proved the boundedness of the solution only for the case of Dirichlet boundary condition or for the case of $N=2$ where one can apply Meyers' theorem on the elliptic equations of type (1.1) to deduce that $\nabla\varphi\in L^p(\Omega)$ for some $p>2$, and therefore to deduce that $u\in W^{2,p/2}\subset C^{2-4/p}$ by the L^p estimate and the Sobolev imbedding theorem. They also established the uniqueness of solutions for the case when the solutions are sufficient "small". The general uniqueness problem, however, is still open.

When $\sigma(u)/k(u)$ is not uniformly bounded away from zero, equations (1.1), (1.2) with Dirichlet boundary data was studied in [4, 5, 6, 15]. The strategy here is to use the transformation found by Diesselhorst in 1900 [7]:

$$\psi = \frac{\varphi^2}{2} + F(u) \quad \text{where} \quad F(u) = \int_0^u \frac{k(s)}{\sigma(s)} ds \tag{1.5}$$

Using this transformation and (1.1), one can write equation (1.2) in a simpler form

$$\nabla(\sigma \nabla \psi) = 0 \tag{1.6}$$

so that one can apply the maximum principle to obtain a L^∞ a priori bound for ψ. Consequently, if one assumes that $F(\infty) = \infty$, one can derive from (1.5) a L^∞ bound for u. Once the L^∞ bound for u is established, the function $\sigma(u(x))$ becomes uniformly bounded from above and away from zero, so that one can apply the classic Hölder estimates to obtain the Hölder continuity for the functions φ and ψ in $\bar{\Omega}$. In [15], Xie and Allegretto also studied equations (1.1)–(1.4) but with the restriction $\Gamma_N^u \subset \Gamma_N^\varphi$. Under this restriction, the boundary condition for ψ is

$$\psi = \frac{\varphi^2}{2} + u^0 \text{ on } \Gamma_D^u, \ \partial_n \psi = H(x, \psi) \equiv \frac{k(u)h(x,u)}{\sigma(u)}\Big|_{u=F^{-1}(\psi - \varphi^2/2)} \text{ on } \Gamma_N^u \tag{1.7}$$

For this type of boundary condition, one can still directly apply the weak maximum principle to get the L^∞ a priori bound for ψ and then use the Hölder estimate to establish the Hölder continuity for the solution.

For the general case of (1.3), (1.4), i.e., without the assumption $\Gamma_N^u \subset \Gamma_N^\varphi$, we cannot use the transformation (1.5) since the boundary condition for ψ involves $\partial_n \varphi$. To overcome this difficulty, we use the transformation

$$\psi = (\varphi - \varphi_0)^2 + F(u) \tag{1.8}$$

Under this transformation, the boundary condition for ψ is of the type (1.7) since $\partial_n(\varphi - \varphi_0)^2 = 0$ on $\partial\Omega$. Although the equation for ψ in the domain Ω is not very good, we can still implement the classical Nash-Moser iteration to establish the L^∞ bounds for ψ.

To establish the Hölder continuity for u in $\bar{\Omega}$, we cannot simply use the function ψ in (1.5) or in (1.8) since the boundary condition for ψ in (1.5) is not pleasant whereas the equation for ψ in (1.8) in the domain Ω is not good. Instead we shall directly work on u and φ. By showing that

$$\iint_{B_R(x)\cap\Omega} \sigma(u)|\nabla\varphi|^2 \le CR^{N-2+\alpha} \quad \forall x \in \bar{\Omega}, \ R > 0$$

for some $\alpha > 0$, we then use the method in [8] to establish the Hölder continuity for u.

The plan of this paper is as follows. In Section 2 we state the problem and our main result. Then we establish the L^∞ estimate in Section 3 and prove the existence of a weak solution of (1.1)–(1.4) in Section 4. We shall establish the Hölder continuity for the weak solution in Section 5. To explain the necessity of our condition, we shall finally give a non-existence and non-uniqueness example in Section 6.

2. Statement of the Problem

It is convenient to introduce a new function v defined by

$$v = F(u) \equiv \int_0^u \frac{k(s)}{\sigma(s)} ds \tag{2.1}$$

Under this transformation, (1.1)-(1.4) are equivalent to

$$-\nabla(a(v)\nabla\varphi) = 0 \quad \text{in } \Omega \tag{2.2}$$

$$-\nabla(a(v)\nabla v) = a(v)|\nabla\varphi|^2 \quad \text{in } \Omega \tag{2.3}$$

$$\varphi = \varphi_0 \quad \text{on } \Gamma_D^\varphi, \quad \partial_n\varphi = 0 \quad \text{on } \Gamma_N^\varphi \tag{2.4}$$

$$v = v_0 \quad \text{on } \Gamma_D^u, \quad \partial_n v + H(x, v) = 0 \quad \text{on } \Gamma_N^u \tag{2.5}$$

where

$$\begin{aligned} a(v) &= \sigma(u)\big|_{u=F^{-1}(v)} \\ v_0 &= F(u_0), \\ H(x,v) &= \frac{k(u)}{\sigma(u)} h(x,u)\Big|_{u=F^{-1}(v)} \end{aligned} \tag{2.6}$$

