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THE INERTIAL FRACTAL SETS FOR NONLINEAR SCHRÖDINGER EQUATIONS*

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Abstract The existence of inertial fractal sets for weakly dissipative Schrödinger equations which possess (E_0, E) type compact attractor is proved. The estimates of the upper bounds of fractal dimension of inertial fractal set are also obtained.

Key Words Schrödinger inertial fractal set.
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1. Introduction

In the study of the inertial manifold of the 2D Navier-Stokes equations (NSE) representing turbulent flows, one finds out that [2] since there exist spectral barriers and spectral gap conditions, the existence of an inertial manifold for 2D NSE is still a mystery. Recently, Eden et al.[3] have studied and discovered that some dissipative evolution equations with real coefficients, for which the (E, E) type compact attractors exist, including 2D NSE, have a kind of set similar to inertial manifold-inertial set. This paper advances the previous results to complex weakly infinite dimensional dynamical system that only possesses (E_0, E) type compact attractors.

2. Main Results

Let $D(A)$, V be two Hilbert spaces, $D(A)$ be dense in V and compactly imbedded into V.

We study

$$\frac{du}{dt} + Au + g(u) = f(x), \quad t > 0, \ x \in \Omega \tag{1}$$

$$u(0) = u_0 \tag{2}$$

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$$u\,|_{\partial\Omega}=0 \tag{3}$$

where Ω is a bounded open set in R^n, $\partial\Omega$ is smooth. A is a positive self adjoint operator with a compact inverse. Let $\{w_n, n=1,2,\cdots\}$ denote the complete set of eigenvectors of A, the corresponding eigenvalues are

$$0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots\nearrow+\infty \tag{4}$$

We assume that the nonlinear semigroup $S(t)$ defined in (1)–(3) possesses a $(D(A), V)$ type compact attractor, namely, there exists a compact $\mathcal{A}$ in V, $\mathcal{A}$ attracts all bounded subsets in $D(A)$ and it is invariant under the action of $S(t)$.

Definition 1 *A compact set M in V is called an inertial fractal set of $(D(A),V)$ type for $(S(t),B)$ if $\mathcal{A}\subseteq M\subseteq B$ and*

1. $S(t)M\subseteq M,\ \forall t\geq 0,$
2. *M has finite fractal dimension, $d_F(M)<\infty$,*
3. *there exist positive constants c_0, c_1 such that*

$$\operatorname{dist}_V(S(t)B,M)\leq c_0e^{-c_1t},\quad \forall t>0$$

where $\operatorname{dist}_V(A,B)=\sup\limits_{x\in A}\inf\limits_{y\in B}|x-y|_V$, B is a positively invariant set for $S(t)$ in V.

Definition 2[3] *If for every $\delta\in\left(0,\dfrac{1}{8}\right)$, there exists an orthogonal projection P_{N_0} of rank equal to N_0 such that for every u and v in B, either*

$$|S(t_*)u-S(t_*)v|_V\leq\delta|u-v|_V \tag{5}$$

or

$$|Q_{N_0}(S(t_*)u-S(t_*)v)|_V\leq|P_{N_0}(S(t_*)u-S(t_*)v)|_V \tag{6}$$

Then we call $S(t)$ is squeezing in B, where $Q_{N_0}=I-P_{N_0}$.

Theorem 1 *Suppose (1)–(3) satisfies the following conditions*

1. *there exists a $(D(A),V)$ type compact attractor $\mathcal{A}$.*
2. *there exists a compact set B in V which is positively invariant for $S(t)$.*
3. *$S(t)$ is squeezing and Lipschitz continuous, that is there exists a bounded function $l(t)$ such that $|S(t)u-S(t)v\,|_V\leq l(t)|u-u|_V$ for every u,v in B.*

Then (1)–(3) admits a $(D(A),V)$ type inertial fractal set M for $(S(t),B)$ and

$$M=\bigcup_{0\leq t\leq t_*}S(t)M_* \tag{7}$$

where

$$M_*=\mathcal{A}\cup\left(\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty}S(t_*)^j\left(E^{(k)}\right)\right) \tag{8}$$

Moreover,

$$d_F(M) \le 1 + N_0 \log\left(1+\sqrt{2}l/\delta\right)/\log\frac{1}{\theta} \tag{9}$$

$$\mathrm{dist}_V(S(t)B, M) \le c_0 e^{-c_1 t} \tag{10}$$

where $\theta, N_0, E^{(k)}$ are defined as in [3], *l is the Lipschitz constant for $S(t_*)$ in B. t_* is a positive constant.*

Proof We utilize that B is compact in V and it is positive invariance for $S(t)$, with B instead of X in [3], note also that $S(t)$ is Lipschitz continuous and has a squeezing property in B, then this theorem is proved by the same manner of the proof of Theorem 1 in [3].

