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SECOND ORDER PARAMETER-UNIFORM CONVERGENCE FOR A FINITE DIFFERENCE METHOD FOR A SINGULARLY PERTURBED LINEAR PARABOLIC SYSTEM

V. FRANKLIN, M. PARAMASIVAM, J.J.H. MILLER, AND S. VALARMATHI

Abstract. A singularly perturbed linear system of second order partial differential equations of parabolic reaction-diffusion type with given initial and boundary conditions is considered. The diffusion term of each equation is multiplied by a small positive parameter. These singular perturbation parameters are assumed to be distinct. The components of the solution exhibit overlapping layers. Shishkin piecewise-uniform meshes are introduced, which are used in conjunction with a classical finite difference discretisation, to construct a numerical method for solving this problem. It is proved that in the maximum norm the numerical approximations obtained with this method are first order convergent in time and essentially second order convergent in the space variable, uniformly with respect to all of the parameters.

Key Words. Singular perturbation problems, parabolic problems, boundary layers, uniform convergence, finite difference scheme, Shishkin mesh.

1. Introduction

The following parabolic initial-boundary value problem is considered for a singularly perturbed linear system of second order differential equations

$$\frac{\partial \vec{u}}{\partial t} - E\frac{\partial^2 \vec{u}}{\partial x^2} + A\vec{u} = \vec{f}, \text{ on } \Omega, \ \vec{u} \text{ given on } \Gamma, \tag{1}$$

where $\Omega = \{(x,t) : 0 < x < 1, 0 < t \le T\}$, $\overline{\Omega} = \Omega \cup \Gamma$, $\Gamma = \Gamma_L \cup \Gamma_B \cup \Gamma_R$ with $\vec{u}(0,t) = \vec{\phi}_L(t)$ on $\Gamma_L = \{(0,t) : 0 \le t \le T\}$, $\vec{u}(x,0) = \vec{\phi}_B(x)$ on $\Gamma_B = \{(x,0) : 0 \le x \le 1\}$, $\vec{u}(1,t) = \vec{\phi}_R(t)$ on $\Gamma_R = \{(1,t) : 0 \le t \le T\}$. Here, for all $(x,t) \in \overline{\Omega}$, $\vec{u}(x,t)$ and $\vec{f}(x,t)$ are column $n-$ vectors, E and $A(x,t)$ are $n \times n$ matrices, $E = \mathrm{diag}(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)$ with $0 < \varepsilon_i \le 1$ for all $i = 1, \ldots, n$. The ε_i are assumed to be distinct and, for convenience, to have the ordering

$$\varepsilon_1 < \cdots < \varepsilon_n.$$

Cases with some of the parameters coincident are not considered here.
The problem (1) can also be written in the operator form

$$\vec{L}\vec{u} = \vec{f} \text{ on } \Omega, \ \vec{u} \text{ given on } \Gamma,$$

where the operator $\vec{L}$ is defined by

$$\vec{L} = I\frac{\partial}{\partial t} - E\frac{\partial^2}{\partial x^2} + A,$$

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where I is the identity matrix. The reduced problem corresponding to (1) is defined by

$$\frac{\partial \vec{u}_0}{\partial t} + A\vec{u}_0 = \vec{f}, \text{ on } \Omega, \ \ \vec{u}_0 = \vec{u} \text{ on } \{(x,0) : 0 < x < 1\}. \tag{2}$$

For a general introduction to parameter-uniform numerical methods for singular perturbation problems, see [7], [9] and [1]. The piecewise-uniform Shishkin meshes $\Omega^{M,N}$ in the present paper have the elegant property that they reduce to uniform meshes when the parameters are not small. The problem posed in the present paper is also considered in [3], where parameter uniform convergence in the maximum norm is proved, which is first order in time and essentially first order in space. The meshes used there are different from those in the present paper. The main result of the present paper is established in [6] for the special case $n = 1$ and in [5] for $n = 2$. The proof in the present paper of first order convergence in the time variable and essentially second order convergence in the space variable, for general n, draws heavily on the analogous result in [8], where a slightly weaker result is proved for a reaction-diffusion system. The final result in the present paper is that the error in the maximum norm is bounded by $C(M^{-1} + (N^{-1}\ln N)^2)$, where C is a constant which is independent of the singular perturbation parameters $\vec{\varepsilon}$ and of the mesh parameters M, N. It is the factor $\ln N$ here, which makes the convergence essentially rather than fully second order, but it has little significance in practice.

The plan of the paper is as follows. In the next three sections both standard and novel bounds on the smooth and singular components of the exact solution are obtained. The sharp estimates for the singular component in Lemma 4.3 are proved by mathematical induction, while interesting orderings of the points $x_{i,j}^{(s)}$ are established in Lemma 4.2. In Section 5 piecewise-uniform Shishkin meshes are introduced. In Section 6 the discrete problem is defined and a discrete maximum principle, discrete stability properties and a comparison principle are established. In Section 7 an expression for the local truncation error is derived and standard estimates are stated. In Section 8 parameter-uniform estimates for the local truncation error of the smooth and singular components are obtained by means of a sequence of lemmas. The section culminates with the statement and proof of the required parameter-uniform error estimate in the maximum norm.

2. Solutions of the continuous problem

Standard theoretical results on the solutions of (1) are stated, without proof, in this section. See [2] and [4] for more details. For all $(x,t) \in \overline{\Omega}$, it is assumed that the components $a_{ij}(x,t)$ of $A(x,t)$ satisfy the inequalities

$$a_{ii}(x,t) > \sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}(x,t)| \text{ for } 1 \le i \le n, \text{ and } a_{ij}(x,t) \le 0 \text{ for } i \neq j \tag{3}$$

and, for some α,

$$0 < \alpha < \min_{\substack{(x,t)\in\overline{\Omega}\\ 1\le i\le n}} \Big(\sum_{j=1}^{n} a_{ij}(x,t)\Big). \tag{4}$$

It is also assumed, without loss of generality, that

$$\max_{1\le i\le n} \sqrt{\varepsilon_i} \le \frac{\sqrt{\alpha}}{6}. \tag{5}$$

The norms $\parallel \vec{V} \parallel = \max_{1\le k\le n} |V_k|$ for any n-vector $\vec{V}$, $\parallel y \parallel_D = \sup\{|y(x,t)| : (x,t) \in D\}$ for any scalar-valued function y and domain D, and $\parallel \vec{y} \parallel = \max_{1\le k\le n} \parallel y_k \parallel$ for any vector-valued function $\vec{y}$ are introduced. When $D = \overline{\Omega}$ or Ω the subscript D is usually dropped. In a compact domain D a function is said to be Hölder continuous of degree λ, $0 < \lambda \le 1$, if, for all $(x_1,t_1),(x_2,t_2) \in D$,

$$|u(x_1,t_1) - u(x_2,t_2)| \le C(|x_1 - x_2|^2 + |t_1 - t_2|)^{\lambda/2}.$$

The set of Hölder continuous functions forms a normed linear space $C^0_\lambda(D)$ with the norm

$$||u||_{\lambda,D} = ||u||_D + \sup_{(x_1,t_1),(x_2,t_2)\in D} \frac{|u(x_1,t_1) - u(x_2,t_2)|}{(|x_1 - x_2|^2 + |t_1 - t_2|)^{\lambda/2}},$$

where $||u||_D = \sup\limits_{(x,t)\in D} |u(x,t)|$. For each integer $k \ge 1$, the subspaces $C^k_\lambda(D)$ of $C^0_\lambda(D)$, which contain functions having Hölder continuous derivatives, are defined as follows

$$C^k_\lambda(D) = \{u : \frac{\partial^{l+m} u}{\partial x^l \partial t^m} \in C^0_\lambda(D) \text{ for } l,m \ge 0 \text{ and } 0 \le l + 2m \le k\}.$$

The norm on $C^0_\lambda(D)$ is taken to be $||u||_{\lambda,k,D} = \max\limits_{0\le l+2m\le k} ||\frac{\partial^{l+m} u}{\partial x^l \partial t^m}||_{\lambda,D}$. For a vector function $\vec{v} = (v_1, v_2, ..., v_n)$, the norm is defined by $||\vec{v}||_{\lambda,k,D} = \max\limits_{1\le i\le n} ||v_i||_{\lambda,k,D}$. Sufficient conditions for the existence, uniqueness and regularity of a solution of (1) are given in the following theorem.

Theorem 2.1. *Assume that* A, $\vec{f} \in C^2_\lambda(\overline{\Omega})$, $\vec{\phi}_L \in C^1(\Gamma_L)$, $\vec{\phi}_B \in C^2(\Gamma_B)$, $\vec{\phi}_R \in C^1(\Gamma_R)$ *and that the following compatibility conditions are fulfilled at the corners* $(0,0)$ *and* $(1,0)$ *of* Γ

$$\vec{\phi}_B(0) = \vec{\phi}_L(0) \quad and \quad \vec{\phi}_B(1) = \vec{\phi}_R(0), \tag{6}$$

$$\begin{aligned} \frac{d\vec{\phi}_L(0)}{dt} - E\frac{d^2\vec{\phi}_B(0)}{dx^2} + A(0,0)\vec{\phi}_B(0) &= \vec{f}(0,0), \\ \frac{d\vec{\phi}_R(0)}{dt} - E\frac{d^2\vec{\phi}_B(1)}{dx^2} + A(1,0)\vec{\phi}_B(1) &= \vec{f}(1,0) \end{aligned} \tag{7}$$

and

$$\begin{aligned} \frac{d^2}{dt^2}\vec{\phi}_L(0) &= E^2\frac{d^4}{dx^4}\vec{\phi}_B(0) - 2EA(0,0)\frac{d^2}{dx^2}\vec{\phi}_B(0) - EA(0,0)\frac{d}{dx}\vec{\phi}_B(0) \\ &\quad -(A^2(0,0) + \frac{\partial A}{\partial t}(0,0) + E\frac{\partial^2 A}{\partial x^2}(0,0))\vec{\phi}_B(0) \\ &\quad -A(0,0)\vec{f}(0,0) + \frac{\partial \vec{f}}{\partial t}(0,0) + E\frac{\partial^2 \vec{f}}{\partial x^2}(0,0), \end{aligned} \tag{8}$$

$$\begin{aligned}\frac{d^2}{dt^2}\vec{\phi}_R(0) &= E^2\frac{d^4}{dx^4}\vec{\phi}_B(1) - 2EA(1,0)\frac{d^2}{dx^2}\vec{\phi}_B(1) - EA(1,0)\frac{d}{dx}\vec{\phi}_B(1)\\ &\quad -(A^2(1,0) + \frac{\partial A}{\partial t}(1,0) + E\frac{\partial^2 A}{\partial x^2}(1,0))\vec{\phi}_B(1)\\ &\quad -A(1,0)\vec{f}(1,0) + \frac{\partial \vec{f}}{\partial t}(1,0) + E\frac{\partial^2 \vec{f}}{\partial x^2}(1,0).\end{aligned} \tag{9}$$

Then there exists a unique solution $\vec{u}$ of (1) *satisfying $\vec{u} \in C^4_\lambda(\overline{\Omega})$.*

It is assumed throughout the paper that all of the assumptions (3) - (9) of this section are fulfilled. Furthermore, C denotes a generic positive constant, which is independent of x, t and of all singular perturbation and discretization parameters. Inequalities between vectors are understood in the componentwise sense.

3. Standard analytical results

The operator $\vec{L}$ satisfies the following maximum principle

Lemma 3.1. *Let assumptions* (3) - (9) *hold. Let $\vec{\psi}$ be any vector-valued function in the domain of $\vec{L}$ such that $\vec{\psi} \geq \vec{0}$ on Γ. Then $\vec{L}\vec{\psi}(x,t) \geq \vec{0}$ on Ω implies that $\vec{\psi}(x,t) \geq \vec{0}$ on $\overline{\Omega}$.*

Proof. Let i^*, x^*, t^* be such that $\psi_{i^*}(x^*,t^*) = \min_i \min_{\overline{\Omega}} \psi_i(x,t)$ and assume that the lemma is false. Then $\psi_{i^*}(x^*,t^*) < 0$. From the hypotheses we have $(x^*,t^*) \notin \Gamma$ and $\frac{\partial^2 \psi_{i^*}}{\partial x^2}(x^*,t^*) \geq 0$. Thus

$$(\vec{L}\vec{\psi})_{i^*}(x^*,t^*) = \frac{\partial \psi_{i^*}}{\partial t}(x^*,t^*) - \varepsilon_{i^*}\frac{\partial^2 \psi_{i^*}}{\partial x^2}(x^*,t^*) + \sum_{j=1}^{n} a_{i^*,j}(x^*,t^*)\psi_j(x^*,t^*) < 0,$$

which contradicts the assumption and proves the result for $\vec{L}$. □

Let $\tilde{A}(x,t)$ be any principal sub-matrix of $A(x,t)$ and $\vec{\tilde{L}}$ the corresponding operator. To see that any $\vec{\tilde{L}}$ satisfies the same maximum principle as $\vec{L}$, it suffices to observe that the elements of $\tilde{A}(x,t)$ satisfy *a fortiori* the same inequalities as those of $A(x,t)$.