Note that in order for $F^{-1}(v)$ to be well-defined for all $v \in R$, it is necessary to assume that

$$\int_0^\infty \frac{k(s)}{\sigma(s)} ds = \infty \tag{2.7}$$

In fact, if this condition is not satisfied, Cimatti [4] has shown in one-dimensional case that the solution may not exist. Therefore, to ensure the existence, one has to assume that the a priori L^∞ bound for v obtained in Section 3 is in the definition range of $F^{-1}(v)$. For simplicity, here we assume that (2.7) is satisfied so that $a(v), H(x,v)$ is globally defined and we can just work on (2.2)-(2.5).

Also note that the function $k(u)$ is not involved in (2.2)-(2.5), so one can generally assume that $k(u)=1$.

We shall make the following assumptions for $\Omega, \varphi_0(\cdot), v_0(\cdot), H(\cdot, v)$, and $a(v)$:

(A1) Ω is a bounded domain with a (piecewise) smooth (C^2) boundary $\partial\Omega$. Γ_D^φ and Γ_D^u are non-empty (piecewise) smooth (C^2) hypersurfaces with smooth $N-2$ dimensional boundaries in $\partial\Omega$.

(A2) The functions φ_0 and v_0 have extensions into $\bar{\Omega}$. The extensions, which are still denoted by φ_0 and v_0, satisfy

$$\|\varphi_0, v_0\|_{C^1(\bar{\Omega})} \leq M_0 \tag{2.8}$$

$$\partial_n\varphi_0 = 0 \quad \text{on} \quad \Gamma_N^\varphi \tag{2.9}$$

$$v_0 \geq 0 \quad \text{on} \quad \bar{\Omega} \tag{2.10}$$

where M_0 is a constant ≥ 1.

(A3) The function $H(x,v)$ is measurable in $x \in \Gamma_N^u$ and continuous in $v \in R^1$ (uniformly for all $x \in \Gamma_N^u$); moreover, H satisfies

$$|H(x,v)| \leq M_1 \quad \forall (x,v) \in \Gamma_N^u \times [0, M_0] \tag{2.11}$$

$$H(x,v) \geq 0 \quad \forall v \geq M_0,\ x \in \Gamma_N^u \tag{2.12}$$

$$H(x,v) \leq 0 \quad \forall v \leq 0,\ x \in \Gamma_N^u \tag{2.13}$$

where M_0 is the constant in (2.8) and M_1 is some positive constant.

(A4) The function $a(v)$ is continuous and positive in R^1 and satisfies the following conditions:

1. There exists a constant $\Sigma > 0$ such that

$$\frac{1}{\Sigma} \leq \frac{a(v+y)}{a(v)} \leq \Sigma \quad \forall v > 0,\ |y| \leq 4M_0^2 \tag{2.14}$$

2. Either

$$\int_0^\infty a(s)ds = \infty \quad \text{and} \quad \lim_{v\to\infty} \frac{1}{a(v)} \int_0^v a(s)ds = \infty \tag{2.15}$$

or

$$\lim_{v\to\infty} \frac{a(v)}{v^2} \int_0^v \frac{1}{a(s)} ds = 0 \tag{2.16}$$

Notice that any (positive) polynomial satisfies (2.15) whereas any (positive and) monotone decreasing function satisfies (2.16).

First of all, we define a (weak) solution of (2.2)–(2.5).

Definition 1 *A pair (v,φ) is called a weak solution of* (2.2)–(2.5) *if*

$$v,\varphi \in H^1(\Omega),\ a(v)H(\cdot,v) \in L^1(\Gamma_N^u),\ a(v)|\nabla\varphi| \in L^2(\Omega),$$

$$\varphi = \varphi_0 \ \text{on}\ \Gamma_D^\varphi, \quad v = v_0 \ \text{on}\ \Gamma_D^u,$$

and

$$\iint a(v)\nabla\varphi\nabla\xi = 0 \quad \forall \xi \in H^1(\Omega),\ \ \xi|_{\Gamma_D^\varphi} = 0 \tag{2.17}$$

$$\iint a(v)\nabla v\nabla\eta - a(v)|\nabla\varphi|^2\eta + \int_{\Gamma_N^u} a(v)H(x,v)\eta = 0$$

$$\forall \eta \in H^1(\Omega)\cap L^\infty(\Omega), \quad \eta|_{\Gamma_D^u} = 0 \tag{2.18}$$

Notice that taking $\xi = \varphi - \varphi_0$ in (2.17) yields $a(v)|\nabla\varphi|^2 \in L^1(\Omega)$, so that (2.18) is well defined. Our main result is the following:

Theorem 1 *Assume that* (A1)–(A4) *are satisfied. Then the system* (2.2)–(2.5) *admits a (weak) solution satisfying*

$$v,\varphi \in C^\alpha(\bar\Omega) \tag{2.19}$$

for some constant $\alpha \in (0,1)$.