Proposition 1 *Suppose that problem* (1)–(3) *possesses a unique global solution, $u \in C_w(R_+, D(A))$, if $u_0 \in D(A)$; $u \in C(R_+, V)$ if $u_0 \in V$. Moreover there exist closed absorbing sets B_0, B_1 in $D(A)$, V respectively. Then nonlinear semigroup $S(t)$ defined by problem* (1)–(3) *possesses a $(D(A), V)$ type compact attractor*

$$\mathcal{A} = \overline{\bigcap_{t\ge 0} S(t)B_0} \tag{11}$$

$\mathcal{A}$ is bounded in $D(A)$. Notation C_w denotes the class of function which is weakly continuous with respect to t in topology of $D(A)$.

Proof From [5] we know $\mathcal{A}$ is a weakly compact attractor in $D(A)$ and attracts all bounded set in $D(A)$ with respect to weak topology in $D(A)$. Since weakly bounded property is equivalent to strongly bounded property in Hilbert space, we deduce that $\mathcal{A}$ is bounded set in $D(A)$, by the compact imbedding of $D(A)$ into V we know $\mathcal{A}$ is compact in V. Suppose B is any bounded set in $D(A)$, since $\mathcal{A}$ attracts every bounded set in $D(A)$ with respect to weak topology in $D(A)$, we can extract a sequence $S(t_n)B$, which weakly converges to $\mathcal{A}$ in $D(A)$ as $t_n \to +\infty$, by the compact imbedding of $D(A)$ into V we know $S(t_n)B$ strongly converges to $\mathcal{A}$ in V, this shows that $\mathcal{A}$ is a $(D(A), V)$ type attractor.

Proposition 2 *There exists $t_0(B_0)$ such that*

$$B = \bigcup_{0\le t\le t_0(B_0)} S(t)B_0 \tag{12}$$

is a compact, positively invariant set in V and is absorbing set for all bounded subset in $D(A)$.

Proof By the definition of B_0 we know that, there exists $t_0(B_0)$ such that

$$S(t)B_0 \subseteq B_0 \quad \text{for} \quad t \ge t_0(B_0)$$

It is easy to check that B defined in (12) satisfies that result of this proposition. Indeed, denoting $t = kt_0(B_0) + t_1$, $0 \le t_1 \le t_0(B_0)$, we have

$$S(t)B = \overline{\bigcup_{t_1\le s\le t_0(B_0)} S(s)S(kt_0(B_0))B_0} \bigcup \overline{\bigcup_{0\le s\le t_1} S(s)S((k+1)t_0(B_0))B_0}$$

$$\subseteq \overline{\bigcup_{t_1\le s\le t_0(B_0)} S(s)B_0} \bigcup \overline{\bigcup_{0\le s\le t_1} S(s)B_0} \subseteq \overline{\bigcup_{0\le s\le t_0(B_0)} S(s)B_0} = B$$

Since $D(A)$ is imbedded into V compactly, we deduce that B is compact in V, absorbability of B is clear.

Proposition 3 *Let $u_1(t), u_2(t)$ be two solutions of problem (1)–(3) with $u_1(0)$, $u_2(0) \in B$ respectively, setting $w(t) = u_1 - u_2$. If*