Lemma 3.2. *Let assumptions* (3) - (9) *hold. If $\vec{\psi}$ is any vector-valued function in the domain of $\vec{L}$, then, for each i, $1 \leq i \leq n$ and $(x,t) \in \overline{\Omega}$,*

$$|\psi_i(x,t)| \leq \max\left\{\| \vec{\psi} \|_\Gamma, \frac{1}{\alpha} \| \vec{L}\vec{\psi} \|\right\}.$$

Proof. Define the two functions

$$\vec{\theta}^{\pm}(x,t) = \max\left\{\| \vec{\psi} \|_\Gamma, \frac{1}{\alpha} \| \vec{L}\vec{\psi} \|\right\}\vec{e} \pm \vec{\psi}(x,t)$$

where $\vec{e} = (1, \ldots, 1)^T$ is the unit column vector. Using the properties of A it is not hard to verify that $\vec{\theta}^{\pm} \geq \vec{0}$ on Γ and $\vec{L}\vec{\theta}^{\pm} \geq \vec{0}$ on Ω. It follows from Lemma 3.1 that $\vec{\theta}^{\pm} \geq \vec{0}$ on $\overline{\Omega}$ as required. □

A standard estimate of the exact solution and its derivatives is contained in the following lemma.

Lemma 3.3. *Let assumptions* (3) - (9) *hold and let $\vec{u}$ be the exact solution of (1). Then, for all $(x,t) \in \overline{\Omega}$ and each $i = 1, \dots, n$,*

$|\frac{\partial^l u_i}{\partial t^l}(x,t)| \le C(||\vec{u}||_\Gamma + \sum_{q=0}^{l} ||\frac{\partial^q \vec{f}}{\partial t^q}||), \;\; l = 0, 1, 2$

$|\frac{\partial^l u_i}{\partial x^l}(x,t)| \le C\varepsilon_i^{\frac{-l}{2}}(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}||), \;\; l = 1, 2$

$|\frac{\partial^l u_i}{\partial x^l}(x,t)| \le C\varepsilon_i^{-1}\varepsilon_1^{\frac{-(l-2)}{2}}(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}|| + ||\frac{\partial^2 \vec{f}}{\partial t^2}|| + \varepsilon_1^{\frac{l-2}{2}}||\frac{\partial^{l-2} \vec{f}}{\partial x^{l-2}}||), \;\; l = 3, 4$

$|\frac{\partial^l u_i}{\partial x^{l-1}\partial t}(x,t)| \le C\varepsilon_i^{\frac{1-l}{2}}(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}|| + ||\frac{\partial^2 \vec{f}}{\partial t^2}||), \;\; l = 2, 3.$

Proof. The bound on $\vec{u}$ is an immediate consequence of Lemma 3.2. Differentiating (1) partially with respect to time once, respectively twice, and applying Lemma 3.2, the bounds on $\frac{\partial \vec{u}}{\partial t}$, respectively $\frac{\partial^2 \vec{u}}{\partial t^2}$ are obtained. To bound $\frac{\partial u_i}{\partial x}$, for each i and (x,t), consider an interval $I = [a, a + \sqrt{\varepsilon}_i], \; a \ge 0$ such that $x \in I$.
Then for some y such that $a < y < a + \sqrt{\varepsilon}_i$ and $t \in (0, T]$

$$\frac{\partial u_i}{\partial x}(y,t) = \frac{u_i(a + \sqrt{\varepsilon_i}, t) - u_i(a,t)}{\sqrt{\varepsilon_i}}$$

$$|\frac{\partial u_i}{\partial x}(y,t)| \le C\varepsilon_i{}^{\frac{-1}{2}}||\vec{u}||. \tag{10}$$

Then for any $x \in I$

$$\frac{\partial u_i}{\partial x}(x,t) = \frac{\partial u_i}{\partial x}(y,t) + \int_y^x \frac{\partial^2 u_i(s,t)}{\partial x^2} ds$$

$$\frac{\partial u_i}{\partial x}(x,t) = \frac{\partial u_i}{\partial x}(y,t) + \varepsilon_i^{-1}\int_y^x \left(\frac{\partial u_i(s,t)}{\partial t} - f_i(s,t) + \sum_{j=1}^{n} a_{ij}(s,t)u_j(s,t) \right) ds$$

$$|\frac{\partial u_i}{\partial x}(x,t)| \le |\frac{\partial u_i}{\partial x}(y,t)| + C\varepsilon_i^{-1}\int_y^x (||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}||)ds.$$

Using (10) in the above equation

$$|\frac{\partial u_i}{\partial x}(x,t)| \le C\varepsilon_i{}^{\frac{-1}{2}}(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}||).$$

Rearranging the terms in (1), it is easy to get

$$|\frac{\partial^2 u_i}{\partial x^2}| \le C\varepsilon_i^{-1}(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}||).$$

Analogous steps are used to get the rest of the estimates. □

The Shishkin decomposition of the exact solution $\vec{u}$ of (1) is $\vec{u} = \vec{v} + \vec{w}$ where the smooth component $\vec{v}$ is the solution of

$$\vec{L}\vec{v} = \vec{f} \text{ in } \Omega, \; \vec{v} = \vec{u}_0 \text{ on } \Gamma \tag{11}$$

and the singular component $\vec{w}$ is the solution of

$$\vec{L}\vec{w} = \vec{0} \text{ in } \Omega, \; \vec{w} = \vec{u} - \vec{v} \text{ on } \Gamma. \tag{12}$$

Theorem 2.1 ensures that $\vec{v}, \vec{w} \in C_\lambda^4(\overline{\Omega})$. For convenience the left and right boundary layers of $\vec{w}$ are separated using the further decomposition $\vec{w} = \vec{w}^L + \vec{w}^R$ where $\vec{L}\vec{w}^L = \vec{0}$ on Ω, $\vec{w}^L = \vec{w}$ on Γ_L, $\vec{w}^L = \vec{0}$ on $\Gamma_B \cup \Gamma_R$ and $\vec{L}\vec{w}^R = \vec{0}$ on

Ω, $\vec{w}^R = \vec{w}$ on Γ_R, $\vec{w}^R = \vec{0}$ on $\Gamma_L \cup \Gamma_B$.
Bounds on the smooth component and its derivatives are contained in

Lemma 3.4. *Let assumptions* (3) - (9) *hold. Then the smooth component* $\vec{v}$ *and its derivatives satisfy, for each* $(x,t) \in \overline{\Omega}$ *and* $i = 1, \dots, n$,

$$\begin{aligned} (a)\ &|\tfrac{\partial^l v_i}{\partial t^l}(x,t)| && \leq && C \text{ for } l = 0,1,2 \\ (b)\ &|\tfrac{\partial^l v_i}{\partial x^l}(x,t)| && \leq && C(1+\varepsilon_i^{1-\frac{l}{2}}) \text{ for } l = 0,1,2,3,4 \\ (c)\ &|\tfrac{\partial^l v_i}{\partial x^{l-1}\partial t}(x,t)| && \leq && C \ \textit{for}\ \ l = 2,3. \end{aligned}$$

Proof. The bound on $\vec{v}$ is an immediate consequence of the defining equations for $\vec{v}$ and Lemma 3.2. Differentiating the equation (11) twice partially with respect to x and applying Lemma 3.2 for $\frac{\partial^2 v_i}{\partial x^2}$, we get

$$|\frac{\partial^2 v_i}{\partial x^2}(x,t)| \leq C(1 + ||\frac{\partial \vec{v}}{\partial x}||). \tag{13}$$

Let

$$\frac{\partial v_{i^*}}{\partial x}(x^*,t^*) = ||\frac{\partial \vec{v}}{\partial x}|| \qquad \text{for some } i = i^*,\ x = x^*,\ t = t^*. \tag{14}$$

Using Taylor expansion, it follows that, for some $y \in [0, 1-x^*]$ and some $\eta \in (x^*, x^*+y)$

$$v_{i^*}(x^*+y,t^*) = v_{i^*}(x^*,t^*) + y\frac{\partial v_{i^*}}{\partial x}(x^*,t^*) + \frac{y^2}{2}\frac{\partial^2 v_{i^*}}{\partial x^2}(\eta,t^*). \tag{15}$$

Rearranging (15) yields

$$\frac{\partial v_{i^*}}{\partial x}(x^*,t^*) = \frac{v_{i^*}(x^*+y,t^*) - v_{i^*}(x^*,t^*)}{y} - \frac{y}{2}\frac{\partial^2 v_{i^*}}{\partial x^2}(\eta,t^*)$$

$$|\frac{\partial v_{i^*}}{\partial x}(x^*,t^*)| \leq \frac{2}{y}||\vec{v}|| + \frac{y}{2}||\frac{\partial^2 \vec{v}}{\partial x^2}||. \tag{16}$$

Using (14) and (16) in (13),

$$|\frac{\partial^2 v_i}{\partial x^2}| \leq C(1 + \frac{2}{y}||\vec{v}|| + \frac{y}{2}||\frac{\partial^2 \vec{v}}{\partial x^2}||).$$

This leads to

$$(1 - \frac{Cy}{2})||\frac{\partial^2 \vec{v}}{\partial x^2}|| \leq C(1 + \frac{2}{y}||\vec{v}||)$$

and so, using (a) with $l = 0$,

$$||\frac{\partial^2 \vec{v}}{\partial x^2}|| \leq C. \tag{17}$$

Using (17) in (16) yields

$$||\frac{\partial \vec{v}}{\partial x}|| \leq C.$$

Repeating the above steps with $\frac{\partial v_i}{\partial t}$, it is easy to get the required bounds on the mixed derivatives. The bounds on $\dfrac{\partial^3 \vec{v}}{\partial x^3}$, $\dfrac{\partial^4 \vec{v}}{\partial x^4}$ are derived by a similar argument. $\square$

4. Improved estimates

The layer functions $B_i^L,\ B_i^R,\ B_i,\ i = 1,\ \dots,\ n,$, associated with the solution $\vec{u}$, are defined on $[0,1]$ by

$$B_i^L(x) = e^{-x\sqrt{\alpha/\varepsilon_i}},\ B_i^R(x) = B_i^L(1-x),\ B_i(x) = B_i^L(x) + B_i^R(x).$$

The following elementary properties of these layer functions, for all $1 \leq i < j \leq n$ and $0 \leq x < y \leq 1$, should be noted:

$$B_i(x) = B_i(1-x). \tag{18}$$

$$B_i^L(x) < B_j^L(x),\ B_i^L(x) > B_i^L(y),\ 0 < B_i^L(x) \leq 1. \tag{19}$$

$$B_i^R(x) < B_j^R(x),\ B_i^R(x) < B_i^R(y),\ 0 < B_i^R(x) \leq 1. \tag{20}$$

$$B_i(x) \text{ is monotone decreasing for increasing } x \in [0, \frac{1}{2}]. \tag{21}$$

$$B_i(x) \text{ is monotone increasing for increasing } x \in [\frac{1}{2}, 1]. \tag{22}$$

$$B_i(x) \leq 2B_i^L(x) \text{ for } x \in [0, \frac{1}{2}],\ B_i(x) \leq 2B_i^R(x) \text{ for } x \in [\frac{1}{2}, 1]. \tag{23}$$

$$B_i^L(2\sqrt{\frac{\varepsilon_i}{\alpha}}\ln N) = N^{-2}. \tag{24}$$

The interesting points $x_{i,j}^{(s)}$ are now defined.

Definition 4.1. *For B_i^L, B_j^L, each $i,j,\ 1 \leq i \neq j \leq n$ and each $s, s > 0$, the point $x_{i,j}^{(s)}$ is defined by*

$$\frac{B_i^L(x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{B_j^L(x_{i,j}^{(s)})}{\varepsilon_j^s}. \tag{25}$$

It is remarked that

$$\frac{B_i^R(1-x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{B_j^R(1-x_{i,j}^{(s)})}{\varepsilon_j^s}. \tag{26}$$

In the next lemma the existence and uniqueness of the points $x_{i,j}^{(s)}$ are shown. Various properties are also established.

