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UNIQUENESS OF GENERALIZED SOLUTIONS FOR A QUASILINEAR DEGENERATE PARABOLIC SYSTEM

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Abstract In this paper we study the uniqueness of generalized solutions for a class of quasilinear degenerate parabolic systems arising from dynamics of biological groups. The results obtained give an answer to a problem posed by A.S. Kalashnikov [1].

Key Words Uniqueness; weak solution; quasilinear degenerate parabolic system.

Classification 35K55, 35K65.

In this paper we consider a quasilinear degenerate parabolic system of the form

$$\frac{\partial u_i}{\partial t} = a_i \Delta u_i^{m_i} + b_i u_1^{p_i} u_2^{q_i} \tag{1}$$

in $Q_T = R^T \times (0, T)$ with the initial condition

$$u_i(x, 0) = u_{0i}(x) \tag{2}$$

for $x \in R^N$, where $m_i \geq 1, p_i \geq 1, q_i \geq 1, T > 0, a_i > 0,\ b_i$ are given real numbers and $u_{0i}(i = 1, 2)$ are bounded measurable functions in R^N.

The system (1) arises from modeling interacting evolution of two biological groups with densities u_1, u_2 (see [1]).

Definition *A vector function (u_1, u_2) with $u_i \in L^\infty(Q_T)$ and $u_i \geq 0$ $(i = 1, 2)$ is a generalized solution of (1)–(2), if (u_1, u_2) satisfies*

$$\iint_{Q_T} (u_i \varphi_{it} + a_i u_i^{m_i} \Delta \varphi_i + b_i u_1^{p_i} u_2^{q_i} \varphi_i)\, dx dt = 0 \tag{3}$$

for all $\varphi_i \in C_0^\infty(Q_T)$, $i = 1, 2$;

$$\lim_{t \to 0} \int_{R^N} \psi_i(u_i(x, t) - u_{0i}(x)) dx = 0 \tag{4}$$

for all $\psi_i \in C_0^\infty(R^N)$, $i = 1, 2$.

A.S. Kalashnikov first studied the system (1) and proved the existence of generalized solutions to the Cauchy problem (1)–(2) (see [1]). However, he was not able to solve the problem of uniqueness, and put forward as an open problem in the paper [1]. Afterwards, A.S. Kalashnikov mentioned this problem again at a symposium [2].

In this paper, we attempt to give an answer to this problem. The main result obtained is the following theorem.

Theorem *Let the vector functions* (u_1, u_2) *and* (v_1, v_2) *be two generalized solutions of the Cauchy problem* (1)–(2). *Then*

$$u_i(x,t) = v_i(x,t), \quad i = 1, 2$$

for a.e. $(x,t) \in Q_T$.

Here the uniqueness is proved for the cases $p_i \geq 1$ and $q_i \geq 1$ $(i = 1, 2)$. The following example shows that these conditions can not be removed in general: both

$$w = 0 \quad \text{and} \quad w = [(1-p)t]^{\frac{1}{1-p}}$$

are generalized solutions of the equation

$$w_t = \Delta w^m + w^p \tag{5}$$

in $Q_T = R^N \times (0, T)$ with the initial condition

$$w(x, 0) = 0 \tag{6}$$

on R^N, where $m > 1$ and $1 > p > 0$. In case $b_i < 0$ $(i = 1, 2)$ the uniqueness seems to be true even if $p_i < 1$ and $q_i < 1$ $(i = 1, 2)$. But we are not able to prove yet.

The result (Theorem) can be extended to more general systems of the form

$$\frac{\partial u_i}{\partial t} = \Delta A_i(u_i) + B_i(u_1, u_2, \cdots, u_n), \quad i = 1, 2, \cdots, n$$

in Q_T, where $A_i : R^1 \to R^1$, $B_i : R^n \to R^1$ $(i = 1, 2, \cdots, n)$ are locally Lipschitz continuous, respectively.

The same problem for more general systems with convection term has been studied by one of the authours and his colleague and the uniqueness of BV solutions has been proved (see [3]).