1. $|w(t)|_V^2 \le ke^{\alpha t}|w(0)|_V^2,$ (13)
2. $\frac{d}{dt}\varphi(Q_N w) + c_0\varphi(Q_N w) \le c_1\lambda_{N+1}^{-2\beta}|Q_N w|_V^2$ (14)

holds, then there exists t_* such that $S(t_*)$ is Lipschitz continuous and squeezing in B where $\varphi(Q_N w)$ satisfies

$$|Q_N w(t)|_V^2 \le k_0\varphi(Q_N w) \le k_1|Q_N w(t)|_V^2 \tag{15}$$

$k_0, k_1, k, c_0, c_1, \alpha, \beta$ are positive constants independent of $w(t)$, $Q_N w = w|_{Q_N V}$ λ_{N+1} is eigenvalue as in (4), N satisfies

$$c_1 k_0 k(\alpha + c_0)^{-1}\lambda_{N+1}^{-2\beta} e^{2\alpha t_*} < \frac{1}{256} \tag{16}$$

t_* satisfies

$$k_1 e^{-c_0 t_*} < \frac{1}{256} \tag{17}$$

Proof *From (14), by Gronwall's inequality we have*

$$\begin{aligned}\varphi(Q_N w) &\le \varphi(Q_N w(0))e^{-c_0 t} + c_1 k\lambda_{N+1}^{-2\beta} e^{\alpha t}|Q_N w(0)|_V^2 e^{-c_0 t}\int_0^t e^{(\alpha+c_0)s}ds \\ &\le k_0^{-1}k_1|Q_N w(0)|_V^2 e^{-c_0 t} + c_1 k(c_0+\alpha)^{-1}\lambda_{N+1}^{-2\beta} e^{2\alpha t}|Q_N w(0)|_V^2 \\ &\le |w(0)|_V^2\left[k_0^{-1}k_1 e^{-c_0 t} + c_1 k(\alpha + c_0)^{-1}\lambda_{N+1}^{-2\beta} e^{2\alpha t}\right]\end{aligned} \tag{18}$$

Let t_* be large enough so that

$$k_1 e^{-c_0 t_*} < \frac{1}{256} \tag{19}$$

Next we choose N large enough so that

$$c_1 k_0 k(\alpha + c_0)^{-1} e^{2\alpha t_*}\lambda_{N+1}^{-2\beta} < \frac{1}{256} \tag{20}$$

From (18)–(20) we obtain that

$$k_0\varphi(Q_N w(t)) \le \frac{1}{2}\delta^2|w(0)|_V^2, \quad \delta \in \left(0, \frac{1}{8}\right) \tag{21}$$

Then from (15) we have

$$|Q_N w(t_*)|_V^2 \le \frac{1}{128}|w(0)|_V^2 \tag{22}$$

So

$$|w(t_*)|_V^2 = |P_N w|_V^2 + |Q_N w|_V^2 < 2|Q_N w(t_*)|_V^2 < \frac{1}{64}|w(0)|_V^2$$

when $|Q_N w(t_*)|_V^2 > |P_N w(t_*)|_V^2$. *It completes the proof of proposition. By Propositions 1-3 and Theorem 1, we immediately have*

Theorem 2 *Suppose that problem (1)–(3) satisfies the conditions of Proposition 3 and there exist bounded and closed absorbing sets* B_0, B_1 *in* $D(A)$, V *respectively.*

Then $(S(t), B)$ *admits a* $(D(A), V)$ *type inertial fractal set* M *and*

$$d_F(M) \le 1 + N_0 \log\left(1 + \sqrt{2}l/\delta\right) / \log\frac{1}{\theta} \tag{23}$$

$$dist_V(S(t)B, M) \le c_0 e^{-c_1 t} \tag{24}$$

where $l = ke^{\alpha t_*}$, $\delta \in \left(0, \frac{1}{8}\right)$, $4\delta < \theta < 1$, c_0, c_1 *are constants,* N_0 *satisfies (20),* t_* *satisfies (19).*

2. Application

We consider

$$i\frac{du}{dt} - Au + g(|u|^2)u + i\gamma u = f(x), \quad (t,x) \in R_+ \times (0,l) \tag{25}$$

$$u(0) = u_0 \tag{26}$$

$$u(0,t) = u(l,t) = 0 \quad (\text{or } u(x,t) = u(x+l,t),\ \forall t > 0,\ x \in R) \tag{27}$$

where $A = -\frac{\partial^2}{\partial x^2}$, $g(s) \in C^2(R_+)$ *satisfies*

$$\lim_{s\to+\infty} \frac{h(s) - wG(s)}{s^3} \le 0 \tag{28}$$

$$\lim_{s\to+\infty} \frac{G_+(s)}{s^3} = 0 \tag{29}$$

$$h(s) = sg(s),\ G(s) = \int_0^s g(\sigma)d\sigma,\ G_+(s) = \max\{0, G(s)\} \tag{30}$$