Lemma 4.2. *For all i,j, such that $1 \leq i < j \leq n$ and $0 < s \leq 3/2$, the points $x_{i,j}^{(s)}$ exist, are uniquely defined and satisfy the following inequalities*

$$\frac{B_i^L(x)}{\varepsilon_i^s} > \frac{B_j^L(x)}{\varepsilon_j^s},\ \ x \in [0, x_{i,j}^{(s)}),\ \ \frac{B_i^L(x)}{\varepsilon_i^s} < \frac{B_j^L(x)}{\varepsilon_j^s},\ x \in (x_{i,j}^{(s)}, 1]. \tag{27}$$

Moreover

$$x_{i,j}^{(s)} < x_{i+1,j}^{(s)},\ \text{if}\ \ i+1 < j\ \ \text{and}\ \ x_{i,j}^{(s)} < x_{i,j+1}^{(s)},\ \ \text{if}\ \ i < j. \tag{28}$$

Also

$$x_{i,j}^{(s)} < 2s\sqrt{\frac{\varepsilon_j}{\alpha}}\ \ \textit{and}\ \ x_{i,j}^{(s)} \in (0, \frac{1}{2})\ \ \text{if}\ \ i < j. \tag{29}$$

Analogous results hold for the B_i^R, B_j^R *and the points* $1 - x_{i,j}^{(s)}$.

Proof. The proof is given in [8]. □

Bounds on the singular components $\vec{w}^L$, $\vec{w}^R$ of $\vec{u}$ and their derivatives are contained in

Lemma 4.3. *Let assumptions* (3) - (9) *hold. Then there exists a constant* C, *such that, for each* $(x,t) \in \overline{\Omega}$ *and* $i = 1, \ldots, n$,

$$
\begin{aligned}
&|\frac{\partial^l w_i^L}{\partial t^l}(x,t)| \le CB_n^L(x), \text{ for } l = 0,1,2,\\
&|\frac{\partial^l w_i^L}{\partial x^l}(x,t)| \le C\sum_{q=i}^n \frac{B_q^L(x)}{\varepsilon_q^{\frac{l}{2}}}, \text{ for } l = 1,2,\\
&|\frac{\partial^3 w_i^L}{\partial x^3}(x,t)| \le C\sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q^{\frac{3}{2}}},\\
&|\frac{\partial^4 w_i^L}{\partial x^4}(x,t)| \le C\frac{1}{\varepsilon_i}\sum_{q=1}^n \frac{B_q^L(x)}{\varepsilon_q}.
\end{aligned} \tag{30}
$$

Analogous results hold for the w_i^R *and their derivatives.*

Proof. To obtain the bound of $\vec{w}^L$, define the functions $\psi_i{}^{\pm}(x,t) = Ce^{\alpha t}B_n^L(x) \pm w_i^L(x,t)$, for each $i = 1,\ldots,n$. It is clear that, for $(x,t) \in \Omega$, $\psi_i{}^{\pm}(0,t)$, $\psi_i{}^{\pm}(x,0)$, $\psi_i{}^{\pm}(1,t)$ and $(\vec{L}\vec{\psi}^{\pm})_i(x,t)$ are non-negative. By Lemma 1, $\psi_i{}^{\pm}(x,t) \ge 0$ for $(x,t) \in \overline{\Omega}$. It follows that $|w_i^L(x,t)| \le Ce^{\alpha t}B_n^L(x)$ or

$$|w_i^L(x,t)| \le CB_n^L(x). \tag{31}$$

To obtain the bound for $\frac{\partial w_i^L}{\partial t}$, define the two functions $\theta_i{}^{\pm}(x,t) = CB_n^L(x) \pm \frac{\partial w_i^L}{\partial t}(x,t)$ for each $i = 1,\ldots,n$. Differentiating the homogeneous equation satisfied by w_i^L, partially with respect to t, and rearranging yields

$$\frac{\partial^2 w_i^L}{\partial t^2} - \varepsilon_i \frac{\partial^3 w_i^L}{\partial x^2 \partial t} + \sum_{j=1}^n a_{ij}\frac{\partial w_j^L}{\partial t} = \frac{-\partial \sum_{j=1}^n a_{ij}}{\partial t} w_j^L, \tag{32}$$

and we get

$$
\begin{gathered}
|(\vec{L}\frac{\partial \vec{w}^L}{\partial t})_i| \le CB_n^L(x)\\
|\frac{\partial w_i^L}{\partial t}(0,t)| \le |\frac{\partial u_i}{\partial t}(0,t)| + |\frac{\partial v_i}{\partial t}(0,t)| \le C_1\\
|\frac{\partial w_i^L}{\partial t}(1,t)| = 0\\
|\frac{\partial w_i^L}{\partial t}(x,0)| = 0
\end{gathered}
$$

as $w_i^L = 0$ on $\Gamma_B \cup \Gamma_R$.
By Lemma 3.2, for a proper choice of C, it follows that

$$|\frac{\partial w_i^L}{\partial t}| \le CB_n^L(x). \tag{33}$$

Now the bound for $\frac{\partial^2 w_i^L}{\partial x \partial t}$ is obtained by using Lemma 3.3 and Lemma 3.4

$$|\frac{\partial^2 w_i^L}{\partial x \partial t}| \leq |\frac{\partial^2 u_i}{\partial x \partial t}| + |\frac{\partial^2 v_i}{\partial x \partial t}|$$

$$|\frac{\partial^2 w_i^L}{\partial x \partial t}| \leq C\varepsilon_i^{\frac{-1}{2}}(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}|| + ||\frac{\partial^2 \vec{f}}{\partial t^2}||).$$

Similarly,

$$|\frac{\partial^3 w_i^L}{\partial x^2 \partial t}| \leq C\varepsilon_i^{-1}(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}|| + ||\frac{\partial^2 \vec{f}}{\partial t^2}||). \tag{34}$$

As before, using suitable barrier functions, it is not hard to verify that

$$|\frac{\partial^{l+m} w_i^L}{\partial x^l \partial t^m}| \leq C\varepsilon_i^{\frac{-l}{2}} B_n^L(x), \ \ l \leq 3, \ m \leq 2 \text{ and } 0 \leq l + 2m \leq 4.$$

Using (31), (33) and (34) in (32), $|\frac{\partial^2 w_i^L}{\partial t^2}| \leq C$. Then defining the barrier function $\Psi_i^{\pm}(x,t) = Ce^{\alpha t} B_n^L(x) \pm \frac{\partial^2 w_i^L}{\partial t^2}(x,t)$ and using Lemma 3.2, the required bound is obtained.

The bounds on $\frac{\partial^l w_i^L}{\partial x^l}$, $l = 1,2,3,4$ and $i = 1,\dots,n$ are now derived by induction on n. For $n = 1$, the result follows from [6]. It is then assumed that the required bounds on $\frac{\partial w_i^L}{\partial x}, \frac{\partial^2 w_i^L}{\partial x^2}, \frac{\partial^3 w_i^L}{\partial x^3}$ and $\frac{\partial^4 w_i^L}{\partial x^4}$ hold for all systems up to order $n-1$. Define $\vec{\tilde{w}}^L = (w_1^L, \dots, w_{n-1}^L)$, then $\vec{\tilde{w}}^L$ satisfies the system

$$\frac{\partial \vec{\tilde{w}}^L}{\partial t} - \tilde{E}\frac{\partial^2 \vec{\tilde{w}}^L}{\partial x^2} + \tilde{A}\vec{\tilde{w}}^L = \vec{g},$$

with

$$\vec{\tilde{w}}^L(0,t) = \vec{\tilde{u}}(0,t) - \vec{\tilde{u}}_0(0,t), \vec{\tilde{w}}^L(1,t) = \vec{\tilde{0}},$$
$$\vec{\tilde{w}}^L(x,0) = \vec{\tilde{u}}(x,0) - \vec{\tilde{u}}_0(x,0) = \vec{\tilde{\phi}}_B(x) - \vec{\tilde{\phi}}_B(x) = \vec{\tilde{0}}.$$

Here, $\tilde{E}$ and $\tilde{A}$ are the matrices obtained by deleting the last row and column from E, A respectively, the components of $\vec{g}$ are $g_i = -a_{i,n}w_n^L$ for $1 \leq i \leq n-1$ and $\vec{\tilde{u}}_0$ is the solution of the reduced problem. Now decompose $\vec{\tilde{w}}^L$ into smooth and singular components to get $\vec{\tilde{w}}^L = \vec{q} + \vec{r}$, where $\vec{L}\vec{q} = \vec{g}, \ \vec{q} = \vec{\tilde{u}}_0$ on Γ, $\vec{L}\vec{r} = \vec{0}, \ \vec{r} = \vec{\tilde{w}}^L - \vec{q}$ on Γ. By induction, the bounds on the derivatives of $\vec{\tilde{w}}^L$ hold. That is, for $i = 1,\dots,n-1$,

$$\left.\begin{array}{lcl} |\frac{\partial w_i^L}{\partial x}| & \leq & C\sum_{q=i}^{n-1} \varepsilon_q^{\frac{-1}{2}} B_q^L(x) \\ |\frac{\partial^2 w_i^L}{\partial x^2}| & \leq & C\sum_{q=i}^{n-1} \varepsilon_q^{-1} B_q^L(x) \\ |\frac{\partial^3 w_i^L}{\partial x^3}| & \leq & C\sum_{q=1}^{n-1} \varepsilon_q^{\frac{-3}{2}} B_q^L(x) \\ |\varepsilon_i\frac{\partial^4 w_i^L}{\partial x^4}| & \leq & C\sum_{q=1}^{n-1} \varepsilon_q^{-1} B_q^L(x). \end{array}\right\} \tag{35}$$

Rearranging the n^{th} equation of the system satisfied by w_n^L yields

$$\varepsilon_n \frac{\partial^2 w_n^L}{\partial x^2} = \frac{\partial w_n^L}{\partial t} + \sum_{j=1}^{n} a_{nj} w_j^L.$$

Using (31) and (33) gives

$$|\frac{\partial^2 w_n^L}{\partial x^2}| \le C\varepsilon_n^{-1}B_n^L(x). \tag{36}$$

Applying the mean value theorem to w_n^L at some $y \in I = (a, a+\sqrt{\varepsilon}_n)$,

$$\frac{\partial w_n^L}{\partial x}(y,t) = \frac{w_n^L(a+\sqrt{\varepsilon}_n, t) - w_n^L(a,t)}{\sqrt{\varepsilon}_n}.$$

Using (31) gives

$$|\frac{\partial w_n^L}{\partial x}(y,t)| \le \frac{C}{\sqrt{\varepsilon}_n}(B_n^L(a+\sqrt{\varepsilon}_n) + B_n^L(a)).$$

So

$$|\frac{\partial w_n^L}{\partial x}(y,t)| \le \frac{C}{\sqrt{\varepsilon}_n}B_n^L(a). \tag{37}$$

Again, for $x \in I$, such that $y < \eta < x$,

$$\frac{\partial w_n^L}{\partial x}(x,t) = \frac{\partial w_n^L}{\partial x}(y,t) + (x-y)\frac{\partial^2 w_n^L}{\partial x^2}(\eta,t). \tag{38}$$

Using (36) and (37) in (38) yields

$$\begin{array}{rcl} |\frac{\partial w_n^L}{\partial x}(x,t)| & \le & C[\varepsilon_n^{\frac{-1}{2}}B_n^L(a) + \varepsilon_n^{\frac{-1}{2}}B_n^L(\eta)] \\ & \le & C\varepsilon_n^{\frac{-1}{2}}B_n^L(a) \\ & = & C\varepsilon_n^{\frac{-1}{2}}B_n^L(x)\frac{B_n^L(a)}{B_n^L(x)} \\ & = & C\varepsilon_n^{\frac{-1}{2}}B_n^L(x)e^{(x-a)\sqrt{\alpha}/\sqrt{\varepsilon}_n} \\ & \le & C\varepsilon_n^{\frac{-1}{2}}B_n^L(x)e^{\sqrt{\varepsilon}_n\sqrt{\alpha}/\sqrt{\varepsilon}_n}. \end{array}$$

Therefore

$$|\frac{\partial w_n^L}{\partial x}(x,t)| \le C\varepsilon_n^{\frac{-1}{2}}B_n^L(x). \tag{39}$$

Now, differentiating the equation satisfied by w_n^L partially with respect to x, and rearranging, gives

$$\varepsilon_n\frac{\partial^3 w_n^L}{\partial x^3} = \frac{\partial^2 w_n^L}{\partial x \partial t} + \sum_{q=1}^{n-1} a_{nq}\frac{\partial w_q^L}{\partial x} + a_{nn}\frac{\partial w_n^L}{\partial x} + \sum_{q=1}^{n}\frac{\partial a_{nq}}{\partial x}w_q^L.$$