In order to prove the theorem we define

$$A_i(x,t) = \begin{cases} a_i \cdot \dfrac{u_i^{m_i}(x,t) - v_i^{m_i}(x,t)}{u_i(x,t) - v_i(x,t)} & \text{if } u_i(x,t) \neq v_i(x,t) \\ 0, & \text{otherwise} \end{cases}$$

$$A_{i,\varepsilon}(x,t) = A_i(x,t) + \varepsilon, \ i = 1, 2$$

$$A_{i,\varepsilon,\rho}(x,t) = (A_{i,\varepsilon} * J_\rho)(x,t), \ i = 1, 2$$

for all $(x,t) \in Q_T$, where

$$0 < \rho < 1, \quad J_\rho \in C^\infty(R^{N+1}), \ \int_{R^{N+1}} J_\rho = 1$$

with

$$\text{supp} J_\rho \subset \{(x,t) : |x| < \rho, |t| < \rho\}$$

Clearly, we have

$$\varepsilon \le A_{i,\varepsilon}(x,t) \le M, \quad i = 1,2$$
$$\varepsilon \le A_{i,\varepsilon,\rho}(x,t) \le M, \quad i = 1,2$$

for all $(x,t) \in Q_T$, where M is a positive constant depending only on the L^∞-norm of u_i and v_i $(i = 1,2)$.

For $\theta_i(x) \in C_0^\infty(R^N)$ $(i = 1,2)$ with $|\theta_i| \le 1$, we choose a positive number R such that

$$\theta_i(x) \in C_0^\infty(B_{R-1}), \quad i = 1,2$$

where

$$B_R = \{x \in R^N : |x| < R\}$$

Now consider the following boundary value problem

$$\frac{\partial \psi_i}{\partial t} + A_{i,\varepsilon,\rho}\Delta\psi_i = 0 \quad \text{in } B_R \times (0,T) \tag{7}$$
$$\psi_i(x,t) = 0, \ (x,t) \in \partial B_R \times (0,T) \tag{8}$$
$$\psi_i(x,T) = \theta_i(x)e^{-|x|}, \ x \in B_R \tag{9}$$

where $i = 1,2$.

It is known that the boundary value problem (7)–(9) has a unique smooth solution $\psi_{i,\varepsilon,\rho}$. In order to prove the uniqueness of solutions for (1)–(2), we need the following lemmas.

Lemma 1 *The solution $\psi_{i,\varepsilon,\rho}$ of the boundary value problem (7)–(9) satisfies the following inequalities*

$$|\psi_{i,\varepsilon,\rho}(x,t)| \le 1, \ (x,t) \in B_R \times (0,T) \tag{10}$$
$$\int_{B_R} |\nabla \psi_{i,\varepsilon,\rho}(x,t)|^2 \le M_1; \ t \in (0,T) \tag{11}$$
$$\int_0^T \int_{B_R} A_{i,\varepsilon,\rho}(x,t)(\Delta\psi_{i,\varepsilon,\rho})^2 dxdt \le M_1 \tag{12}$$

where M_1 is a positive constant depending on θ_i $(i = 1,2)$ but independent of ε, ρ and R.

Proof The inequality (10) follows from the Maximum principle. In order to prove (11) and (12), we multiply (7) by $\Delta\psi_{i,\varepsilon,\rho}$ and integrate in $B_R \times (0,T)$ to obtain

$$\int_t^T \int_{B_R} \left\{(\Delta\psi_{i,\varepsilon,\rho})\frac{\partial}{\partial t}(\psi_{i,\varepsilon,\rho}) + A_{i,\varepsilon,\rho}[\Delta\psi_{i,\varepsilon,\rho}]^2\right\} dxdt = 0$$

We compute

$$\int_t^T \int_{B_R} (\Delta\psi_{i,\varepsilon,\rho}) \frac{\partial}{\partial t}\psi_{i,\varepsilon,\rho} dx d\tau = -\frac{1}{2}\int_{B_R} |\nabla [\theta_i(x)e^{-|x|}]|^2 dx + \frac{1}{2}\int_{B_R} |\nabla \psi_{i,\varepsilon,\rho}(x,t)|^2 dx$$

and we have

$$\frac{1}{2}\int_{B_R} |\nabla \psi_{i,\varepsilon,\rho}(x,t)|^2 dx + \int_t^T \int_{B_R} A_{i,\varepsilon,\rho}[\Delta\psi_{i,\varepsilon,\rho}]^2 dx dt = \frac{1}{2}\int_{B_R} \left|\nabla \left[\theta_i(x)e^{-|x|}\right]\right|^2 dx$$

which implies (11) and (12). Thus Lemma 1 is proved.