The proof will be given is Sections 3–5.

Remark 2.1 For simplicity, here we have assumed that Γ_D^u is non-empty. This assumption, however, can be replaced by some monotonicity condition on $H(x,v)$ (see also Remark 3.1).

Remark 2.2 The functions $k(u)$ and $\sigma(u)$ quoted in [15] have the form

$$\sigma(u) = Au^{\gamma}e^{-C/Bu}, \quad k(u) = (D + Eu + Fu^2)^{-2}$$

where A, B, C, D, E and F are positive constants. Clearly, after the transformation (2.1), the function $a(v)$ defined in (2.6) satisfies the assumption (A4).

Remark 2.3 With a boot strap argument, one can start from the regularity (2.19) to establish higher regularity for the solution provided that $a(v)$ is smooth.

Reamrk 2.4 If $\sigma(u)$ vanishes beyond some temperature $u^* > 0$ but satisfies

$$\int_0^{u^*} \frac{k(s)}{\sigma(s)} ds = \infty$$

then the function $F^{-1}(v)$ is well-defined for $v \in [0,\infty)$, so that Theorem 1 still holds.

Note that when $k = 1$ and $\sigma \sim u^{\beta}$ with $\beta \in (0,1)$, the assumption (A4) is satisfied. However, when $\sigma(u) \sim u$, the assumption (A4) is not satisfied since in this case $a(v) \sim e^v$. Therefore, as a test for the necessity of the condition (A4), we consider the case when σ is linear. More precisely, we consider the following problem:

$$(u\varphi_x)_x = 0, \quad x \in (0,1) \tag{2.20}$$

$$-u_{xx} = u\varphi_x^2, \quad x \in (0,1) \tag{2.21}$$

$$\varphi(0) = 0, \quad \varphi(1) = V \tag{2.22}$$

$$u_x(0) = 0, \quad u_x(1) + u(1) = 0 \tag{2.23}$$

$$u(x) > 0, \quad x \in [0,1] \tag{2.24}$$

For this system, we proved the following:

Theorem 2 *There exists a unique positive constant* V_0 *such that* (2.20)–(2.24) *has a solution if and only if* $V = \pm V_0$. *Moreover, if* (u,φ) *is a solution of* (2.20)–(2.24) *with* $V = \pm V_0$, *then for any* $k > 0$, *the pair* (ku,φ) *is also a solution to* (2.20)–(2.24)

The proof will be given in Section 6.

3. L^∞ Estimates

In this section, we shall always assume that (A1)–(A4) are satisfied.

Lemma 1 *Let (v,φ) be a weak solution of (2.2)–(2.5). Then one has the following bounds:*

$$\sup_{\Omega}|\varphi| \le \max_{\Gamma_D^\varphi}|\varphi_0| \tag{3.1}$$

$$\inf_{\Omega} v \ge 0 \tag{3.2}$$

This lemma is a consequence of the weak maximum principle. In fact, take $\xi = \max\{\varphi - \max_{\Gamma_D^\varphi}|\varphi_0|, 0\}$ in (2.17) and $\eta = \max\{\min\{v,0\},-1\}$ in (2.18), one deduces that $a(v)|\nabla\xi|^2 = 0$ and $a(v)|\nabla\eta|^2 = 0$. Therefore $\xi = \eta = 0$ since $a(v) > 0$ a.e. in Ω and $\xi,\eta \in H^1(\Omega)$. One then obtains the inequality $\sup_{\Omega}\varphi \le \max_{\Gamma_D^\varphi}|\varphi_0|$ and the inequality (3.2). Similarly, one can prove that $\inf_{\Omega}\varphi \ge -\sup_{\Gamma_D^\varphi}|\varphi_0|$. The assertions of Lemma 1 thus follows.

Now we shall establish the upper bound for v. To express the idea simpler, in the sequel we shall write equations and boundary conditions in the classical way in stead of in the integral identities as in the definition 1. Although it looks formal, it can be verified rigorously in the distribution sense by taking appropriate test functions in the definition 1.