The bounds on w_n^L and (35) then give

$$|\frac{\partial^3 w_n^L}{\partial x^3}| \le C\sum_{q=1}^{n}\varepsilon_q^{\frac{-3}{2}}B_q^L(x).$$

Similarly

$$|\varepsilon_n\frac{\partial^4 w_n^L}{\partial x^4}| \le C\sum_{q=1}^{n}\varepsilon_q^{-1}B_q^L(x).$$

Using the bounds on w_n^L, $\frac{\partial w_n^L}{\partial x}$, $\frac{\partial^2 w_n^L}{\partial x^2}$, $\frac{\partial^3 w_n^L}{\partial x^3}$ and $\frac{\partial^4 w_n^L}{\partial x^4}$, it is seen that $\vec{g}$, $\frac{\partial \vec{g}}{\partial x}$, $\frac{\partial^2 \vec{g}}{\partial x^2}$, $\frac{\partial^3 \vec{g}}{\partial x^3}$, $\frac{\partial^4 \vec{g}}{\partial x^4}$ are bounded by $CB_n^L(x)$, $C\frac{B_n^L(x)}{\sqrt{\varepsilon}_n}$, $C\frac{B_n^L(x)}{\varepsilon_n}$, $C\sum_{q=1}^{n}\frac{B_q^L(x)}{\varepsilon_q^{\frac{3}{2}}}$, $C\varepsilon_n^{-1}\sum_{q=1}^{n}\frac{B_q^L(x)}{\varepsilon_q}$ respectively. Introducing the functions $\vec{\psi}^{\pm}(x,t) = CB_n^L(x)\vec{e} \pm \vec{q}(x,t)$, it is easy to see that $\vec{\psi}^{\pm}(0,t) = C\vec{e} \pm \vec{q}(0,t) \geq \vec{0}$, $\vec{\psi}^{\pm}(1,t) = CB_n^L(1)\vec{e} \pm \vec{0} \geq \vec{0}$, $\vec{\psi}^{\pm}(x,0) = CB_n^L(x)\vec{e} \pm \vec{0} \geq \vec{0}$ and

$$(\vec{L}\vec{\psi}^{\pm})_i(x,t) = C(-\varepsilon_i\frac{\alpha}{\varepsilon_n} + \sum_{j=1}^{n} a_{ij})B_n^L(x) \pm CB_n^L(x) \geq 0,\ as - \frac{\varepsilon_i}{\varepsilon_n} \geq -1.$$

Applying Lemma 1, it follows that $||\vec{q}(x,t)|| \leq CB_n^L(x)$. Defining barrier functions $\vec{\theta}^{\pm}(x,t) = C\varepsilon_n^{\frac{-l}{2}} B_n^L(x)\vec{e} \pm \frac{\partial^l \vec{q}}{\partial x^l}$, $l = 1,2$ and using Lemmas 3.3 and 3.4 for the problem satisfied by $\vec{q}$, the bounds required for $\frac{\partial \vec{q}}{\partial x}$ and $\frac{\partial^2 \vec{q}}{\partial x^2}$ are obtained. Then it is easy to derive the bounds for $\frac{\partial^l \vec{q}}{\partial x^l}$, $l = 3,4$ from the defining equation of $\vec{q}$. By induction, the following bounds for $\vec{r}$ are obtained for $i = 1,\dots,n-1$,

$$|\frac{\partial r_i}{\partial x}| \leq C\left[\frac{B_i^L(x)}{\sqrt{\varepsilon_i}} + \dots + \frac{B_{n-1}^L(x)}{\sqrt{\varepsilon}_{n-1}}\right],$$

$$|\frac{\partial^2 r_i}{\partial x^2}| \leq C\left[\frac{B_i^L(x)}{\varepsilon_i} + \dots + \frac{B_{n-1}^L(x)}{\varepsilon_{n-1}}\right],$$

$$|\frac{\partial^3 r_i}{\partial x^3}| \leq C\left[\frac{B_1^L(x)}{\varepsilon_1^{\frac{3}{2}}} + \dots + \frac{B_{n-1}^L(x)}{\varepsilon_{n-1}^{\frac{3}{2}}}\right],$$

$$|\varepsilon_i\frac{\partial^4 r_i}{\partial x^4}| \leq C\left[\frac{B_1^L(x)}{\varepsilon_1} + \dots + \frac{B_{n-1}^L(x)}{\varepsilon_{n-1}}\right].$$

Combining the bounds for the derivatives of q_i and r_i it follows that, for $i = 1,2,\dots,n$

$$|\frac{\partial^l w_i^L}{\partial x^l}| \leq |\frac{\partial^l q_i}{\partial x^l}| + |\frac{\partial^l r_i}{\partial x^l}| \leq C\sum_{q=i}^{n}\frac{B_q^L(x)}{\varepsilon_q^{\frac{l}{2}}} \ \text{ for } \ l = 1,2,$$

$$|\frac{\partial^3 w_i^L}{\partial x^3}| \leq C\sum_{q=1}^{n}\frac{B_q^L(x)}{\varepsilon_q^{\frac{3}{2}}},$$

$$|\varepsilon_i\frac{\partial^4 w_i^L}{\partial x^4}| \leq C\sum_{q=1}^{n}\frac{B_q^L(x)}{\varepsilon_q}.$$

Recalling the bounds on the derivatives of w_n^L completes the proof of the lemma for the system of n equations.
A similar proof of the analogous results for the boundary layer functions w_i^R holds. $\square$

In the following lemma sharper estimates of the smooth component are presented.

Lemma 4.4. *Let assumptions* (3) - (9) *hold. Then the smooth component* $\vec{v}$ *of the solution* $\vec{u}$ *of* (1) *satisfies for all* $i = 1,\cdots,n$ *and all* $(x,t) \in \overline{\Omega}$

$$|\frac{\partial^l v_i}{\partial x^l}(x,t)| \leq C\left(1 + \sum_{q=i}^{n}\frac{B_q(x)}{\varepsilon_q^{\frac{l}{2}-1}}\right) \text{ for } l = 0,1,2,3.$$

Proof. Define two barrier functions

$$\vec{\psi}^{\pm}(x,t) \ = \ C[1 + B_n(x)]\vec{e} \pm \frac{\partial^l \vec{v}}{\partial x^l}(x,t), \ \ l = 0,1,2 \ \text{ and } \ (x,t) \in \overline{\Omega}.$$

Using Lemma 3.4, it follows that, for a proper choice of C, with $\vec{v} = \vec{u}_0$ on Γ,

$$\psi_i^{\pm}(0,t) \; = \; C \; \pm \; \frac{\partial^l v_i}{\partial x^l}(0,t) \geq 0$$

$$\psi_i^{\pm}(1,t) \; = \; C \; \pm \; \frac{\partial^l v_i}{\partial x^l}(1,t) \geq 0$$

$$\psi_i^{\pm}(x,0) \; = \; C[1 + B_n(x)] \; \pm \; \frac{\partial^l v_i}{\partial x^l} \; \geq \; 0$$

and $(\vec{L}\vec{\psi}^{\pm})_i(x,t) \geq 0$.
By Lemma 3.1

$$|\frac{\partial^l v_i}{\partial x^l}(x,t)| \leq C[1 + B_n(x)] \;\text{ for }\; l = 0,1,2. \tag{40}$$

Consider the equation

$$(\vec{L}(\frac{\partial^2 \vec{v}}{\partial x^2}))_i \; = \; \frac{\partial^2 f_i}{\partial x^2} - 2\frac{\partial \sum_{j=1}^n a_{ij}}{\partial x}\frac{\partial v_j}{\partial x} - \frac{\partial^2 \sum_{j=1}^n a_{ij}}{\partial x^2} v_j \tag{41}$$

with

$$\frac{\partial^2 v_i}{\partial x^2}(0,t) \; = \; 0, \frac{\partial^2 v_i}{\partial x^2}(1,t) \; = \; 0, \frac{\partial^2 v_i}{\partial x^2}(x,0) \; = \; \frac{\partial^2 \phi_{B,i}(x)}{\partial x^2}. \tag{42}$$

For convenience, let $\vec{p}$ denote $\frac{\partial^2 \vec{v}}{\partial x^2}$. Then

$$\vec{\mathrm{L}}\vec{p} = \vec{g} \;\text{ with }\; \vec{p}(0,t) = \vec{0}, \;\; \vec{p}(1,t) = \vec{0}, \;\; \vec{p}(x,0) = \vec{s} \tag{43}$$

where

$$g_i = \frac{\partial^2 f_i}{\partial x^2} - 2\frac{\partial \sum_{j=1}^n a_{ij}}{\partial x}\frac{\partial v_j}{\partial x} - \sum_{j=1}^n \frac{\partial^2 a_{ij}}{\partial x^2} v_j \;\text{ and }\; s_i = \frac{\partial^2 \phi_{B,i}(x)}{\partial x^2}.$$

Let $\vec{q}$ and $\vec{r}$ be the smooth and singular components of $\vec{p}$ given by

$$\vec{L}\vec{q} = \vec{g} \;\text{ with }\; \vec{q}(0,t) = \vec{p}_0(0,t), \;\; \vec{q}(1,t) = \vec{p}_0(1,t), \;\; \vec{q}(x,0) = \vec{p}(x,0) \tag{44}$$

where $\vec{p}_0$ is the solution of the reduced problem

$$\frac{\partial \vec{p}_0}{\partial t} + A\vec{p}_0 = \vec{g} \;\text{ with }\; \vec{p}_0(x,0) = \vec{p}(x,0) = \vec{s}.$$

Now,

$$\vec{L}\vec{r} = \vec{0}, \;\text{ with }\; \vec{r}(0,t) = -\vec{q}(0,t), \; \vec{r}(1,t) = -\vec{q}(1,t), \; \vec{r}(x,0) = \vec{0}. \tag{45}$$

Using Lemma 3.4 and Lemma 4.3, it follows that, for $i = 1, \ldots, n$ and $(x,t) \in \overline{\Omega}$,

$$|\frac{\partial q_i}{\partial x}(x,t)| \leq C$$

and

$$|\frac{\partial r_i}{\partial x}(x,t)| \leq C[\frac{B_i(x)}{\sqrt{\varepsilon_i}} + \cdots + \frac{B_n(x)}{\sqrt{\varepsilon_n}}].$$

Hence, for $(x,t) \in \overline{\Omega}$ and $i = 1, \ldots, n$,

$$|\frac{\partial^3 v_i}{\partial x^3}| = |\frac{\partial p_i}{\partial x}| \leq C[1 + \frac{B_i(x)}{\sqrt{\varepsilon_i}} + \cdots + \frac{B_n(x)}{\sqrt{\varepsilon_n}}]. \tag{46}$$

Then (40) and (46), for $l = 0, 1, 2, 3$ and $(x, t) \in \overline{\Omega}$, lead to

$$|\frac{\partial^l v_i}{\partial x^l}| \le C[1 + \varepsilon_i^{1-\frac{l}{2}} B_i(x) + \cdots + \varepsilon_n^{1-\frac{l}{2}} B_n(x)].$$

□

Remark : It is interesting to note that the above estimate reduces to the estimate of the smooth component of the solution of the scalar problem given in [7] when $n = 1$.

5. The Shishkin mesh

A piecewise uniform Shishkin mesh with $M \times N$ mesh-intervals is now constructed. Let $\Omega_t^M = \{t_k\}_{k=1}^M$, $\Omega_x^N = \{x_j\}_{j=1}^{N-1}$, $\overline{\Omega}_t^M = \{t_k\}_{k=0}^M$, $\overline{\Omega}_x^N = \{x_j\}_{j=0}^N$, $\Omega^{M,N} = \Omega_t^M \times \Omega_x^N$, $\overline{\Omega}^{M,N} = \overline{\Omega}_t^M \times \overline{\Omega}_x^N$ and $\Gamma^{M,N} = \Gamma \cap \overline{\Omega}^{M,N}$. The mesh $\overline{\Omega}_t^M$ is chosen to be a uniform mesh with M mesh-intervals on $[0, T]$. The mesh $\overline{\Omega}_x^N$ is a piecewise-uniform mesh on $[0, 1]$ obtained by dividing $[0, 1]$ into $2n + 1$ mesh-intervals as follows

$$[0, \sigma_1] \cup \cdots \cup (\sigma_{n-1}, \sigma_n] \cup (\sigma_n, 1 - \sigma_n] \cup (1 - \sigma_n, 1 - \sigma_{n-1}] \cup \cdots \cup (1 - \sigma_1, 1].$$

The n parameters σ_r, which determine the points separating the uniform meshes, are defined by $\sigma_0 = 0$, $\sigma_{n+1} = \frac{1}{2}$ and, for $r = 1, \ldots, n$,

$$\sigma_r = \min\left\{\frac{\sigma_{r+1}}{2}, 2\sqrt{\frac{\varepsilon_r}{\alpha}} \ln N\right\}. \tag{47}$$

Clearly

$$0 < \sigma_1 < \ldots < \sigma_n \le \frac{1}{4}, \qquad \frac{3}{4} \le 1 - \sigma_n < \ldots < 1 - \sigma_1 < 1.$$

Then, on the sub-interval $(\sigma_n, 1 - \sigma_n]$ a uniform mesh with $\frac{N}{2}$ mesh-intervals is placed, on each of the sub-intervals $(\sigma_r, \sigma_{r+1}]$ and $(1 - \sigma_{r+1}, 1 - \sigma_r]$, $r = 1, \ldots, n-1$, a uniform mesh of $\frac{N}{2^{n-r+2}}$ mesh-intervals is placed and on both of the sub-intervals $[0, \sigma_1]$ and $(1 - \sigma_1, 1]$ a uniform mesh of $\frac{N}{2^{n+1}}$ mesh-intervals is placed. In practice it is convenient to take

$$N = 2^{n+p+1} \tag{48}$$

for some natural number p. It follows that, for $2 \le r \le n$, in the sub-interval $[\sigma_{r-1}, \sigma_r]$ there are $N/2^{n-r+3} = 2^{r+p-2}$ mesh-intervals and in each of $[0, \sigma_1]$ and $[\sigma_1, \sigma_2]$ there are $N/2^{n+1} = 2^p$. This construction leads to a class of 2^n piecewise uniform Shishkin meshes $\Omega^{M,N}$. Note that these meshes are not the same as those constructed in [3].