Lemma 2 *The solution $\psi_{i,\varepsilon,\rho}$ of the boundary problem (7)–(9) satisfies*

$$\psi_{i,\varepsilon,\rho}(x,t) \le M_2 e^{-|x|} \tag{13}$$

for all $(x,t) \in B_R \times (0,T)$, where M_2 is a positive constant depending only on T and the L^∞-norm of u_i and v_i.

Proof We consider the following functions

$$w_i^{\pm}(x,t) = \mp\psi_{i,\varepsilon,\rho}(x,t) + e^{1-|x|+\gamma(T-t)}, \quad i = 1,2$$

where $\gamma > 0$ will be determined later.

From (7)–(9) we have

$$w_i^{\pm}(x,t) \ge 0$$

on $|x| = 1$ and $|x| = R$ and

$$w_i^{\pm}(x,T) = \mp\theta_i e^{-|x|} + e^{1-|x|+\gamma(T-T)} = (\mp\theta_i + e)e^{-|x|} \ge 0$$

and

$$\frac{\partial}{\partial t}\left(w_i^{\pm}\right) + A_{i,\varepsilon,\rho}\Delta w_i^{\pm} = \frac{\partial}{\partial t}e^{1-|x|+\gamma(T-t)} + A_{i,\varepsilon,\rho}\Delta e^{1-|x|+\gamma(T-t)}$$
$$= e^{1-|x|+\gamma(T-t)}\left\{A_{i,\varepsilon,\rho} - (N-1)A_{i,\varepsilon,\rho}|x|^{-1} - \gamma\right\}$$

Therefore, we can choose γ depending only on L^∞-norm of u_i and v_i such that

$$\frac{\partial}{\partial t}w_i^{\pm} + A_{i,\varepsilon,\rho}\Delta w_i^{\pm} < 0$$

for all $(x,t) \in (B_R \backslash B_1) \times (0,T)$. Applying the comparison principle, we have

$$w_i^{\pm}(x,t) \ge 0$$

for $(x,t) \in (B_R \backslash B_1) \times (0,T)$.

Namely

$$\mp\psi_{i,\varepsilon,\rho}(x,t) + e^{1-|x|+\gamma(T-t)} \ge 0$$

which implies (13).

Lemma 3 *There exists a positive constant M_3 independent of ε, ρ and R such that*

$$|\nabla \psi_{i,\varepsilon,\rho}| \leq M_3 e^{-R} \tag{14}$$

on $\partial B_R \times (0,T)$.

Proof We consider the following functions

$$z_i^{\pm}(x,t) = \mp\psi_{i,\varepsilon,\rho}(x,t) + K_1 e^{-R}[e^{K_2(|x|-R)} - 1]$$

for all $(x,t) \in B_R \times (0,T)$. Clearly, we have

$$z_i^{\pm}(x,t) = 0$$

for $(x,t) \in \partial B_R \times (0,T)$ and

$$z_i^{\pm}(x,T) = \mp\theta_i e^{-|x|} + K_1 e^{-R}[e^{K_2(|x|-R)} - 1] < 0$$

for $x \in B_R \backslash B_{R-1}$.

Using Lemma 2, we can choose K_1 and K_2 large enough such that

$$z_i^{\pm}(x,t) = \mp\psi_{i,\varepsilon,\rho}(x,t) + K_1 e^{-R}[e^{-K_2} - 1] < 0$$

for $|x| = R-1$.