Introduce a function ψ defined by

$$\psi = (\varphi - \varphi_0)^2 + v \tag{3.3}$$

Since $\partial_n\varphi_0 = \partial_n\varphi = 0$ on Γ_n^φ (in the distribution sense), it follows that $\partial_n(\varphi-\varphi_0)^2 = 0$ on $\partial\Omega$ and therefore ψ satisfies the boundary condition:

$$\partial_n\psi = \partial_n v = -H(x, \psi - (\varphi-\varphi_0)^2) \quad \text{on} \quad \Gamma_N^u \tag{3.4}$$

One can compute

$$\begin{aligned} -\nabla(a(v)\nabla\psi) &= -2\nabla((\varphi-\varphi_0)a\nabla\varphi) + 2\nabla(a(\varphi-\varphi_0)\nabla\varphi_0) - \nabla(a\nabla v) \\ &= -a|\nabla\varphi|^2 + 2\nabla(a(\varphi-\varphi_0)\nabla\varphi_0) + 2a\nabla\varphi\nabla\varphi_0 \end{aligned}$$

by utilizing (2.2) and (2.3). Using Cauchy's inequality for the last term, one obtains the inequality:

$$-\nabla(a\nabla\psi) \le a|\nabla\varphi_0|^2 + 2\nabla(a(\varphi-\varphi_0)\nabla\varphi_0) \tag{3.5}$$

To establish the upper bound for ψ, we first consider the case when $a(v)$ satisfies (2.15). The second equation in (2.15) implies that for any $\varepsilon > 0$, there exists a positive constant C_ε such that

$$a(v) \le \varepsilon\int_0^v a(s)ds + C_\varepsilon \qquad \forall v > 0 \tag{3.6}$$

Set $M_2 = 4M_0^2 + M_0$. Then in view of the definition of ψ in (3.3), the boundedness of φ in (3.1) and the boundedness $v = v_0$ on Γ_D^u, we have

$$\psi \le M_2 \quad \text{on } \Gamma_D^u \tag{3.7}$$

Define Ψ as

$$\Psi = A(\psi) \equiv \int_{M_2}^{\psi} a(s)ds \tag{3.8}$$

it follows from (3.7), (3.4), and (3.5) that Ψ satisfies

$$\Psi \le 0 \quad \text{on } \Gamma_D^u \tag{3.9}$$

$$\partial_n \Psi = -a(\psi)H(x,v)|_{v=A^{-1}(\Psi)-(\varphi-\varphi_0)^2} \tag{3.10}$$

$$-\nabla\Big(\frac{a(v)}{a(\psi)}\nabla\Psi\Big) \le a(v)|\nabla\varphi_0|^2 + 2\nabla(a(v)(\varphi-\varphi_0)\nabla\varphi_0 \tag{3.11}$$

(in the distribution sense).

By the routine truncation and approximation process if necessary, we shall assume that $\Psi \in L^\infty(\Omega)$, so that the following proof is valid without further explanation.

Denote $\max\{\Psi, 0\}$ by Ψ_+ and let $p \ge 2$ be any constant. Multiplying (3.11) by Ψ_+^{p-1} and integrating by parts, one obtains

$$\begin{aligned}(p-1)\iint_\Omega \frac{a(v)}{a(\psi)}\Psi_+^{p-2}|\nabla\Psi_+|^2 + \int_{\Gamma_N^u} a(v)\Psi_+^{p-1}H(x,v)\\ \le \iint_\Omega \frac{a(v)}{a(\psi)}|\nabla\varphi_0|^2 a(\psi)\Psi_+^{p-1} - 2\frac{a(v)}{a(\psi)}(\varphi-\varphi_0)\nabla\varphi_0 a(\psi)\Psi_+^{p-2}\nabla\Psi_+\end{aligned} \tag{3.12}$$

Here we have used the fact that $(\varphi-\varphi_0)\partial_n\varphi_0 = 0$ on $\partial\Omega$.

Note that if $\Psi_+ > 0$, then $\psi > 4M_0^2 + M_0$ and $v = \psi - (\varphi-\varphi_0)^2 > M_0$, so that by (2.12), $H(x,v) \ge 0$. That is, the second term on the left-hand side of (3.12) is non-negative. Using (2.8), (2.14), and (3.6) in (3.12), one gets

$$\iint_\Omega \Psi_+^{p-2}|\nabla\Psi_+|^2 \le C\iint_\Omega (\varepsilon\Psi_+ + C_\varepsilon)\Psi_+^{p-2}(\Psi_+ + |\nabla\Psi_+|)$$

Using Cauchy's inequality, one finds that

$$\iint_\Omega \Psi_+^{p-2}|\nabla\Psi_+|^2 \le C\iint_\Omega (\varepsilon\Psi_+^p + \varepsilon^{1-p}C_\varepsilon^p) \tag{3.13}$$

Since $\Psi_+|_{\Gamma_D^u} = 0$, the Sobolev inequality implies that

$$\iint_\Omega \Psi_+^2 \le C(\Omega,\Gamma_D^u)\iint_\Omega |\nabla\Psi_+|^2 \tag{3.14}$$

Substituting (3.13) (with $p = 2$) into the right-hand side and taking ε small enough, one obtains the L^2 estimate

$$\|\Psi_+\|_{L^2(\Omega)} \le C \tag{3.15}$$

The L^∞ estimate for Ψ_+ then follows from (3.13), (3.15), and the Nash-Moser iteration technique. For reader's convenience, we give the proof below.

