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FORCED OSCILLATION FOR CERTAIN NONLINEAR DELAY PARABOLIC EQUATIONS*

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Abstract In this paper we investigate the forced oscillation of the solution of a class of nonlinear parabolic equations with continuous distributed deviating arguments.
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In this paper we consider the following nonlinear parabolic equation with continuous distributed deviating arguments

$$\text{(E)} \qquad u_t = a(t)\Delta u - \int_a^b q(x,t,\xi)F[u(x,g(t,\xi))]d\sigma(\xi) + h(x,t),\ (x,t)\in\Omega\times R_+$$

where Ω is a bounded domain in R^n with piecewise smooth boundary $\partial\Omega$; $R_+ = [0,+\infty)$, $u=u(x,t)$, Δ is the Laplacian in R^n; $a(t)\in C(R_+,R_+)$, $q(x,t,\xi)\in C(\overline{\Omega}\times R_+\times[a,b],R_+)$, $F(u)\in C(R,R)$; $g(t,\xi)\in C(R_+\times[a,b],R)$, $g(t,\xi)\le t$, $\xi\in[a,b]$; $g(t,\xi)$ is a nondecreasing function with respect to t and ξ, respectively; and $\lim\limits_{t\to+\infty}\min\limits_{\xi\in[a,b]}\{g(t,\xi)\}=+\infty$; $\sigma(\xi)\in([a,b],R)$ is nondecreasing in ξ; the forcing term $h(x,t)\in C(\overline{\Omega}\times R_+,R)$; the integral in (E) is Stieltjes integral.

We consider three kinds of boundary conditions:

$$\text{(B}_1\text{)} \qquad u=\varphi,\quad (x,t)\in\partial\Omega\times R_+$$

$$\text{(B}_2\text{)} \qquad \frac{\partial u}{\partial N}=\psi,\quad (x,t)\in\partial\Omega\times R_+$$

$$\text{(B}_3\text{)} \qquad \frac{\partial u}{\partial N}+\mu u=0,\quad (x,t)\in\partial\Omega\times R_+$$

where N is the unit exterior normal vector to $\partial\Omega$, φ and ψ are continuous functions on $\partial\Omega\times R_+$, and μ is a nonnegative continuous function on $\partial\Omega\times R_+$.

Some papers have been published concerning the oscillation theory of certain classes of delay parabolic equations. We mention here the work [1]–[4] and their references.

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Concerning forced oscillation of delay parabolic equations, only the work of N. Yoshida [4] is known. The case with discrete distributed deviating arguments all have been considered in those papers. However, it seems that very little is known about the work of the case with continuous distributed deviating arguments. We know only about the works of certain classes of ordinary differential equations with continuous distributed deviating arguments, e.g. see [5], [6]. In this paper we discuss the forced oscillation of the solution of the partial differential equation (E) with continuous distributed deviating arguments. Some oscillatory criteria are obtained for Equation (E) satisfying (B_1), (B_2) and (B_3), respectively.

Definition *The solution $u(x,t)$ of Equation (E) satisfying certain boundary condition is called oscillating in the domain $\Omega \times R_+$ if for each positive number τ there exists a point $(x_0, t_0) \in \Omega \times [\tau, +\infty)$ such that the condition $u(x_0, t_0) = 0$ holds.*

The following fact will be used:

The smallest eigenvalue α_0 of the Dirichlet problem

$$\begin{cases} \Delta u + \alpha u = 0 & \text{in } \Omega \\ u \mid_{\partial\Omega} = 0 \end{cases}$$

is positive and the corresponding eigenfunction $\Phi(x)$ is positive in Ω.

Lemma 1 *Let the following condition hold:*

(H_1) *$F(u)$ is a positive and convex function in the segment $(0, +\infty)$.*

If $u(x,t)$ is a positive solution of the problem (E), (B_1) in the domain $\Omega \times [\tau, +\infty)$, $\tau \geq 0$, then the function

$$X(t) = \left[\int_\Omega \Phi(x)dx\right]^{-1} \cdot \int_\Omega u(x,t)\Phi(x)dx \tag{1}$$

satisfies the inequality

$$(\mathrm{I}_1) \qquad X'(t) + \alpha_0 a(t)X(t) + \int_a^b Q(t,\xi)F[X(g(t,\xi))]d\sigma(\xi) \leq H(t)$$

where $Q(t,\xi) = \min\limits_{x\in\overline{\Omega}} q(x,t,\xi)$,

$$H(t) = \left[\int_\Omega \Phi(x)dx\right]^{-1} \cdot \left[-a(t)\int_{\partial\Omega} \varphi \frac{\partial \Phi}{\partial N} dS + \int_\Omega h(x,t)\Phi(x)dx\right] \tag{2}$$

dS is an areal element of $\partial\Omega$.