From the above construction it is clear that the transition points $\{\sigma_r, 1-\sigma_r\}_{r=1}^n$ are the only points at which the mesh-size can change and that it does not necessarily change at each of these points. The following notation is introduced: if $x_j = \sigma_r$, then $h_r^- = x_j - x_{j-1}, h_r^+ = x_{j+1} - x_j$, $J = \{\sigma_r : h_r^+ \ne h_r^-\}$. In general, for each point x_j in the mesh-interval $(\sigma_{r-1}, \sigma_r]$,

$$x_j - x_{j-1} = 2^{n-r+3} N^{-1}(\sigma_r - \sigma_{r-1}). \tag{49}$$

Also, for $x_j \in (\sigma_n, \frac{1}{4}]$, $x_j - x_{j-1} = N^{-1}(1 - 4\sigma_n)$ and for $x_j \in (0, \sigma_1]$, $x_j - x_{j-1} = 2^{n+1} N^{-1} \sigma_1$. Thus, for $1 \le r \le n$, the change in the mesh-size at the point $x_j = \sigma_r$

is

$$h_r^+ - h_r^- = 2^{n-r+3}N^{-1}(d_r - d_{r-1}), \tag{50}$$

where

$$d_r = \frac{\sigma_{r+1}}{2} - \sigma_r \tag{51}$$

with the convention $d_0 = 0$. Notice that $d_r \geq 0$, that $\Omega^{M,N}$ is a classical uniform mesh when $d_r = 0$ for all $r = 1 \ \dots \ n$ and, from (47), that

$$\sigma_r \leq C\sqrt{\varepsilon_r}\ln N, \quad 1 \leq r \leq n. \tag{52}$$

It follows from (49) and (52) that for r = 1, ... ,n,

$$h_r^- + h_r^+ \leq C\sqrt{\varepsilon_{r+1}}N^{-1}\ln N. \tag{53}$$

Also

$$\sigma_r = 2^{-(s-r+1)}\sigma_{s+1} \text{ when } d_r = \cdots = d_s = 0, \ 1 \leq r \leq s \leq n. \tag{54}$$

The results in the following lemma are used later.

Lemma 5.1. *Assume that $d_r > 0$ for some $r, 1 \leq r \leq n$. Then the following inequalities hold*

$$B_r^L(1-\sigma_r) \leq B_r^L(\sigma_r) = N^{-2}. \tag{55}$$

$$x_{r-1,r}^{(s)} \ \leq \ \sigma_r - h_r^- \text{ for } 0 < s \leq 2, 1 < r \leq n. \tag{56}$$

$$B_q^L(\sigma_r - h_r^-) \leq CB_q^L(\sigma_r) \ \text{ for } \ 1 \leq r \leq q \leq n. \tag{57}$$

$$\frac{B_q^L(\sigma_r)}{\sqrt{\varepsilon_q}} \leq C\frac{1}{\sqrt{\varepsilon_r}\ln N} \ \text{ for } \ 1 \leq q \leq n, \ \ 1 \leq r \leq n. \tag{58}$$

Analogous results hold for B_r^R.

Proof. The proof is given in [8]. □

6. The discrete problem

In this section a classical finite difference operator with an appropriate Shishkin mesh is used to construct a numerical method for (1), which is shown later to be first order parameter-uniform in time and essentially second order parameter-uniform in the space variable.

The discrete initial-boundary value problem is now defined on any mesh by the finite difference method

$$D_t^-\vec{U} - E\delta_x^2\vec{U} + A\vec{U} = \vec{f} \ \text{ on } \ \Omega^{M,N}, \ \ \vec{U} = \vec{u} \ \text{ on } \ \Gamma^{M,N}. \tag{59}$$

This is used to compute numerical approximations to the exact solution of (1). It is assumed henceforth that the mesh is a Shishkin mesh, as defined in the previous section. Note that (59), can also be written in the operator form

$$\vec{L}^{M,N}\vec{U} \ = \ \vec{f} \ \text{ on } \ \Omega^{M,N}, \ \ \vec{U} = \vec{u} \ \text{ on } \ \Gamma^{M,N},$$

where

$$\vec{L}^{M,N} \ = \ ID_t^- - E\delta_x^2 + A$$

and D_t^-, δ_x^2, D_x^+ and D_x^- are the difference operators

$$D_t^-\vec{U}(x_j,t_k) \ = \ \frac{\vec{U}(x_j,t_k) - \vec{U}(x_j,t_{k-1})}{t_k - t_{k-1}},$$

$$\begin{aligned}
\delta_x^2\vec{U}(x_j,t_k) &= \frac{D_x^+\vec{U}(x_j,t_k)-D_x^-\vec{U}(x_j,t_k)}{(x_{j+1}-x_{j-1})/2},\\
D_x^+\vec{U}(x_j,t_k) &= \frac{\vec{U}(x_{j+1},t_k)-\vec{U}(x_j,t_k)}{x_{j+1}-x_j},\\
D_x^-\vec{U}(x_j,t_k) &= \frac{\vec{U}(x_j,t_k)-\vec{U}(x_{j-1},t_k)}{x_j-x_{j-1}}.
\end{aligned}$$

For any function $\vec{Z}$ defined on the Shishkin mesh $\overline{\Omega}^{M,N}$, we define $||\vec{Z}|| = \max\limits_i \max\limits_{j,k} |Z_i(x_j,t_k)|$.
The following discrete results are analogous to those for the continuous case.

Lemma 6.1. *Let assumptions* (3) - (9) *hold. Then, for any vector-valued mesh function* $\vec{\Psi}$*, the inequalities* $\vec{\Psi} \geq \vec{0}$ on $\Gamma^{M,N}$ *and* $\vec{L}^{M,N}\vec{\Psi} \geq \vec{0}$ *on* $\Omega^{M,N}$ *imply that* $\vec{\Psi} \geq \vec{0}$ *on* $\overline{\Omega}^{M,N}$.

Proof. Let i^*, j^*, k^* be such that $\Psi_{i^*}(x_{j^*},t_{k^*}) = \min_i \min_{j,k} \Psi_i(x_j,t_k)$ and assume that the lemma is false. Then $\Psi_{i^*}(x_{j^*},t_{k^*}) < 0$. From the hypotheses we have $j^* \neq 0,\ N$ and $\Psi_{i^*}(x_{j^*},t_{k^*}) - \Psi_{i^*}(x_{j^*},t_{k^*-1}) \leq 0$, $\Psi_{i^*}(x_{j^*},t_{k^*}) - \Psi_{i^*}(x_{j^*-1},t_{k^*}) \leq 0$, $\Psi_{i^*}(x_{j^*+1},t_{k^*}) - \Psi_{i^*}(x_{j^*},t_{k^*}) \geq 0$, so $D_t^-\Psi_{i^*}(x_{j^*},t_{k^*}) \leq 0$, $\delta_x^2\Psi_{i^*}(x_{j^*},t_{k^*}) > 0$. It follows that

$$\begin{aligned}
\left(\vec{L}^{M,N}\vec{\Psi}\right)_{i^*}(x_{j^*},t_{k^*}) \quad &= \quad D_t^-\Psi_{i^*}(x_{j^*},t_{k^*}) - \varepsilon_{i*}\delta_x^2\Psi_{i^*}(x_{j^*},t_{k^*})\\
&\quad + \sum_{q=1}^{n} a_{i^*,\,q}(x_{j^*},t_{k^*})\Psi_q(x_{j^*},t_{k^*}) \ < \ 0,
\end{aligned}$$

which is a contradiction, as required. □

An immediate consequence of this is the following discrete stability result.

Lemma 6.2. *Let assumptions* (3) - (9) *hold. Then, for any vector-valued mesh function* $\vec{\Psi}$ *on* $\overline{\Omega}^{M,N}$ *and* $i = 1, \ldots, n$,

$$|\Psi_i(x_j,t_k)| \ \leq \ \max\left\{||\vec{\Psi}||_{\Gamma^{M,N}}, \frac{1}{\alpha}||\vec{L}^{M,N}\vec{\Psi}||\right\}.$$

Proof. Define the two functions

$$\vec{\Theta}^{\pm}(x_j,t_k) = \max\{||\vec{\Psi}||_{\Gamma^{M,N}}, \frac{1}{\alpha}||L^{\vec{M},N}\vec{\Psi}||\}\vec{e} \pm \vec{\Psi}(x_j,t_k)$$

where $\vec{e} = (1,\ \ldots\ ,1)$. Using the properties of A it is not hard to verify that $\vec{\Theta}^{\pm} \geq \vec{0}$ on $\Gamma^{M,N}$ and $\vec{L}^{M,N}\vec{\Theta}^{\pm} \geq \vec{0}$ on $\Omega^{M,N}$. It follows from Lemma 6.1 that $\vec{\Theta}^{\pm} \geq \vec{0}$ on $\overline{\Omega}^{M,N}$. □

The following comparison principle will be used in the proof of the error estimate.

Lemma 6.3. *Assume that, for each* $i = 1,\ \ldots\ ,n$*, the vector-valued mesh functions* $\vec{\Phi}$ *and* $\vec{Z}$ *satisfy*

$$|Z_i| \leq \Phi_i \ \text{ on } \ \Gamma^{M,N} \ \text{ and } \ |(\vec{L}^{M,N}\vec{Z})_i| \leq (\vec{L}^{M,N}\vec{\Phi})_i \ \text{ on } \ \Omega^{M,N}.$$

Then, for each $i = 1,\ \ldots\ ,n$,

$$|Z_i| \leq \Phi_i \ \text{ on } \ \overline{\Omega}^{M,N}.$$

Proof. Define the two mesh functions $\vec{\Psi}^{\pm}$ by

$$\vec{\Psi}^{\pm} = \vec{\Phi} \pm \vec{Z}.$$

Then, for each $i = 1, \ \dots \ , n, \quad \Psi_i^{\pm}$ satisfies

$$\Psi_i^{\pm} \geq 0 \ \text{ on } \ \Gamma^{M,N} \ \text{ and } \ (\vec{L}^{M,N}\vec{\Psi}^{\pm})_i \geq 0 \ \text{ on } \ \Omega^{M,N}.$$

The result follows from an application of Lemma 6.1. □

7. The local truncation error

From Lemma 6.2, it is seen that in order to bound the error $\vec{U} - \vec{u}$, it suffices to bound $\vec{L}^{M,N}(\vec{U} - \vec{u})$. But this expression satisfies, for $(x_j, t_k) \in \Omega^{M,N}$,

$$\begin{aligned} &\vec{L}^{M,N}(\vec{U} - \vec{u}) = \vec{L}^{M,N}(\vec{U}) - \vec{L}^{M,N}(\vec{u}) = \\ &\vec{f} - \vec{L}^{M,N}(\vec{u}) = \vec{L}(\vec{u}) - \vec{L}^{M,N}(\vec{u}) = (\vec{L} - \vec{L}^{M,N})\vec{u}. \end{aligned}$$

It follows that

$$\vec{L}^{M,N}(\vec{U} - \vec{u}) = (\frac{\partial}{\partial t} - D_t^-)\vec{u} - E(\frac{\partial^2}{\partial x^2} - \delta_x^2)\vec{u}.$$