Let ε be fixed, say $\varepsilon = 1$, and let $\Psi_\varepsilon \equiv \max\{\Psi_+, C_\varepsilon\}$.

Recall the Sobolev imbedding

$$\|\Psi_\varepsilon^{p/2}\|_{L^{2^*}(\Omega)} \le C(\Omega)\|\Psi_\varepsilon^{p/2}\|_{H^1(\Omega)}$$
$$\le C(\Omega)(\|\nabla\Psi_+^{p/2}\|_{L^2(\Omega)} + \|\Psi_\varepsilon^{p/2}\|_{L^2(\Omega)}) \tag{3.16}$$

where

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N > 2 \\ \text{any } q < \infty & \text{if } N \le 2 \end{cases}$$

Substituting (3.13) into the right-hand side of (3.16) yields

$$\|\Psi_\varepsilon\|_{L^{\mu p}(\Omega)} \le (Cp)^{1/p}\|\Psi_\varepsilon\|_{L^p(\Omega)} \quad \forall p \ge 2$$

where $\mu = \dfrac{N}{N-2}$ if $N > 2$ and $\mu = 2$ if $N \le 2$. Therefore, successively apply the above inequality with $p_0 = 2, p_n = p_0\mu^n$ for $n = 1, 2, \cdots$, one obtains the following:

$$\|\Psi_\varepsilon\|_{L^\infty(\Omega)} = \lim_{n\to\infty}\|\Psi_\varepsilon\|_{L^{2\mu^n}(\Omega)} \le \prod_{i=0}^{\infty}(C2\mu^i)^{1/(2\mu^i)}\|\Psi_\varepsilon\|_{L^2(\Omega)} \le \tilde{C}$$

Transferring back to the function v via (3.8) and (3.3) and using the first equation in (2.15), one obtains that there exists a positive constant M such that

$$\|v\|_{L^\infty(\Omega)} \le M \tag{3.17}$$

In the case when $a(v)$ satisfies (2.16) we can use a similar argument to establish (3.17) and here we just sketch the proof. Set $\psi_{M_2} = \max\{\psi, M_2\}$ and multiply inequality (3.5) by $\xi = \displaystyle\int_{M_2}^{\psi_{M_2}} \frac{s^{p-2}}{a(s)}ds$. After integrating by parts, using (2.8), (2.14), and (2.16), and following the same procedure as before, one gets

$$\iint_\Omega \psi_{M_2}^{p-2}|\nabla\psi_{M_2}|^2 \le C\iint_\Omega (\varepsilon\psi_{M_2}^p + \varepsilon^{1-p}C_\varepsilon^p)$$

The Nash-Moser iteration then yields the L^∞ bound for ψ_{M_2}, which also provides an upper bound for v.

In summary, we have proven the following:

Lemma 2 *Let (v, φ) be a weak solution of (2.2)–(2.5). Then there exists a positive constant M such that*

$$\|\varphi, v\|_{L^\infty(\Omega)} \le M \tag{3.18}$$

Remark 3.1 The only place we need to use the assumption Γ_D^n being nonempty is to derive the L^2 estimate (3.15) where we have to use Sobolev's inequality (3.14).

Clearly, we can drop this assumption and still get the L^∞ estimate if we make some assumption on the growth of $H(x,\cdot)$ and in deriving the L^2 estimate (3.15) we do not drop the non-negative term $\int_{\Gamma_N^u} a(v)\Psi_+^{p-1}H(x,v)$ in (3.12).

4. Existence of a Weak Solution

With the a priori L^∞ estimate, the existence of weak solutions now follows from the routine truncation procedure.

Let $m>0$ be an arbitrary constant. Define a function a_m by

$$a_m(s)=\begin{cases} a(0) & \text{if } s<0 \\ a(s) & \text{if } s\in[0,m] \\ a(m) & \text{if } s>m \end{cases}$$

Then a_m is uniformly bounded from above and away from zero. Applying the result of [11], we know that there exists a (weak) solution (v^m,φ^m) to (2.2)–(2.5) where $a(v)$ is replaced by $a_m(v)$.

Notice that the constant Σ and the limit in the assumption (A4) can be made uniformly in m so that the L^∞ a priori estimate obtained in Section 3 can be made independent of m; i.e., there exists a positive constant M independent of m such that $\|v^m,\varphi^m\|_{L^\infty(\Omega)}\le M$. Therefore if we let $m>M$, we have $a_m(v^m)=a(v^m)$; that is, (v^m,φ^m) is actually a solution (2.2)–(2.5). This establishes the existence of a weak solution.