Clearly,

$$\frac{\partial}{\partial t} z_i^{\pm} + A_{i,\varepsilon,\rho}\Delta z_i^{\pm} = K_1 e^{-R} \cdot e^{K_2(|x|-R)} A_{i,\varepsilon,\rho}\left[K_2^2 + K_2 \frac{N-1}{|x|}\right] > 0$$

Therefore, by maximum principle, we have

$$z_i^{\pm}(x,t) \leq 0$$

for $(x,t) \in (B_R \backslash B_{R-1}) \times (0,T)$, and

$$\frac{\partial z_i^{\pm}}{\partial \nu} \geq 0$$

on $\partial B_R \times (0,T)$, where ν is the outward normal to ∂B_R. This implies

$$\mp \frac{\partial}{\partial \nu}\psi_{i,\varepsilon,\rho} \geq -K_1 K_2 e^{-R}$$

on $\partial B_R \times (0,T)$, and (14) is proved.

Lemma 4 *Let (u_1, u_2) be a generalized solution of the Cauchy problem (1)–(2). Then for a.e. s,t with $0 < s < t < T$ and all $\varphi_i \in C^\infty(0,T; C_0^\infty(R^N))$ $(i=1,2)$ we have*

$$\int_{R^N} \varphi_i(x,t)u_i(x,t)dx - \int_{R^N} \varphi_i(x,s)u_i(x,s)dx$$
$$= \int_s^t \int_{R^N} \left(u_i \frac{\partial \varphi_i}{\partial t} + a_i u_i^{m_i}\Delta\varphi_i + b_i u_i^{p_i} u_{\bar{i}}^{q_i}\varphi_i\right) dxdt, \quad i = 1,2$$

The proof is similar to that given in [4].

Proof of Theorem We choose $\eta_\alpha \in C_0^\infty(B_R)$ such that

$$0 \le \eta_\alpha \le 1 \ in\ B_R;\ \eta_\alpha = 1\ in\ B_{R-\alpha}$$
$$|\nabla \eta_\alpha| \le \frac{\beta}{\alpha};\ |\Delta \eta_\alpha| \le \frac{\beta}{\alpha^2}$$

where $0 < \alpha < R$ and β is a positive constant independent of R and α.

Applying Lemma 4 we have

$$\begin{aligned}\int_{B_R} \eta_\alpha \psi_{i,\varepsilon,\rho}(u_i - v_i)(x,t)dx = &\int_{B_R} \eta_\alpha \psi_{i,\varepsilon,\rho}(u_i - v_i)(x,s)dx \\ &+ \int_s^t \int_{B_R} \eta_\alpha \frac{\partial}{\partial t}\psi_{i,\varepsilon,\rho}(u_i - v_i)(x,\lambda)dxd\lambda \\ &+ \int_s^t \int_{B_R} a_i\left(u_i^{m_i} - v_i^{m_i}\right)\Delta(\eta_\alpha \psi_{i,\varepsilon,\rho})(x,\lambda)dxd\lambda \\ &- \int_s^t \int_{B_R} b_i\left(u_1^{p_i}u_2^{q_i} - v_1^{p_i}v_2^{q_i}\right)(\eta_\alpha \psi_{i,\varepsilon,\rho})(x,\lambda)dxd\lambda\end{aligned}$$

for a.e. s,t with $0 < s < t < T$, where $\psi_{i,\varepsilon,\rho}$ is a solution of the boundary value problem (7)–(9) with $T = t$.

Let $s \to 0$. Then we obtain

$$\begin{aligned}&\int_{B_R} \eta_\alpha \psi_{i,\varepsilon,\rho}(u_i - v_i)(x,t)dx \\ &= \int_0^t \int_{B_R} \eta_\alpha \left[a_i\left(u_i^{m_i} - v_i^{m_i}\right) - A_{i,\varepsilon,\rho}(u_i - v_i)\right]\Delta \psi_{i,\varepsilon,\rho}(x,\lambda)dxd\lambda \\ &\quad - \int_0^t \int_{B_R} b_i\left(u_1^{p_i}u_2^{q_i} - v_1^{p_i}v_2^{q_i}\right)(\eta_\alpha \psi_{i,\varepsilon,\rho})(x,\lambda)dxd\lambda \\ &\quad + \int_0^t \int_{B_R} a_i\left(u_i^{m_i} - v_i^{m_i}\right)\{2\nabla \eta_\alpha \cdot \nabla \psi_{i,\varepsilon,\rho} + \psi_{i,\varepsilon,\rho}\Delta \eta_\alpha\}(x,\lambda)dxd\lambda \\ &\overset{\text{def}}{=} I_1 + I_2 + I_3 \qquad (15)\end{aligned}$$