Let $\vec{V}, \vec{W}^L, \vec{W}^R$ be the discrete analogues of $\vec{v}, \vec{w}^L, \vec{w}^R$ respectively. Then, similarly,

$$\begin{aligned} \vec{L}^{M,N}(\vec{V} - \vec{v}) &= (\frac{\partial}{\partial t} - D_t^-)\vec{v} - E(\frac{\partial^2}{\partial x^2} - \delta_x^2)\vec{v}, \\ \vec{L}^{M,N}(\vec{W}^L - \vec{w}^L) &= (\frac{\partial}{\partial t} - D_t^-)\vec{w}^L - E(\frac{\partial^2}{\partial x^2} - \delta_x^2)\vec{w}^L, \\ \vec{L}^{M,N}(\vec{W}^R - \vec{w}^R) &= (\frac{\partial}{\partial t} - D_t^-)\vec{w}^R - E(\frac{\partial^2}{\partial x^2} - \delta_x^2)\vec{w}^R, \end{aligned}$$

and so, for each $i = 1, \ \dots \ , n$,

$$|(\vec{L}^{M,N}(\vec{V} - \vec{v}))_i| \leq |(\frac{\partial}{\partial t} - D_t^-)v_i| + |\varepsilon_i(\frac{\partial^2}{\partial x^2} - \delta_x^2)v_i|, \tag{60}$$

$$|(\vec{L}^{M,N}(\vec{W^L} - \vec{w}^L))_i| \leq |(\frac{\partial}{\partial t} - D_t^-)w_i^L| + |\varepsilon_i(\frac{\partial^2}{\partial x^2} - \delta_x^2)w_i^L|, \tag{61}$$

$$|(\vec{L}^{M,N}(\vec{W^R} - \vec{w}^R))_i| \leq |(\frac{\partial}{\partial t} - D_t^-)w_i^R| + |\varepsilon_i(\frac{\partial^2}{\partial x^2} - \delta_x^2)w_i^R|. \tag{62}$$

Thus, the smooth and singular components of the local truncation error can be treated separately. Note that, for any smooth function ψ and for each $(x_j, t_k) \in \Omega^{M,N}$, the following distinct estimates of the local truncation error hold:

$$|(\frac{\partial}{\partial t} - D_t^-)\psi(x_j, t_k)| \ \leq \ C(t_k - t_{k-1}) \max_{s \, \in \, [t_{k-1}, t_k]} |\frac{\partial^2 \psi}{\partial t^2}(x_j, s)|, \tag{63}$$

$$|(\frac{\partial}{\partial x} - D_x^+)\psi(x_j, t_k)| \ \leq \ C(x_{j+1} - x_j) \max_{s \, \in \, [x_j, x_{j+1}]} |\frac{\partial^2 \psi}{\partial x^2}(s, t_k)|, \tag{64}$$

$$|(\frac{\partial^2}{\partial x^2} - \delta_x^2)\psi(x_j, t_k)| \ \leq \ C \max_{s \, \in \, I_j} |\frac{\partial^2 \psi}{\partial x^2}(s, t_k)|, \tag{65}$$

$$|(\frac{\partial^2}{\partial x^2} - \delta_x^2)\psi(x_j, t_k)| \ \leq \ C(x_{j+1} - x_{j-1}) \max_{s \in I_j} |\frac{\partial^3 \psi}{\partial x^3}(s, t_k)|. \tag{66}$$

Furthermore, if $x_j \notin J$, then

$$|(\frac{\partial^2}{\partial x^2} - \delta_x^2)\psi(x_j, t_k)| \ \leq \ C(x_{j+1} - x_{j-1})^2 \max_{s \in I_j} |\frac{\partial^4 \psi}{\partial x^4}(s, t_k)|. \tag{67}$$

Here $I_j = [x_{j-1}, x_{j+1}]$.

8. Error estimate

The proof of the error estimate is broken into two parts. In the first a theorem concerning the smooth part of the error is proved. Then the singular part of the error is considered. A barrier function is now constructed, which is used in both parts of the proof.

For each $x_j = \sigma_r \in J$, introduce a piecewise linear polynomial θ_r on $\overline{\Omega}$, defined by

$$\theta_r(x) = \begin{cases} \dfrac{x}{\sigma_r}, \ \ 0 \leq x \leq \sigma_r. \\ 1, \ \ \sigma_r < x < 1 - \sigma_r. \\ \dfrac{1-x}{\sigma_r}, \ \ 1 - \sigma_r \leq x \leq 1. \end{cases}$$

It is not hard to verify that for any $x_j \in \Omega^{M,N}$

$$(\vec{L}^{M,N} \theta_r \vec{e})_i(x_j) \geq \begin{cases} \alpha \theta_r(x_j), \ \text{ if } \ x_j \notin J \\ \alpha + \dfrac{2\varepsilon_i}{\sigma_r(h_r^- + h_r^+)}, \ \text{ if } \ x_j \in J, \ x_j \in \{\sigma_r, 1 - \sigma_r\}. \end{cases} \tag{68}$$

Now, define the barrier function $\vec{\Phi}$ by

$$\vec{\Phi}(x_j, t_k) = C[M^{-1} + (N^{-1} \ln N)^2 + (N^{-1} \ln N)^2 \sum_{\{r:\, \sigma_r \in J\}} \theta_r(x_j)]\vec{e}, \tag{69}$$

where C is any sufficiently large constant.

Then, on $\Omega^{M,N}$, $\vec{\Phi}$ satisfies

$$0 \leq \Phi_i(x_j, t_k) \leq C(M^{-1} + (N^{-1} \ln N)^2), \ \ 1 \leq i \leq n. \tag{70}$$

Also, for $x_j \notin J$,

$$(\vec{L}^{M,N} \vec{\Phi})_i(x_j, t_k) \geq C(M^{-1} + (N^{-1} \ln N)^2) \tag{71}$$

and, for $x_j \in J, \ x_j \in \{\sigma_r, 1 - \sigma_r\}$, using (52), (53) and (68),

$$(\vec{L}^{M,N} \vec{\Phi})_i(x_j, t_k) \geq C(M^{-1} + (N^{-1} \ln N)^2 + \frac{\varepsilon_i}{\sqrt{\varepsilon_r \varepsilon_{r+1}}} N^{-1}). \tag{72}$$

The following theorem gives the estimate for the smooth component of the error.

Theorem 8.1. *Let assumptions* (3) - (9) *hold. Let $\vec{v}$ denote the smooth component of the exact solution from (1) and $\vec{V}$ the smooth component of the discrete solution from (59). Then*

$$||\vec{V} - \vec{v}|| \leq C(M^{-1} + (N^{-1} \ln N)^2). \tag{73}$$

Proof. By the comparison principle in Lemma 6.3 it suffices to show that, for all i, j, k and some C,

$$|(\vec{L}^{M,N}(\vec{V} - \vec{v}))_i(x_j, t_k)| \leq (\vec{L}^{M,N} \vec{\Phi})_i(x_j, t_k). \tag{74}$$

For each mesh point x_j there are two possibilities: either $x_j \notin J$ or $x_j \in J$.

If $x_j \notin J$, apply Lemma 3.4(a) with $l = 2$ and (63) to the t-derivative and apply Lemma 3.4(b) with $l = 4$ and (67) to the x- derivative to get

$$\begin{aligned} |(\vec{L}^{M,N}(\vec{V}-\vec{v}))_i(x_j,t_k)| &\leq C[t_k - t_{k-1} + (x_{j+1}-x_{j-1})^2] \\ &\leq C(M^{-1} + (N^{-1}\ln N)^2). \end{aligned} \tag{75}$$

Then (71) and (75) imply (74).
On the other hand, if $x_j \in J$, then $x_j \in \{\sigma_r, 1-\sigma_r\}$, for some $r, 1 \leq r \leq n$. Here the argument for $x_j = \sigma_r$ is given. For $x_j = 1-\sigma_r$ it is analogous.
If $x_j = \sigma_r \in J$, apply Lemma 3.4(a) with $l = 2$ and (63) to the t-derivative, and apply Lemma 4.4 with $l = 3$ and (66) to the x- derivative to get

$$|(\vec{L}^{M,N}(\vec{V}-\vec{v}))_i(x_j,t_k)| \leq C[t_k - t_{k-1} + \varepsilon_i(x_{j+1}-x_{j-1})(1+\sum_{q=i}^{n}\frac{B_q(x_{j-1})}{\sqrt{\varepsilon_q}})],$$

so, since $x_{j-1} = \sigma_r - h_r^-$,

$$|(\vec{L}^{M,N}(\vec{V}-\vec{v}))_i(x_j,t_k)| \leq C[M^{-1} + \varepsilon_i N^{-1}(1+\sum_{q=i}^{n}\frac{B_q(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}})]. \tag{76}$$

For each r, $1 \leq r \leq n$ there are at most two possibilities: either $i \geq r$ or $i \leq r-1$. If $i \geq r$, then $\sum_{q=i}^{n}\frac{B_q(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}} \leq \frac{C}{\sqrt{\varepsilon_i}} \leq \frac{C}{\sqrt{\varepsilon_r}}$. Substituting this into (76) gives

$$|(\vec{L}^{M,N}(\vec{V}-\vec{v}))_i(x_j,t_k)| \leq C[M^{-1} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r}}N^{-1}]. \tag{77}$$

(72) and (77) imply (74).
If $i \leq r-1$, which arises only if $r \geq 1$, there are two possibilities: either $d_r > 0$ or $d_r = 0$ and $d_{r-1} > 0$, because the case $d_r = d_{r-1} = 0$ cannot occur for $x_j = \sigma_r \in J$. Since $x_{j-1} = \sigma_r - h_r^-$ and $\sigma_r - h_r^- < \frac{1}{2}$, $B_q(x_{j-1}) = B_q(\sigma_r - h_r^-) = B_q^L(\sigma_r - h_r^-) + B_q^R(\sigma_r - h_r^-) \leq 2B_q^L(\sigma_r - h_r^-)$. Then $\sum_{q=i}^{n}\frac{B_q(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}} \leq 2\sum_{q=i}^{n}\frac{B_q^L(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}}$.
If $d_r > 0$, then using (27) in Lemma 4.2 and (56) in Lemma 5.1 give $\frac{B_q^L(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}} \leq \frac{B_r^L(\sigma_r - h_r^-)}{\sqrt{\varepsilon_r}}$ for $1 \leq q \leq r$. Hence $\sum_{q=i}^{n}\frac{B_q(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}} \leq \frac{C}{\sqrt{\varepsilon_r}}$. Substituting this into (76) gives

$$|(\vec{L}^{M,N}(\vec{V}-\vec{v}))_i(x_j,t_k)| \leq C[M^{-1} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r}}N^{-1}]. \tag{78}$$

(72) and (78) imply (74).
If $d_r = 0$ and $d_{r-1} > 0$ then using (27) and the fact that $\sigma_r - h_r^- \geq \sigma_{r-1} \geq x_{q,r-1}$, $1 \leq q \leq r-2$ give $\frac{B_q^L(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}} \leq \frac{B_{r-1}^L(\sigma_r - h_r^-)}{\sqrt{\varepsilon_{r-1}}}$ for $1 \leq q \leq r-1$. Hence

$$\begin{aligned} \sum_{q=i}^{n}\frac{B_q^L(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}} &\leq C\sum_{q=r-1}^{n}\frac{B_q^L(\sigma_{r-1})}{\sqrt{\varepsilon_q}} \leq C[\frac{B_{r-1}^L(\sigma_{r-1})}{\sqrt{\varepsilon_{r-1}}} + \frac{1}{\sqrt{\varepsilon_r}}] \\ &\leq C[\frac{N^{-2}}{\sqrt{\varepsilon_{r-1}}} + \frac{1}{\sqrt{\varepsilon_r}}]. \end{aligned}$$

Substituting this into (76) gives

$$\begin{aligned} |(L^{M,N}(\vec{V}-\vec{v}))_i(x_j,t_k)| &\leq C[M^{-1} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r}}N^{-1} + \frac{\varepsilon_i}{\sqrt{\varepsilon_{r-1}}}N^{-3}] \\ &\leq C[M^{-1} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r}}N^{-1}]. \end{aligned} \tag{79}$$

(72) and (79) imply (74). This completes the proof. $\square$

In order to estimate the singular component of the error the following four lemmas are required.

Lemma 8.2. *Assume that $x_j \notin J$. Let assumptions* (3) - (9) *hold. Then, on $\Omega^{M,N}$, for each $1 \leq i \leq n$, the following estimates hold*

$$|(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j, t_k)| \leq C(M^{-1} + \frac{(x_{j+1} - x_{j-1})^2}{\varepsilon_1}). \tag{80}$$

An analogous result holds for the $\vec{W}^R - \vec{w}^R$.