Proof Suppose that $u(x,t)$ is a positive solution of the problem (E), (B_1) in $\Omega \times [\tau, +\infty)$, $\tau \geq 0$. Note that

$$\lim_{t\to+\infty} \min_{\xi\in[a,b]} \{g(t,\xi)\} = +\infty$$

so there exists a $t_1 \geq \tau$ such that

$$u(x, g(t,\xi)) > 0, \quad t \geq t_1, \ \xi \in [a,b]$$

Multiplying both sides of Equation (E) by the eigenfunction $\Phi(x)$, integrating with respect to x over the domain Ω and using the formula for differentiating under the integral, we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega u\Phi(x)dx =& a(t)\int_\Omega \Delta u\Phi(x)dx - \int_\Omega\int_a^b q(x,t,\xi)F[u(x,g(t,\xi))]\Phi(x)d\sigma(\xi)dx\\ &+\int_\Omega h(x,t)\Phi(x)dx, \quad t\geq t_1 \end{aligned}\tag{3}$$

Using Green's formula, we have

$$\begin{aligned}\int_\Omega \Delta u\Phi(x)dx &= \int_{\partial\Omega}\Phi\frac{\partial u}{\partial N}dS - \int_{\partial\Omega}u\frac{\partial \Phi}{\partial N}dS + \int_\Omega u\Delta\Phi dx\\ &= -\int_{\partial\Omega}\varphi\frac{\partial\Phi}{\partial N}dS - \alpha_0\int_\Omega u\Phi dx, \quad t\geq t_1\end{aligned}\tag{4}$$

From (H_1) and Jensen's inequality, we obtain

$$\begin{aligned}&\int_a^b\int_\Omega F[u(x,g(t,\xi))]\Phi(x)dxd\sigma(\xi)\\ &\geq \int_a^b\left[\int_\Omega\Phi(x)dx\cdot F\left(\frac{\int_\Omega u(x,g(t,\xi))\Phi(x)dx}{\int_\Omega\Phi(x)dx}\right)\right]d\sigma(\xi), \quad t\geq t_1\end{aligned}\tag{5}$$

Combining (3)–(5), with (2) we see that the function $X(t)$ defined by (1) satisfies the inequality (I_1).

The proof of Lemma 1 is complete.

Theorem 1 *Assume that* (H_1) *holds, and that:*

(H_2) $\qquad F(-u) = -F(u) \quad for\ u\in(0,+\infty)$

If the functional differential inequalities

(I_1) $$X'(t)+\alpha_0 a(t)X(t)+\int_a^b Q(t,\xi)F[X(g(t,\xi))]d\sigma(\xi)\leq H(t)$$

and

(I_2) $$X'(t)+\alpha_0 a(t)X(t)+\int_a^b Q(t,\xi)F[X(g(t,\xi))]d\sigma(\xi)\leq -H(t)$$

have no eventually positive solution, then every solution of the problem (E), (B_1) *is oscillatory in* $\Omega\times R_+$.

Proof Suppose that $u(x,t)$ is a nonoscillatory solution of the problem (E), (B_1). If $u(x,t)>0$, $(x,t)\in\Omega\times[\tau,+\infty)$, for some $\tau\geq 0$, then from Lemma 1 it follows that the function defined by (1) is an eventually positive solution of the inequality (I_1), which contradicts the condition of the theorem.

If $u(x,t) < 0, (x,t) \in \Omega \times [\tau, +\infty)$, then set

$$u^*(x,t) = -u(x,t), \quad (x,t) \in \Omega \times [\tau, +\infty)$$

By (H_2), it is easy to check that $u^*(x,t)$ is a positive solution of the problem

$$\begin{cases} u_t = a(t)\Delta u - \int_a^b q(x,t,\xi)F[u(x,g(t,\xi))]d\sigma(\xi) - h(x,t), & (x,t) \in \Omega \times R_+ \\ u(x,t) = -\varphi(x,t), \quad (x,t) \in \partial\Omega \times R_+ \end{cases}$$

From Lemma 1 it follows that

$$X^*(t) = \left[\int_\Omega \Phi(x)dx\right]^{-1} \int_\Omega u^*(x,t)\Phi(x)dx$$

is an eventually positive solution of the inequality (I_2), which contradicts the conditions of the theorem as well. This completes the proof of Theorem 1.