Let $D(A) = H^2(0,l) \cap H_0^1(0,l)$, $V = H_0^1(0,l)$ $\Big($*or*

$$D(A) = \left\{v \in H^2_{\text{loc}}(R), v(x+l) = v(x), \forall x \in R\right\}$$

$$V = \left\{v \in H^1_{\text{loc}}(R), v(x+l) = v(x),\ \forall x \in R\right\}\Big)$$

The norm of u *in* $D(A)$ *and* V *is defined by*

$$\|u\|_{D(A)} = \left[|u|_0^2 + l^2|u_x|_0^2 + l^4|u_{xx}|_0^2\right]^{1/2} \tag{31}$$

$$\|u\|_V = \left[|u|_0^2 + l^2|u_x|_0^2\right]^{1/2},\ |\cdot|_0 \text{ is the norm of } L^2(0,l) \tag{32}$$

Proposition 4[5] *Suppose that* g *satisfies (28)–(29),* $f \in L^2(0,l)$, *then for problem (25)–(27), we have the following statements.*

1. *If* $u_0 \in V$ $(D(A))$, *then there exists unique solution* u *and* $u \in C(R_+, V) \cap L^\infty(R_+, V)$. $(u \in C(R_+, D(A)) \cap L^\infty(R_+, D(A)))$

2. *There exists bounded closed absorbing set* B_0, B_1 *respectively* $B_0 = \{u \in D(A), \|u\|_{D(A)} \le \rho_{\infty,2}\}$, $B_1 = \{u \in V, \|u\|_V \le \rho_{\infty,1}\}$.

From Proposition 2,

$$B = \overline{\bigcup_{0 \le t \le t_0(B_0)} S(t)B_0} \tag{33}$$

is a positively invariant compact convex set in V. In order to prove the existence of inertial fractal set of $\{S(t), B\}$, according to Theorem 2, the condition of Proposition 3 must be checked.

Let u_1, u_2 be two solutions for problem (25)–(27), $u_1(0), u_2(0) \in B$, we set $w(t) = u_1(t) - u_2(t)$, then $w(t)$ satisfies

$$i\frac{\partial w}{\partial t} - Aw + g\left(|u_1|^2\right)u_1 - g\left(|u_2|^2\right)u_2 + i\gamma w = 0 \tag{34}$$

By using

$$\frac{1}{2}\frac{d}{dt}|w_x|_0^2 = \mathrm{Im}(iw_t - Aw, Aw) \tag{35}$$

and

$$\frac{1}{2}\frac{d}{dt}|w|_0^2 = \mathrm{Im}(iw_t - Aw, w) \tag{36}$$

we have

$$\frac{1}{2}\frac{d}{dt}\|w\|_V^2 = \mathrm{Im}\left(g\left(|u_2|^2\right)u_2 - g\left(|u_1|^2\right)u_1, w + l^2 Aw\right) - \gamma\|w\|_V^2 \tag{37}$$

Note that $g(s) \in C^2(R_+)$, so

$$g\left(|u_2|^2\right)u_2 - g\left(|u_1|^2\right)u_1 = g\left(|\xi|^2\right)w + g'\left(|\xi|^2\right)\xi^2\overline{w} + g'\left(|\xi|^2\right)|\xi|^2 w \tag{38}$$

where $\xi = \tau u_1 + (1-\tau)u_2 \in B$, $\tau \in (0,1)$. (notice that B is convex)
By using (38) we have

$$\mathrm{Im}\left(g\left(|u_2|^2\right)u_2 - g\left(|u_1|^2\right)u_1, w\right) = \mathrm{Im}\left(g'\left(|\xi|^2\right)\xi^2\overline{w}, w\right)$$

Note that $|\xi|_{L^\infty} \le \rho_{\infty,2}$; $|g'\left(|\xi|^2\right)| \le C$, we have

$$\mathrm{Im}\left(g\left(|u_2|^2\right)u_2 - g\left(|u_1|^2\right)u_1, w\right) \le c\rho_{\infty,2}^2|w|_0^2 \tag{39}$$