5. Hölder Estimates

The L^∞ estimate obtained in Section 3 and the continuity assumption on $a(v)$ imply that there exists a constant $\sigma^*>0$ such that

$$1/\sigma^*\le a(v(x))\le\sigma^* \qquad \forall x\in\Omega \tag{5.1}$$

We shall now use (5.1) to establish the Hölder continuity for the weak solution of (2.2)–(2.5).

Lemma 3 *There exist a constant $\alpha(0,1)$ and a constant $C_\alpha>0$ such that*

$$\|\varphi\|_{C^\alpha(\Omega)}\le C_\alpha \tag{5.2}$$

$$\iint_{B_R(x)\cap\Omega} a(v(y))|\nabla\varphi(y)|^2\le C_\alpha R^{N-2+2\alpha} \qquad \forall x\in\overline{\Omega}, R>0 \tag{5.3}$$

where $B_R(x)$ is a ball in $\boldsymbol{R}^N$ centered at x with radius R.

Proof The first estimate follows from (5.1) and the classical Hölder estimates [14]. We need only to prove (5.3).

Let $x \in \overline{\Omega}$ and $R > 0$ be given. We consider two cases:

(i) $\operatorname{dist}(x, \Gamma_D^\varphi) > 2R$;

(ii) $\operatorname{dist}(x, \Gamma_D^\varphi) \le 2R$.

Here $\operatorname{dist}(x, A)$ denotes the distance from x to the set A.

First we consider the case (i). Denote by $\bar{\varphi}$ the average of φ on the set $B_{2R}(x) \cap \Omega$. Let $\zeta \in C_0^\infty(\mathbf{R}^N)$ be a cut-off function satisfying $\zeta = 0$ in $\mathbf{R}^N \setminus B_{2R}(x), \zeta = 1$ in $B_R(x), 0 \le \zeta \le 1$ in $\mathbf{R}^N$, and $|\nabla\zeta| \le 2/R$. Then, $(\varphi - \bar{\varphi})\zeta^2 \in H^1(\Omega)$ and $(\varphi - \bar{\varphi})\zeta^2 = 0$ on Γ_D. Taking ξ in (2.17) to be $(\varphi - \bar{\varphi})\zeta^2$ yields

$$\begin{aligned} 0 &= \int_\Omega a\nabla\varphi\nabla((\varphi - \bar{\varphi})\zeta^2) = \int_\Omega a|\nabla\varphi|^2\zeta^2 + 2a\zeta(\varphi - \bar{\varphi})\nabla\varphi\nabla\zeta \\ &\ge \frac{1}{2}\int_\Omega a|\nabla\varphi|^2\zeta^2 - 2\int_\Omega a(\varphi - \bar{\varphi})^2|\nabla\zeta|^2 \\ &\ge \frac{1}{2}\int_{B_R(x)\cap\Omega} a|\nabla\varphi|^2 - CR^{N-2+2\alpha}\|\varphi\|^2_{C^\alpha(\Omega)} \end{aligned}$$

Inequality (5.3) thus follows.

It the case (ii), notice that both φ and φ_0 are Hölder continuous, they coincide on Γ_D^φ, and $\operatorname{dist}(x, \Gamma_D^\varphi) \le 2R$, so that

$$|\varphi(y) - \varphi_0(y)| \le CR^\alpha \qquad \forall y \in B_{2R}(x)$$

It follows that

$$\begin{aligned} 0 &= \int_\Omega a\nabla\varphi\nabla((\varphi - \varphi_0)\zeta^2) \\ &\ge \int_\Omega \Big(\frac{1}{2}a|\nabla\varphi|^2\zeta^2 - a|\nabla\varphi_0|^2\zeta^2 - 4(\varphi - \varphi_0)^2|\nabla\zeta|^2\Big) \\ &\ge \Big(\frac{1}{2}\int_{B_R(x)\cap\Omega} a|\nabla\varphi|^2\Big) - CR^{2N-2+2\alpha} \end{aligned}$$

i.e., (5.3) holds. This completes the proof of Lemma 3.

In the following, we shall assume that $\alpha \in (0, 1/2)$ so that $2\alpha \in (0, 1)$.

We need the following lemma to establish the Hölder continuity for the function v.

Lemma 4 *Let*

$$\Gamma(x - \xi) = \begin{cases} \omega_n|x - \xi|^{2-N} & \text{if } N > 2 \\ -(2\pi)^{-1}\ln|x - \xi| & \text{if} N = 2 \end{cases}$$

be the fundamental solution of the Δ *in* $\mathbf{R}^N$ *and let* $w_f(x)$ *be the harmonic potential*

$$w_f \equiv \int_{\mathbf{R}^N} \Gamma(x - \xi)f(\xi)d\xi, \qquad x \in \mathbf{R}^N$$

where

$$f = \begin{cases} a|\nabla\varphi|^2 & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

Then there exists a constant C depending only on the constant C_α in Lemma 3, such that

$$\|w_f\|_{C^{2\alpha}(R^N)} \le C$$

Proof The proof is much the same as the potential analysis in [8, 9]. For reader's convenience, we sketch the proof below. For simplicity, we assume that $N > 2$.