Remark 1 N. Yoshida [1], D.P. Mishev and D.D. Bainov [2] have considered boundary condition of the type

$$(B_1^*) \qquad u = 0, \quad (x,t) \in \partial\Omega \times R_+$$

but have not considered the boundary condition (B_1). (B_1^*) is *a* special case of (B_1).

Theorem 2 *Assume that* (H_1) *and* (H_2) *hold. If*

$$\liminf_{t\to+\infty} \int_A^t \left[-a(y)\int_{\partial\Omega} \varphi\frac{\partial\Phi}{\partial N}dS + \int_\Omega h(x,y)\Phi(x)dx\right]dy = -\infty \tag{6}$$

$$\limsup_{t\to+\infty} \int_A^t \left[-a(y)\int_{\partial\Omega} \varphi\frac{\partial\Phi}{\partial N}dS + \int_\Omega h(x,y)\Phi(x)dx\right]dy = +\infty \tag{7}$$

for every sufficiently large number $A > 0$, *then every solution of the problem* (E), (B_1) *is oscillatory in* $\Omega \times R_+$.

Proof By Theorem 1 we shall show that the inequalities (I_1) and (I_2) have no eventually positive solution. Suppose that $X(t)$ is an eventually positive solution of the inequality (I_1). Then there exists a $t_1 > 0$ such that

$$X(t) > 0, \quad X(g(t,\xi)) > 0, \quad t \geq t_1, \quad \xi \in [a,b]$$

From (I_1) it follows that

$$X'(t) \leq \left[\int_\Omega \Phi(x)dx\right]^{-1} \cdot \left[-a(t)\int_{\partial\Omega} \varphi\frac{\partial\Phi}{\partial N}dS + \int_\Omega h(x,y)\Phi(x)dx\right], \quad t \geq t_1$$

Integrating both sides of the above last inequality from t_1 to t, we get

$$X(t) - X(t_1) \leq \left[\int_\Omega \Phi(x)dx\right]^{-1} \int_{t_1}^t \left[-a(y)\int_{\partial\Omega} \varphi\frac{\partial\Phi}{\partial N}dS + \int_\Omega h(x,y)\Phi(x)dx\right]dy, \quad t \geq t_1$$

Using (6), we have

$$\liminf_{t\to+\infty} X(t) = -\infty$$

Thus we can see that $X(t)$ has no lower bounds, which contradicts the fact that $X(t) > 0$ for all $t \geq t_1$.

Because of (7) it's easy to see that

$$\liminf_{t\to+\infty} \int_A^t \left[a(y)\int_{\partial\Omega} \varphi \frac{\partial \Phi}{\partial N} dS - \int_\Omega h(x,y)\Phi(x)dx\right] dy$$
$$= -\limsup_{t\to+\infty} \int_A^t \left[-a(y)\int_{\partial\Omega} \varphi \frac{\partial \Phi}{\partial N} dS + \int_\Omega h(x,y)\Phi(x)dx\right] dy = -\infty$$

so we can prove that the inequality (I_2) has no eventually positive solution by the analogous arguments as in the above proof. The proof of Theorem 2 is complete.

Setting $\varphi \equiv 0$ in Theorem 2, then we can obtain the following corollary.

Corollary 1 *Assume that* (H_1) *and* (H_2) *hold. If*

$$\liminf_{t\to+\infty} \int_A^t \int_\Omega h(x,y)\Phi(x)dxdy = -\infty \tag{8}$$

$$\limsup_{t\to+\infty} \int_A^t \int_\Omega h(x,y)\Phi(x)dxdy = +\infty \tag{9}$$

for every sufficiently large number $A > 0$, then every solution of the problem (E), (B_1^*) *is oscillatory in $\Omega \times R_+$.*

Theorem 3 *Assume that* (H_1) *and* (H_2) *hold. If the functional differential inequalities*

(I_3)
$$Y'(t) + \int_a^b Q(t,\xi)F[Y(g,(t,\xi))]d\sigma(\xi) \leq G(t)$$

and

(I_4)
$$Y'(t) + \int_a^b Q(t,\xi)F[Y(g,(t,\xi))]d\sigma(\xi) \leq -G(t)$$

have no eventually positive solution, then every solution of the problem (E), (B_2) *is oscillatory in $\Omega \times R_+$, where*

$$G(t) = |\Omega|^{-1}\left[a(t)\int_{\partial\Omega} \psi dS + \int_\Omega h(x,t)dx\right], \quad |\Omega| = \int_\Omega dx$$

The proof of Theorem 3 is similar to that of Theorem 1 and hence is omitted.