Also

$$\begin{aligned}\mathrm{Im}\left(g\left(|u_2|^2\right)u_2 - g\left(|u_1|^2\right)u_1, Aw\right) =& \mathrm{Im}(g\left(|\xi|^2\right)w \\ &+ g'\left(|\xi|^2\right)\xi^2 w + g'\left(|\xi|^2\right)|\xi|^2 w, Aw)\end{aligned}$$

$$=\mathrm{Im}\left(4g'\left(|\xi|^2\right)\mathrm{Re}\left(\xi\bar{\xi}_x\right)w,w_x\right)+\mathrm{Im}\left(g'|\xi|^2\right)\xi^2\overline{w}_x,w_x\Big)$$
$$+\mathrm{Im}\left(2g''\left(|\xi|^2\right)\mathrm{Re}\left(\xi\bar{\xi}_x\right)\overline{w}+2g'\left(|\xi|^2\right)\xi\xi_x\overline{w}+2g''\left(|\xi|^2\right)\mathrm{Re}(\xi\xi_x)|\xi|^2w,w_x\right) \quad (40)$$

Now using $|\xi_x|_{L^\infty}\le|\xi_x|_1^{1/2}|\xi|_0^{1/2}\le\rho_{\infty,2}$ and L^∞ estimates on $g'\left(|\xi|^2\right)$, $g''\left(|\xi|^2\right)$, we infer that

$$\left|\mathrm{Im}\left(g\left(|u_2|^2\right)u_2-g\left(|u_1|^2\right)u_1,Aw\right)\right|\le c_3|w_x|_0^2+c_4'\int_0^l|w||w_x|dx\le c_4\|w\|_V^2 \quad (41)$$

Combining (37), (39), with (41), we have

$$\frac{d}{dt}\|w(t)\|_V^2\le c_6\|w(t)\|_V^2 \quad (42)$$

Then

$$\|w(t)\|_V^2\le e^{c_6t}\|w(0)\|_V^2 \quad (43)$$

where c_6 depends only on $\gamma,\rho_{\infty,2},g,g',g''$.

Let

$$\varphi(w)=\int_0^l\left\{|w_x|^2-g\left(|\xi|^2\right)|w|^2-2g'\left(|\xi|^2\right)\mathrm{Re}(\xi\overline{w})^2\right\}dx \quad (44)$$

Using the fact that $|\xi|_{L^\infty}\le\rho_{\infty,2}$ for $\xi\in B$, $g\left(|\xi|^2\right),g'\left(|\xi|^2\right)\le c$, we have

$$\|Q_Nw(t)\|_V^2\le2l^2\varphi(Q_Nw)\le2c_2l^2\|Q_Nw(t)\|_V^2 \quad (45)$$

when

$$\lambda_{N+1}\ge l^{-2}+2c_0+4c_0\rho_{\infty,2}^2 \quad (46)$$

where $c_2=\max\left\{l^{-2},c_0+2c_0\rho_{\infty,2}^2\right\}$.

We multiply (34) by $\gamma\overline{w}$ and integrate on $(0,l)$, take the real part. Next we multiply again (34) by $-\overline{w}_t$ and integrate on $(0,l)$ take the real part, then we obtain respectively

$$\mathrm{Im}\int_0^l\gamma\overline{w}_tw\,dx-\int_0^l\gamma|w_x|^2dx+\gamma\int_0^lg'\left(|\xi|^2\right)|\xi|^2|w|^2dx+\gamma\int_0^lg\left(|\xi|^2\right)|w|^2dx$$
$$+\mathrm{Re}\int_0^l\gamma g'\left(|\xi|^2\right)\mathrm{Re}(\xi\overline{w})^2dx=0 \quad (47)$$

$$\frac{1}{2}\frac{d}{dt}\left[\int_0^l|w_x|^2dx-\int_0^l\left[2g'\left(|\xi|^2\right)\mathrm{Re}(\xi\overline{w})^2+g\left(|\xi|^2\right)|w|^2\right]dx\right]$$
$$+\gamma\mathrm{Im}\int_0^lw\overline{w}_tdx=r(t,w) \quad (48)$$

where

$$r(t,w)=-\frac{1}{2}\int_0^l\left[|w|^2\frac{\partial}{\partial t}g\left(|\xi|^2\right)+\mathrm{Re}(\xi\overline{w})\frac{\partial}{\partial t}g'\left(|\xi|^2\right)\right.$$

$$+ 2g'\left(|\xi|^2\right)\mathrm{Re}(\xi\overline{w})\mathrm{Re}\left(\xi_t\overline{w}\right)]dx \tag{49}$$