We compute

$$\begin{aligned}I_1 = &\int_0^t \int_{B_R} \eta_\alpha \left[a_i\left(u_i^{m_i} - v_i^{m_i}\right) - A_{i,\varepsilon,\rho}(u_i - v_i)\right]\Delta \psi_{i,\varepsilon,\rho}(x,\lambda)dxd\lambda \\ = &\int_0^t \int_{B_R} \eta_\alpha (u_i - v_i)(A_{i\varepsilon} - A_{i,\varepsilon,\rho})\Delta \psi_{i,\varepsilon,\rho}(x,\lambda)dxd\lambda \\ &- \varepsilon \int_0^t \int_{B_R} \eta_\alpha (u_i - v_i)\Delta \psi_{i,\varepsilon,\rho}(x,\lambda)dxd\lambda \\ \le &\left\{\int_0^t \int_{B_R} [\eta_\alpha (u_i - v_i)\Delta \psi_{i,\varepsilon,\rho}(x,\lambda)]^2 dxd\lambda\right\}^{\frac12} \cdot \left\{\int_0^t \int_{B_R} |A_{i\varepsilon} - A_{i,\varepsilon,\rho}|^2(x,\lambda)dxd\lambda\right\}^{\frac12}\end{aligned}$$

$$+\varepsilon\left\{\int_0^t\int_{B_R}[\eta_\alpha(u_i-v_i)]^2(x,\lambda)dxd\lambda\right\}^{\frac{1}{2}}\cdot\left\{\int_0^t\int_{B_R}[\Delta\psi_{i,\varepsilon,\rho}(x,\lambda)]^2dxd\lambda\right\}^{\frac{1}{2}}$$

Using Lemma 1 and noting the definitions of $A_{i,\varepsilon,\rho}$ and $A_{i,\varepsilon}$, we conclude

$$|I_1|\le C\varepsilon^{-\frac{1}{2}}\left\{\int_0^t\int_{B_R}|A_{i,\varepsilon}-A_{i,\varepsilon,\rho}|^2(x,\lambda)dxd\lambda\right\}^{\frac{1}{2}}+C\varepsilon^{\frac{1}{2}}$$

where C is a positive constant independent of ε and ρ.

Let $\rho\to 0$ and $\varepsilon\to 0$. Then we have

$$|I_1|\to 0 \tag{16}$$

Noting that $p_i\ge 1, q_i\ge 1$ $(i=1,2)$ and Lemma 2, we obtain

$$\begin{aligned}|I_2|&=\left|-\int_0^t\int_{B_R}b_i\,(u_1^{p_i}u_2^{q_i}-v_1^{p_i}v_2^{q_i})\,(\eta_\alpha\psi_{i,\varepsilon,\rho})(x,\lambda)dxd\lambda\right|\\&\le C\int_0^t\int_{B_R}(|u_1-v_1|+|u_2-v_2|)e^{-|x|}dxd\lambda\end{aligned}\tag{17}$$

where C is a positive constant depending only on p_i, q_i, b_i and L^∞-norms of u_i and v_i $(i=1,2)$.