Proof. Since $x_j \notin J$, from (67) and Lemma 4.3, it follows that

$$\begin{aligned} |(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j, t_k)| &= |(((\tfrac{\partial}{\partial t} - D_t^-) - E(\tfrac{\partial^2}{\partial x^2} - \delta_x^2))\vec{w}^L)_i(x_j, t_k)| \\ &\leq C(M^{-1} + (x_{j+1} - x_{j-1})^2 \max_{s \in I_j} \sum_{q=1}^{n} \frac{B_q^L(s)}{\varepsilon_q}) \\ &\leq C(M^{-1} + \tfrac{(x_{j+1} - x_{j-1})^2}{\varepsilon_1}) \end{aligned}$$

as required. □

The following decompositions of the singular components w_i^L are used in the next lemma

$$w_i^L = \sum_{m=1}^{r+1} w_{i,m}, \tag{81}$$

where the components $w_{i,m}$ are defined by

$$w_{i,r+1} = \begin{cases} p_i^{(s)} & \text{on } [0, x_{r,r+1}^{(s)}) \\ w_i^L & \text{otherwise} \end{cases}$$

and, for each m, $r \geq m \geq 2$,

$$w_{i,m} = \begin{cases} p_i^{(s)} & \text{on } [0, x_{m-1,m}^{(s)}) \\ w_i^L - \displaystyle\sum_{q=m+1}^{r+1} w_{i,q} & \text{otherwise} \end{cases}$$

and

$$w_{i,1} = w_i^L - \sum_{q=2}^{r+1} w_{i,q} \text{ on } [0,1].$$

Here the polynomials $p_i^{(s)}$, for $s = 3/2$ and $s = 1$, are defined by

$$p_i^{(3/2)}(x,t) = \sum_{q=0}^{3} \frac{\partial^q w_i^L}{\partial x^q}(x_{r,r+1}^{(3/2)}, t) \frac{(x - x_{r,r+1}^{(3/2)})^q}{q!}$$

and

$$p_i^{(1)}(x,t) = \sum_{q=0}^{4} \frac{\partial^q w_i^L}{\partial x^q}(x_{r,r+1}^{(1)}, t) \frac{(x - x_{r,r+1}^{(1)})^q}{q!}.$$

Notice that the decomposition (81) depends on the choice of the polynomials $p_i^{(s)}$ and that the $x_{i,j}^{(s)}$ are defined by (25). The following lemma provides estimates of the derivatives of the components in the decomposition (81).

Lemma 8.3. *Assume that* $d_r > 0$ *for some* r, $1 \le r \le n$. *Let assumptions* (3) - (9) *hold. Then, for each* $1 \le i \le n$, *the components in the decomposition* (81) *satisfy the following estimates for each* q *and* r, $1 \le q \le r$, *and all* $(x_j, t_k) \in \Omega^{M,N}$,

$$|\frac{\partial^2 w_{i,q}}{\partial x^2}(x_j,t_k)| \le C \min\{\frac{1}{\varepsilon_q}, \frac{1}{\varepsilon_i}\} B^L_q(x_j),$$
$$|\frac{\partial^3 w_{i,q}}{\partial x^3}(x_j,t_k)| \le C \min\{\frac{1}{\varepsilon_i\sqrt{\varepsilon_q}}, \frac{1}{\varepsilon_q^{3/2}}\} B^L_q(x_j),$$
$$|\frac{\partial^3 w_{i,r+1}}{\partial x^3}(x_j,t_k)| \le C \min\{\sum_{q=r+1}^n \frac{B^L_q(x_j)}{\varepsilon_i\sqrt{\varepsilon_q}}, \sum_{q=r+1}^n \frac{B^L_q(x_j)}{\varepsilon_q^{3/2}}\},$$
$$|\frac{\partial^4 w_{i,q}}{\partial x^4}(x_j,t_k)| \le C \frac{B^L_q(x_j)}{\varepsilon_i\varepsilon_q},$$
$$|\frac{\partial^4 w_{i,r+1}}{\partial x^4}(x_j,t_k)| \le C \sum_{q=r+1}^n \frac{B^L_q(x_j)}{\varepsilon_i\varepsilon_q}.$$

Analogous results hold for the w^R_i *and their derivatives.*

Proof. Consider first the decomposition (81) corresponding to the polynomials $p_i^{(3/2)}$.

From the above definitions it follows that, for each m, $1 \le m \le r$, $w_{i,m} = 0$ on $[x^{(3/2)}_{m,m+1}, 1]$.

To establish the bounds on the third derivatives it is seen that: for $x \in [x^{(3/2)}_{r,r+1}, 1]$, Lemma 4.3 and $x \ge x^{(3/2)}_{r,r+1}$ imply that

$$|\frac{\partial^3 w_{i,r+1}}{\partial x^3}(x,t)| = |\frac{\partial^3 w^L_i}{\partial x^3}(x,t)| \le C \sum_{q=1}^n \frac{B^L_q(x)}{\varepsilon_q^{3/2}} \le C \sum_{q=r+1}^n \frac{B^L_q(x)}{\varepsilon_q^{3/2}};$$

for $x \in [0, x^{(3/2)}_{r,r+1}]$, Lemma 4.3 and $x \le x^{(3/2)}_{r,r+1}$ imply that

$$|\frac{\partial^3 w_{i,r+1}}{\partial x^3}(x,t)| = |\frac{\partial^3 w^L_i}{\partial x^3}(x^{(3/2)}_{r,r+1}, t)|$$
$$\le C \sum_{q=1}^n \frac{B^L_q(x^{(3/2)}_{r,r+1})}{\varepsilon_q^{3/2}} \le C \sum_{q=r+1}^n \frac{B^L_q(x^{(3/2)}_{r,r+1})}{\varepsilon_q^{3/2}} \le C \sum_{q=r+1}^n \frac{B^L_q(x)}{\varepsilon_q^{3/2}};$$

and for each $m = r, \ \ldots \ , 2$, it follows that

for $x \in [x^{(3/2)}_{m,m+1}, 1]$, $\frac{\partial^3 w_{i,m}}{\partial x^3} = 0$;
for $x \in [x^{(3/2)}_{m-1,m}, x^{(3/2)}_{m,m+1}]$, Lemma 4.3 implies that

$$|\frac{\partial^3 w_{i,m}}{\partial x^3}(x,t)| \le |\frac{\partial^3 w^L_i}{\partial x^3}(x,t)| + \sum_{q=m+1}^{r+1} |\frac{\partial^3 w_{i,q}}{\partial x^3}(x,t)|$$
$$\le C \sum_{q=1}^n \frac{B^L_q(x)}{\varepsilon_q^{3/2}} \le C \frac{B^L_m(x)}{\varepsilon_m^{3/2}}, \text{ using (27)};$$

for $x \in [0, x^{(3/2)}_{m-1,m}]$, Lemma 4.3 and $x \le x^{(3/2)}_{m-1,m}$ imply that

$$|\frac{\partial^3 w_{i,m}}{\partial x^3}(x,t)| = |\frac{\partial^3 w^L_i}{\partial x^3}(x^{(3/2)}_{m-1,m}, t)|$$
$$\le C \sum_{q=1}^n \frac{B^L_q(x^{(3/2)}_{m-1,m})}{\varepsilon_q^{3/2}} = C \frac{B^L_m(x^{(3/2)}_{m-1,m})}{\varepsilon_m^{3/2}} \le C \frac{B^L_m(x)}{\varepsilon_m^{3/2}}, \text{ using (25) and (27)};$$

for $x \in [x^{(3/2)}_{1,2}, 1]$, $\frac{\partial^3 w_{i,1}}{\partial x^3} = 0$;

for $x \in [0, x_{1,2}^{(3/2)}]$, Lemma 4.3 implies that

$$|\frac{\partial^3 w_{i,1}}{\partial x^3}(x,t)| \le |\frac{\partial^3 w_i^L}{\partial x^3}(x,t)| + \sum_{q=2}^{r+1} |\frac{\partial^3 w_{i,q}}{\partial x^3}(x,t)| \le C \sum_{q=1}^{n} \frac{B_q^L(x)}{\varepsilon_q^{3/2}} \le C \frac{B_1^L(x)}{\varepsilon_1^{3/2}}.$$

For the bounds on the second derivatives note that, for each m, $1 \le m \le r$: for $x \in [x_{m,m+1}^{(3/2)}, 1]$, $\frac{\partial^2 w_{i,m}}{\partial x^2} = 0$;
for $x \in [0, x_{m,m+1}^{(3/2)}]$,

$$\int_x^{x_{m,m+1}^{(3/2)}} \frac{\partial^3 w_{i,m}}{\partial x^3}(s,t)ds = \frac{\partial^2 w_{i,m}}{\partial x^2}(x_{m,m+1}^{(3/2)}, t) - \frac{\partial^2 w_{i,m}}{\partial x^2}(x,t) = -\frac{\partial^2 w_{i,m}}{\partial x^2}(x,t)$$

and so

$$|\frac{\partial^2 w_{i,m}}{\partial x^2}(x,t)| \le \int_x^{x_{m,m+1}^{(3/2)}} |\frac{\partial^3 w_{i,m}}{\partial x^3}(s,t)|ds \le \frac{C}{\varepsilon_m^{3/2}} \int_x^{x_{m,m+1}^{(3/2)}} B_m^L(s)ds \le C \frac{B_m^L(x)}{\varepsilon_m}.$$

This completes the proof of the estimates for $s = 3/2$.
Secondly, consider the decomposition (81) corresponding to the polynomials $p_i^{(1)}$. From the above definitions it follows that, for each m, $1 \le m \le r$, $w_{i,m} = 0$ on $[x_{m,m+1}^{(1)}, 1]$.
To establish the bounds on the fourth derivatives it is seen that:
for $x \in [x_{r,r+1}^{(1)}, 1]$, Lemma 4.3, (27) and $x \ge x_{r,r+1}^{(1)}$ imply that

$$|\varepsilon_i \frac{\partial^4 w_{i,r+1}}{\partial x^4}(x,t)| = |\varepsilon_i \frac{\partial^4 w_i^L}{\partial x^4}(x,t)| \le C \sum_{q=1}^{n} \frac{B_q^L(x)}{\varepsilon_q} \le C \sum_{q=r+1}^{n} \frac{B_q^L(x)}{\varepsilon_q};$$

for $x \in [0, x_{r,r+1}^{(1)}]$, Lemma 4.3, (27) and $x \le x_{r,r+1}^{(1)}$ imply that

$$|\varepsilon_i \frac{\partial^4 w_{i,r+1}}{\partial x^4}(x,t)| = |\varepsilon_i \frac{\partial^4 w_i^L}{\partial x^4}(x_{r,r+1}^{(1)}, t)| \le \sum_{q=1}^{n} \frac{B_q^L(x_{r,r+1}^{(1)})}{\varepsilon_q}$$

$$\le C \sum_{q=r+1}^{n} \frac{B_q^L(x_{r,r+1}^{(1)})}{\varepsilon_q} \le C \sum_{q=r+1}^{n} \frac{B_q^L(x)}{\varepsilon_q};$$

and for each $m = r, \ \dots \ , 2$, it follows that

for $x \in [x_{m,m+1}^{(1)}, 1]$, $\frac{\partial^4 w_{i,m}}{\partial x^4} = 0$;
for $x \in [x_{m-1,m}^{(1)}, x_{m,m+1}^{(1)}]$, Lemma 4.3 implies that

$$|\varepsilon_i \frac{\partial^4 w_{i,m}}{\partial x^4}(x,t)| \le |\varepsilon_i \frac{\partial^4 w_i^L}{\partial x^4}(x,t)| + \sum_{q=m+1}^{r+1} |\varepsilon_i \frac{\partial^4 w_{i,q}}{\partial x^4}(x,t)|$$

$$\le C \sum_{q=1}^{n} \frac{B_q^L(x)}{\varepsilon_q} \le C \frac{B_m^L(x)}{\varepsilon_m}, \ \text{using (27);}$$

for $x \in [0, x_{m-1,m}^{(1)}]$, Lemma 4.3 and $x \le x_{m-1,m}^{(1)}$ imply that

$$|\varepsilon_i \frac{\partial^4 w_{i,m}}{\partial x^4}(x,t)| = |\varepsilon_i \frac{\partial^4 w_i^L}{\partial x^4}(x_{m-1,m}^{(1)}, t)| \le C \sum_{q=1}^{n} \frac{B_q^L(x_{m-1,m}^{(1)})}{\varepsilon_q}$$

$$\le C \frac{B_m^L(x_{m-1,m}^{(1)})}{\varepsilon_m} \le C \frac{B_m^L(x)}{\varepsilon_m}, \ \text{using (25) and (27);}$$

for $x \in [x^{(1)}_{1,2}, 1]$, $\frac{\partial^4 w_{i,1}}{\partial x^4} = 0$;
for $x \in [0, x^{(1)}_{1,2}]$, Lemma 4.3 implies that

$$|\varepsilon_i \frac{\partial^4 w_{i,1}}{\partial x^4}(x,t)| \le |\varepsilon_i \frac{\partial^4 w^L_i}{\partial x^4}(x,t)| + \sum_{q=2}^{r+1} |\varepsilon_i \frac{\partial^4 w_{i,q}}{\partial x^4}(x,t)|$$

$$\le C \sum_{q=1}^{n} \frac{B^L_q(x)}{\varepsilon_q} \le C \frac{B^L_1(x)}{\varepsilon_1}.$$