We deduce from (47) and (48) that

$$\frac{1}{2}\frac{d}{dt}\varphi(w) + \gamma|w_x|_0^2 - R(t,w) = r(t,w) \tag{50}$$

where

$$R(t,w) = \gamma\int_0^l g'\left(|\xi|^2\right)|\xi|^2|w|^2dx + \gamma\int_0^l g\left(|\xi|^2\right)|w|^2dx$$
$$+ \gamma\mathrm{Re}\int_0^l g'\left(|\xi|^2\right)\mathrm{Re}(\xi\overline{w})^2dx \tag{51}$$

Note that $\xi \in B$, so $|R(t,w)| \le c(\rho_{\infty,2},\gamma)|w|_0^2$. (52)

Returning to (25) we have

$$iu_t = Au - g(|u|^2)u - i\gamma u - f$$

According to $u \in B, f \in L^2(0,l)$, $u \in L^\infty(R_+, D(A))$, we have $u_t \in L^2(0,l)$, and $\xi_t \in L^2(0,l)$ that is $|\xi_t|_0 \le c(\rho_{\infty,2},|f|_0)$. Therefore we obtain

$$|r(t,w)| \le c_1\int_0^l |w|^2|\xi_t|dx \le c_1|\xi_t|\|w\|_{L^4}^2 \le c|w_x|_0^{1/2}|w|_0^{3/2}$$
$$\le \frac{\gamma}{2}|w_x|_0^2 + \frac{3}{4}c^{4/3}(2\gamma)^{-1/3}|w|_0^2 \tag{53}$$

Substituting (52), (53) into (50), we obtain

$$\frac{d}{dt}\varphi(w) + \gamma|w_x|_0^2 \le c_3|w|_0^2 \tag{54}$$

where $c_3 = c_1c(\rho_{\infty,2},|f|_0) + 3c^{4/3}/4\sqrt[3]{2\gamma}$, c_1, c are constants which do not depend on w, constant $c(\rho_{\infty,2},|f|_0)$ depends only on $\rho_{\infty,2},|f|_0$.

Using

$$\left|\int_0^l g\left(|\xi|^2\right)|w|^2dx + 2\int_0^l g'\left(|\xi|^2\right)\mathrm{Re}(\xi\overline{w})^2dx\right| \le c_4|w|_0^2$$

we obtain

$$\frac{d}{dt}\varphi(Q_Nw) + \gamma\varphi(Q_Nw) \le c_5\lambda_{N+1}^{-1}\|Q_Nw\|_V^2 \tag{55}$$

where $c_5 = c_3 + c_4$, N satisfies (45).

Finally, we have

Theorem 3 *Suppose that $u_1(t), u_2(t)$ be two solutions of the problem (25)–(27), $f \in L^2(0,l)$, $u_0 \in D(A)$, then under the conditions (28)–(29), there exists the inertial fractal set M of $(S(t),B)$ in V and*

$$d_F(M) \le c\max\left\{\sqrt{l^{-2} + 2c_0 + 4c_0^2\rho_{\infty,2}^2}, 16e^{c_5t_*}c_5^{1/2}(c_5+\gamma)^{-1/2}\right\} + 2 \tag{56}$$

$$\mathrm{dist}_V(S(t)B, M) \leq c_0 e^{-c_1 t} \tag{57}$$

t_* satisfies $2c_2 l^2 e^{-\gamma t_*} < \dfrac{1}{256}$, constants c_0, c_1, c, c_6, do not depend on w, c_2 is defined by (46).

Proof By Propositions 4, (43) and (55), we deduce that there exists an inertial fractal set for $(S(t), B)$, where N_0, satisfy

$$\lambda_{N_0+1} \geq \max\left\{2c_0 + 4c_0^2\rho_{\infty,2}^2 + l^{-2}, 256c_5 e^{2c_6 t_*}(\gamma + c_5)^{-1}\right\} \tag{58}$$

We choose $\delta < \dfrac{1}{8}$, $4\delta < \theta < 1$, and observe $\lambda_{N_0+1} \geq c_0 N_0^2$ in (23), the complex space is regarded as the product of the two real spaces, we obtain the (56), (57) follows with (24).

Remark If we solve t_* from the inequality $2c_2 l^2 e^{-\gamma t_*} \leq \dfrac{1}{256}$, then (56) will not contain t_*.

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