Now, we discuss I_3.

$$\begin{aligned}I_3&=\int_0^t\int_{B_R}a_i\,(u_i^{m_i}-v_i^{m_i})\,\{2\nabla\eta_\alpha\cdot\nabla\psi_{i,\varepsilon,\rho}+\psi_{i,\varepsilon,\rho}\Delta\eta_\alpha\}(x,\lambda)dxd\lambda\\&=\int_0^t\int_{B_R}a_i\,(u_i^{m_i}-v_i^{m_i})\,\{2\nabla\eta_\alpha\cdot\nabla\psi_{i,\varepsilon,\rho}\}(x,\lambda)dxd\lambda\\&\quad+\int_0^t\int_{B_R}a_i\,(u_i^{m_i}-v_i^{m_i})\,\psi_{i,\varepsilon,\rho}\Delta\eta_\alpha(x,\lambda)dxd\lambda\\&\overset{\text{def}}{=}I_{31}+I_{32}\end{aligned}\tag{18}$$

From the definitions of $\psi_{i,\varepsilon,\rho}$ and η_α we have

$$\begin{aligned}|I_{31}|&=2\left|\int_0^t\int_{B_R}a_i\,(u_i^{m_i}-v_i^{m_i})\,\{\nabla\eta_\alpha\cdot\nabla\psi_{i,\varepsilon,\rho}\}(x,\lambda)dxd\lambda\right|\\&\le C\alpha^{-1}\int_0^t\int_{B_R\setminus B_{R-\alpha}}|\nabla\psi_{i,\varepsilon,\rho}|(x,\lambda)dxd\lambda\\&\le CR^{N-1}\sup_{B_R\setminus B_{R-\alpha}}|\nabla\psi_{i,\varepsilon,\rho}|\end{aligned}$$

It follows from Lemma 3 that

$$\lim_{\alpha\to 0}|I_{31}|\le CR^{N-1}e^{-R} \tag{19}$$

where C is a positive constant independent of R,ε and ρ.

On the other hand, we have

$$|I_{32}| = \left|\int_0^t \int_{B_R} a_i\left(u_i^{m_i} - v_i^{m_i}\right)\psi_{i,\varepsilon,\rho}\Delta\eta_\alpha dx d\lambda\right|$$
$$\le C\alpha^{-2}\int_0^t \int_{B_R\setminus B_{R-\alpha}} |\psi_{i,\varepsilon,\rho}| dx d\lambda$$
$$\le C\alpha^{-1}R^{N-1}\sup_{B_R\setminus B_{R-\alpha}} |\psi_{i,\varepsilon,\rho}|$$

It follows from Lemma 3 that

$$\lim_{\alpha\to 0} |I_{32}| \le CR^{N-1}e^{-R} \tag{20}$$

where C is a positive constant independent of R, ε and ρ.

Let $\alpha \to 0$. Then, it follows from (15)–(20) that

$$\int_{B_R} \theta_i(x)e^{-|x|}(u_i - v_i)(x,t)dx$$
$$\le |I_1| + CR^{N-1}e^{-R} + C\int_0^t \int_{B_R}(|u_1 - v_1| + |u_2 - v_2|)e^{-|x|}dx d\lambda$$

where C is a positive constant independent of R, ε and ρ.

Let $\rho \to 0, \varepsilon \to 0, \alpha \to 0$ and $R \to +\infty$. Then we have

$$\int_{R^N} \theta_i(x)e^{-|x|}(u_i - v_i)(x,t)dx \le C\int_0^t \int_{R^N}(|u_1 - v_1| + |u_2 - v_2|)e^{-|x|}dx d\lambda$$

for a.e. $t \in (0,T)$ and all $\theta_i(x) \in C_0^\infty(R^N)$ with $|\theta_i| \le 1$ $(i = 1,2)$, where C is a positive constant independent of t and θ_i $(i = 1,2)$.

This implies

$$\int_{R^N}(|u_1(x,t) - v_1(x,t)| + |u_2(x,t) - v_2(x,t)|)e^{-|x|}dx$$
$$\le 2C\int_0^t \int_{R^N}(|u_1(x,\lambda) - v_1(x,\lambda)| + |u_2(x,\lambda) - v_2(x,\lambda)|)e^{-|x|}dx d\lambda$$

Using the Gronwall inequality, we obtain

$$\int_{R^N}(|u_1(x,t) - v_1(x,t)| + |u_2(x,t) - v_2(x,t)|)e^{-|x|}dx = 0$$

for a.e. $t \in (0,T)$.

Therefore, we have

$$u_i = v_i,\ i = 1,2$$

a.e. in Q_T. Thus the Theorem is proved.

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