Let $x, y \in R^N$ be any two points such that $x \ne y$. Set $d = |x - y|$. Then, one has

$$|w_f(x) - w_f(y)| \le \iint_{B_{2d}(x)} \Gamma(x-\xi)|f(\xi)| + \iint_{B_{2d}(y)} \Gamma(y-\xi)|f(\xi)| + \iint_{R^N \setminus B_d(\frac{x+y}{2})} |\Gamma(x-\xi) - \Gamma(y-\xi)||f(\xi)| \equiv I_1 + I_2 + I_3$$

Since (5.3) implies that

$$\int_{B_R(x)} |f(x)| \le C R^{N-2+2\alpha} \qquad \forall x \in R^N, R > 0 \tag{5.4}$$

one can estimate I_1 by

$$I_1 = \sum_{i=0}^{\infty} \iint_{B_{\frac{d}{2^i}}(0)\setminus B_{\frac{d}{2^{i+1}}}(0)} \Gamma(\xi)|f(x-\xi)|$$
$$\le C \sum_{i=1}^{\infty} \Big(\frac{2^{i+1}}{d}\Big)^{N-2} \iint_{B_{d/2^i}(0)} |f(x-\xi)|$$
$$\le C \sum_{i=1}^{\infty} \Big(\frac{2^{i+1}}{d}\Big)^{N-2} \Big(\frac{d}{2^i}\Big)^{(N-2+2\alpha)} \le C d^{2\alpha}$$

Similarly, one can obtain $|I_2| \le C d^{2\alpha}$.

To estimate I_3, recall that

$$|\Gamma(x-\xi) - \Gamma(y-\xi)| \le C|x-y||\xi - z|^{1-N} \quad \text{if } |\xi - z| \ge |x-y|$$

where $z = (x+y)/2$. It follows that

$$|I_3| \le C \iint_{R^N \setminus B_d(z)} \frac{|x-y|}{|\xi - z|^{N-1}} |f(\xi)|$$
$$= C d \sum_{i=0}^{\infty} \iint_{B_{2^{i+1}d} \setminus B_{2^i d}} \frac{1}{\xi^{N-1}} |f(z-\xi)| d\xi$$
$$\le C d \sum_{i=0}^{\infty} \Big(\frac{1}{2^i d}\Big)^{N-1} (2^{i+1} d)^{(N-2+2\alpha)}$$
$$\le C d^{2\alpha} \quad (\text{since } 2\alpha < 1)$$

where in the second inequality we have used (5.4). Lemma 4 thus follows.

Now we are ready to prove the Hölder continuity for v.

Lemma 5 *There exist a positive constant C and a constant $\beta \in (0,1)$ such that*

$$\|v\|_{C^\beta(\overline{\Omega})} \leq C$$

Proof Introduce $w(x) = \int_0^{v(x)} a(s)ds$. It suffices to prove the Hölder continuity of w. Decompose w into the the sum of v_1 and v_2 which are, respectively, the solution of the following problems:

$$\Delta v_1 = f - f_\Omega \quad \text{in } \Omega \tag{5.5}$$

$$\partial_n v_1 = 0 \quad \text{on } \partial\Omega \tag{5.6}$$

$$\int_{\partial\Omega} v_1 = 0 \tag{5.7}$$

and

$$\Delta v_2 = f_\Omega \qquad \text{in } \Omega$$

$$v_2 = \int_0^{v_0(\cdot)} a(s)ds - v_1 \quad \text{on } \Gamma_D^u$$

$$\partial_n v_2 = g(\cdot) \qquad \text{on } \Gamma_N^u$$

where $f = a(v)|\nabla\varphi|^2$, f_Ω is the average of f on Ω, and $g = -H(x, v(x))$.

Let $G(x,\xi) = \Gamma(x-\xi) + h(x,\xi)$ be the Green's function of the Laplace operator Δ corresponding to the Neumann boundary condition (5.6); namely, for each $x \in \Omega$, $h(x,\cdot)$ is harmonic in Ω and satisfies the boundary condition $\partial_{n_\xi} h(x,\xi) = -\partial_{n_\xi}\Gamma(x-\xi) + c$ on $\partial\Omega$ where c is the average of $\partial_{n_\xi}\Gamma(x-\xi)$ on $\partial\Omega$.