Theorem 4 *Assume that* (H_1) *and* (H_2) *hold. If*

$$\liminf_{t\to+\infty} \int_A^t \left[a(y)\int_{\partial\Omega} \psi(x,y)dS + \int_\Omega h(x,y)dx\right] dy = -\infty \tag{10}$$

$$\limsup_{t\to+\infty} \int_A^t \left[a(y)\int_{\partial\Omega} \psi(x,y)dS + \int_\Omega h(x,y)dx\right] dy = +\infty \tag{11}$$

for every sufficiently large number $A > 0$, then every solution of the problem (E), (B_2) *is oscillatory in $\Omega \times R_+$.*

The proof of Theorem 4 is similar to that of Theorem 2.

Remark 2 N. Yoshida [1], D.P. Mishev and D. D. Bainov [2] have not considered the boundary condition (B_2).

Theorem 5 *Assume that* (H_1) *and* (H_2) *hold. If the functional differential inequalities*

$$(I_5) \qquad Z'(t) + \int_a^b Q(t,\xi)F[Z(g(t,\xi))]d\sigma(\xi) \le |\Omega|^{-1}\int_\Omega h(x,t)dx$$

$$(I_6) \qquad Z'(t) + \int_a^b Q(t,\xi)F[Z(g(t,\xi))]d\sigma(\xi) \le -|\Omega|^{-1}\int_\Omega h(x,t)dx$$

have no eventually positive solution, then every solution of the problem (E), (B_3) *is oscillatory in $\Omega \times R_+$.*

We can prove Theorem 5 by the analogous arguments as in the proof of Theorem 1.

Theorem 6 *Assume that* (H_1) *and* (H_2) *hold. If*

$$\liminf_{t\to+\infty} \int_A^t \int_\Omega h(x,y)dxdy = -\infty \tag{12}$$

$$\limsup_{t\to+\infty} \int_A^t \int_\Omega h(x,y)dxdy = +\infty \tag{13}$$

for every sufficiently large number $A > 0$, then every solution of the problem (E), (B_3) is oscillatory in $\Omega \times R_+$.

The proof of Theorem 6 is similar to that of Theorem 2.

Example Consider the equation

$$u_t = u_{xx} - 2\int_{-\pi}^0 u(x,t+\xi)d\xi + e^t \cos x\,[(3+e^{-\pi})\sin t - e^{-\pi}\cos t],$$
$$(x,t) \in \left(0, \frac{\pi}{2}\right) \times R_+ \tag{14}$$

and a boundary condition of the type (B_2):

$$u_x(0,t) = 0, \quad u_x\left(\frac{\pi}{2}, t\right) = -e^t \sin t, \quad t \in R_+ \tag{15}$$

Here $n = 1, \Omega = \left(0, \frac{\pi}{2}\right)$, $a(t) = 1$, $q(x,t,\xi) = 2$, $F(u) = u$, $g(t,\xi) = t + \xi$, $h(x,t) = e^t \cos x\,[(3+e^{-\pi})\sin t - e^{-\pi}\cos t]$, $\int_{\partial\Omega} \psi dS = -e^t \sin t$. It is easy to see that

$$\int_{t_0}^t \left[a(y)\int_{\partial\Omega} \psi(x,y)dS + \int_\Omega h(x,y)dx\right]dy$$
$$= \int_{t_0}^t \left[-e^y \sin y + e^y((3+e^{-\pi})\sin y - e^{-\pi}\cos y)\right]dy$$

$$= e^t[\sin t - (1 + e^{-\pi}) \cos t] - e^{t_0}[\sin t_0 - (1 + e^{-\pi}) \cos t_0]$$

Hence all the conditions of Theorem 4 are fulfilled. Then, from Theorem 4 it follows that every solution of the problem (14), (15) is oscillatory in $\left(0, \frac{\pi}{2}\right) \times R_+$. In fact, $u(x,t) = e^t \sin t \cos x$ is such a solution.

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