By Green's formula, one has

$$v_1 = \iint_\Omega G(x,\xi)f(\xi) = w_f + \iint_\Omega h(x,\xi)f(\xi) = w_f + w_f^1$$

Notice that $h(x,\xi)$ is smooth when x is in a compact subset of Ω, so that w_f^1 is smooth in any compact subset of Ω. When x is near the boundary $\partial\Omega$, the singularity of $h(x,\xi)$ is similar to $\Gamma(x-\xi^*)$ where ξ^* is the reflection of ξ with respect to (the tangent plane of) $\partial\Omega$. Therefore use the same proof as in Lemma 4, we can show that w_f^1 is Hölder continuous with Hölder exponent 2α near the boundary $\partial\Omega$. It follows that $v_1 \in C^{2\alpha}(\overline{\Omega})$.

As a consequence of the Hölder continuity of v_1 in $\overline{\Omega}$, the boundary value of v_2 is Hölder continuous on Γ_D^u. Since $\partial_n v_2|_{\Gamma_N^u} = g(x)$ is bounded, the classical Hölder estimate then implies the v_2 is in $C^\beta(\overline{\Omega})$ where $\beta \in (0, 2\alpha]$ is a constant depending on Γ_D^u.

The relation $\int_0^v a(s)ds = w = v_1 + v_2$ then yields the assertion of the lemma.

Combining the results of Section 4, Lemma 3, and Lemma 5, Theorem 1 follows.

6. A Non-existence and Non-uniqueness Example

Now we shall solve the system (2.20)–(2.24).

Notice that if (u,φ) is a solution to (2.20)–(2.24), then for every $k>0$, (ku,φ) is also a solution, so by (2.24), we can always scale the solution such that

$$u(1)=1 \tag{6.1}$$

Since $u_x(1)+u(1)=0$, one gets

$$u_x(1)=-1 \tag{6.2}$$

The equation $(u\varphi_x)_x=0$ implies that

$$u\varphi_x=I \tag{6.3}$$

where I is a constant, which physically denotes the electrical current. Substituting this relation into the equation $u_{xx}+u\varphi_x^2=0$, one gets

$$(u_x+I\varphi)_x=0$$

which, together with the boundary condition $u_x(0)=\varphi(0)=0$, yields

$$u_x+I\varphi=0 \quad \forall x\in[0,1] \tag{6.4}$$

Using (6.2) and the boundary condition $\varphi(1)=V$, we find the relation

$$I=1/V \tag{6.5}$$

To find u, we substitute $\varphi_x^2=I^2/u^2=1/(V^2u^2)$ into the differential equation for u. This gives

$$u_{xx}+\frac{1}{V^2u}=0$$

Multiplying this equation by u_x, integrating over $(x,1)$, and using (6.1), (6.2), we get

$$\frac{u_x^2}{2}+\frac{1}{V^2}\ln\left(e^{-\frac{V^2}{2}}u\right)=0$$

Since, by the maximum principle, we have $u>1$ for all $x\in[0,1)$, this equation implies that $u_x\neq 0$. It follows that

$$u_x=-\frac{1}{|V|}\sqrt{2\ln(e^{V^2/2}u^{-1})} \tag{6.6}$$

Solving this ODE, we obtain that the solution u is implicitly given by

$$\int_1^{u(x)}\frac{ds}{\sqrt{\ln(e^{V^2/2}s^{-1})}}=\frac{\sqrt{2}}{|V|}(1-x) \quad \forall x\in[0,1] \tag{6.7}$$

Using (6.4), (6.5), and (6.6), one gets

$$\varphi = \frac{V}{|V|}\sqrt{2\ln(e^{V^2/2}u^{-1})} \tag{6.8}$$

Equation (6.6) and the boundary condition $u_x(0) = 0$ imply that $u(0) = e^{V^2/2}$, so, by (6.7), V has to satisfy the equation

$$\int_1^{e^{V^2/2}} \frac{ds}{\sqrt{\ln(e^{v^2/2}s^{-1})}} = \frac{\sqrt{2}}{|V|}. \tag{6.9}$$

Up to now we have shown that if (u, φ) solves (2.20)–(2.24), then after an appropriate scaling on u, the solution is (uniquely) given by (6.7) and (6.8), and the constant V has to satisfy the equation (6.9). On the other hand, one can directly verify that if V satisfies (6.9), then the functions u and φ given by (6.7) and (6.8) form a solution to (2.20)–(2.24). Therefore, to complete the proof of Theorem 2, we need only to show that equation (6.9) has a unique positive solution.

Assume that $V > 0$. Then equation (6.9) can be written as

$$F(v) = 0$$

where

$$F(v) = \int_1^{e^{V^2/2}} \frac{ds}{\sqrt{\ln(e^{V^2/2}s^{-1})}} - \frac{\sqrt{2}}{|V|} = e^{V^2/2}\int_{e^{-V^2/2}}^{1} \frac{dt}{\sqrt{-\ln t}} - \frac{\sqrt{2}}{V}$$

Observe that $F(0+) = -\infty, F(\infty) = \infty$, and $F'(v) > 0$ for all $v \in (0, \infty)$, so one concludes that there exists one and only one solution to equation (6.9). This completes the proof of Theorem 2.

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