For the bounds on the second and third derivatives note that, for each m, $1 \le m \le r$:
for $x \in [x^{(1)}_{m,m+1}, 1]$, $\frac{\partial^2 w_{i,m}}{\partial x^2} = 0 = \frac{\partial^3 w_{i,m}}{\partial x^3}$;
for $x \in [0, x^{(1)}_{m,m+1}]$,

$$\int_x^{x^{(1)}_{m,m+1}} \varepsilon_i \frac{\partial^4 w_{i,m}}{\partial x^4}(s,t)ds = \varepsilon_i \frac{\partial^3 w_{i,m}}{\partial x^3}(x^{(1)}_{m,m+1}, t) - \varepsilon_i \frac{\partial^3 w_{i,m}}{\partial x^3}(x,t) = -\varepsilon_i \frac{\partial^3 w_{i,m}}{\partial x^3}(x,t)$$

and so

$$|\varepsilon_i \frac{\partial^3 w_{i,m}}{\partial x^3}(x,t)| \le \int_x^{x^{(1)}_{m,m+1}} |\varepsilon_i \frac{\partial^4 w_{i,1}}{\partial x^4}(s,t)|ds \le \frac{C}{\varepsilon_m} \int_x^{x^{(1)}_{m,m+1}} B^L_m(s)ds \le C \frac{B^L_m(x)}{\sqrt{\varepsilon}_m}.$$

In a similar way, it can be shown that

$$|\varepsilon_i \frac{\partial^2 w_{i,m}}{\partial x^2}(x,t)| \le C B^L_m(x).$$

The proof for the w^R_i and their derivatives is similar. □

Lemma 8.4. *Assume that $d_r > 0$ for some r, $1 \le r \le n$. Let assumptions* (3) - (9) *hold. Then, if $x_j \notin J$,*

$$|(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j, t_k)| \le C[M^{-1} + B^L_r(x_{j-1}) + \frac{(x_{j+1} - x_{j-1})^2}{\varepsilon_{r+1}}] \tag{82}$$

and if $x_j \in J$

$$|(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j, t_k)| \le C[M^{-1} + N^{-2} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r \varepsilon_{r+1}}} N^{-1}]. \tag{83}$$

Analogous results hold for the $\vec{W^R} - \vec{w^R}$.

Proof. By (63) and Lemma 4.3

$$|\varepsilon_i(\frac{\partial}{\partial t} - D^-_t) w^L_i(x_j, t_k)| \le C(t_k - t_{k-1}). \tag{84}$$

Suppose first that $x_j \notin J$. Then, by (65), (67) and Lemma 8.3
(85)

$$\begin{aligned}
&|\varepsilon_i(\tfrac{\partial^2}{\partial x^2} - \delta^2_x) w^L_i(x_j, t_k)| \\
&\le C[\textstyle\sum_{q=1}^r \max_{s \in I_j} |\varepsilon_i \frac{\partial^2 w_{i,q}}{\partial x^2}(s, t_k)| + (x_{j+1} - x_{j-1})^2 \max_{s \in I_j} |\varepsilon_i \frac{\partial^4 w_{i,r+1}}{\partial x^4}(s, t_k)|] \\
&\le C[\textstyle\sum_{q=1}^r \min\{1, \frac{\varepsilon_i}{\varepsilon_q}\} B^L_q(x_{j-1}) + (x_{j+1} - x_{j-1})^2 \sum_{q=r+1}^n \frac{B^L_q(x_{j-1})}{\varepsilon_q}] \\
&\le C[B^L_r(x_{j-1}) + \tfrac{(x_{j+1} - x_{j-1})^2}{\varepsilon_{r+1}}].
\end{aligned}$$

Then (82) follows from (84) and (85).
Suppose now that $x_j = \sigma_r \in J$

(an analogous argument holds if $x_j = 1 - \sigma_r \in J$). Then, by Lemma 8.3 and the expressions (65) and (66),

$$|\varepsilon_i(\frac{\partial^2}{\partial x^2} - \delta_x^2)w_i^L(x_j,t_k)|$$
$$\le C[\sum_{q=1}^{r} \max_{s\in I_j} |\varepsilon_i \frac{\partial^2 w_{i,q}}{\partial x^2}(s,t_k)| + (x_{j+1}-x_{j-1})\max_{s\in I_j} |\varepsilon_i \frac{\partial^3 w_{i,r+1}}{\partial x^3}(s,t_k)|]$$
$$\le C[\sum_{q=1}^{r} \min\{1,\frac{\varepsilon_i}{\varepsilon_q}\} B_q^L(\sigma_r - h_r^-) + (h_r^- + h_r^+)\sum_{q=r+1}^{n} \min\{1,\frac{\varepsilon_i}{\varepsilon_q}\}\frac{B_q^L(\sigma_r-h_r^-)}{\sqrt{\varepsilon_q}}].$$

When $i \ge r+1$ replace both minima by the upper bound 1 and get, using (19), (57),(58),(24) and (53),

$$|\varepsilon_i(\frac{\partial^2}{\partial x^2} - \delta_x^2)w_i^L(x_j,t_k)| \le C[B_r^L(\sigma_r - h_r^-) + (h_r^- + h_r^+)\sum_{q=r+1}^{n}\frac{B_q^L(\sigma_r)}{\sqrt{\varepsilon_q}}]$$
$$\le C[B_r^L(\sigma_r) + \frac{h_r^-+h_r^+}{\sqrt{\varepsilon_r}\ln N}] \le C[N^{-2} + \sqrt{\frac{\varepsilon_{r+1}}{\varepsilon_r}}N^{-1}] \le C[N^{-2} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r\varepsilon_{r+1}}}N^{-1}],$$

which is (83) for this case. On the other hand, when $i \le r$ replace both minima by the upper bound $\frac{\varepsilon_i}{\varepsilon_q}$ and get, using Lemma 5.1,

$$|\varepsilon_i(\frac{\partial^2}{\partial x^2} - \delta_x^2)w_i^L(x_j,t_k)|$$
$$\le C[\varepsilon_i\frac{B_r^L(\sigma_r-h_r^-)}{\varepsilon_r} + (h_r^- + h_r^+)\varepsilon_i\sum_{q=r+1}^{n}\frac{B_q^L(\sigma_r)}{\varepsilon_q^{3/2}}]$$
$$\le C[\frac{\varepsilon_i}{\varepsilon_r}N^{-2} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r\varepsilon_{r+1}}}N^{-1}] \le C[N^{-2} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r\varepsilon_{r+1}}}N^{-1}],$$

which is (83) for this case. The proof for $\vec{W}^R - \vec{w}^R$ is similar. □

Lemma 8.5. *Let assumptions* (3) - (9) *hold. Then, on* $\Omega^{M,N}$*, for each* $1 \le i \le n$*, the following estimates hold*

$$|(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j,t_k)| \le C(M^{-1} + B_n^L(x_{j-1})). \tag{86}$$

An analogous result holds for $\vec{W}^R - \vec{w}^R$.

Proof. From (65) and Lemma 4.3, for each $i = 1,\dots,n$, it follows that on $\Omega^{M,N}$,

$$\begin{aligned} |(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j,t_k)| &= |(((\frac{\partial}{\partial t} - D_t^-) - E(\frac{\partial^2}{\partial x^2} - \delta_x^2))\vec{w}^L)_i(x_j,t_k)| \\ &\le C(M^{-1} + \varepsilon_i\sum_{q=i}^{n}\frac{B_q^L(x_{j-1})}{\varepsilon_q}) \\ &\le C(M^{-1} + B_n^L(x_{j-1})). \end{aligned}$$

The proof for $\vec{W}^R - \vec{w}^R$ is similar. □

The following theorem gives the estimate of the singular component of the error.

Theorem 8.6. *Let assumptions* (3) - (9) *hold. Let* $\vec{w}$ *denote the singular component of the exact solution from (1) and* $\vec{W}$ *the singular component of the discrete solution from (59). Then*

$$||\vec{W} - \vec{w}|| \le C(M^{-1} + (N^{-1}\ln N)^2). \tag{87}$$

Proof. Since $\vec{w} = \vec{w}^L + \vec{w}^R$, it suffices to prove the result for $\vec{w}^L$ and $\vec{w}^R$ separately. Here it is proved for $\vec{w}^L$; a similar proof holds for $\vec{w}^R$.
By the comparison principle in Lemma 6.3 it suffices to show that, for all i,j,k, and some constant C,

$$|(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j,t_k)| \le (\vec{L}^{M,N}\vec{\Phi})_i(x_j,t_k). \tag{88}$$

This is proved for each mesh point $x_j \in (0,1)$ by considering separately the 4 kinds of subinterval (a) $(0,\sigma_1)$, (b) $[\sigma_1,\sigma_2)$, (c) $[\sigma_m,\sigma_{m+1})$ for some m, $2 \le m \le n-1$

and (d) $[\sigma_n, 1)$.
(a) Clearly $x_j \notin J$ and $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_1}N^{-1}\ln N$. Then, Lemma 8.2 and (71) give (88).
(b) There are 2 possibilities: (b1) $d_1 = 0$ and (b2) $d_1 > 0$.
(b1) Since $\sigma_1 = \frac{\sigma_2}{2}$ and the mesh is uniform in $(0, \sigma_2)$ it follows that $x_j \notin J$, and $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_1}N^{-1}\ln N$. Then Lemma 8.2 and (71) give (88).
(b2) Either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$ then $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_2}N^{-1}\ln N$ and by Lemma 5.1 $B_1^L(x_{j-1}) \le B_1^L(\sigma_1 - h_1^-) \le CN^{-2}$, so Lemma 8.4 (82) with $r = 1$ and (71) give (88).
On the other hand, if $x_j \in J$, then Lemma 8.4 (83) with $r = 1$ and (72) give (88).
(c) There are 3 possibilities: (c1) $d_1 = d_2 = \cdots = d_m = 0$, (c2) $d_r > 0$ and $d_{r+1} = \ldots = d_m = 0$ for some r, $1 \le r \le m-1$ and (c3) $d_m > 0$.
(c1)Since $\sigma_1 = C\sigma_{m+1}$ and the mesh is uniform in $(0, \sigma_{m+1})$, it follows that $x_j \notin J$ and $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_1}N^{-1}\ln N$. Then Lemma 8.2 and (71) give (88).
(c2) Either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$ then $\sigma_{r+1} = C\sigma_{m+1}$, $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_{m+1}}N^{-1}\ln N$ and by Lemma 5.1 $B_r^L(x_{j-1}) \le B_r^L(\sigma_m - h_m^-) \le B_r^L(\sigma_r - h_r^-) \le CN^{-2}$. Thus Lemma 8.4 (82) and (71) give (88).
On the other hand, if $x_j \in J$, then $x_j = \sigma_m$, so Lemma 8.4 (83) with $r = m$ and (72) give (88).
(c3) Either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$ then $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_{m+1}}N^{-1}\ln N$ and by Lemma 5.1 $B_m^L(x_{j-1}) \le B_m^L(\sigma_m - h_m^-) \le CN^{-2}$, so Lemma 8.4 (82) with $r = m$ and (71) give (88).
On the other hand, if $x_j = \sigma_m$, so Lemma 8.4 (83) with $r = m$ and (72) give (88).
(d) There are 3 possibilities: (d1) $d_1 = \ldots = d_n = 0$, (d2) $d_r > 0$ and $d_{r+1} = \ldots = d_n = 0$ for some r, $1 \le r \le n-1$ and (d3) $d_n > 0$.
(d1) Since $\sigma_1 = C$ and the mesh is uniform in $(0, 1)$, it follows that $x_j \notin J$, $\frac{1}{\sqrt{\varepsilon_1}} \le C\ln N$ and $x_{j+1} - x_{j-1} \le CN^{-1}$. Then Lemma 8.2 and (71) give (88).
(d2) Either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$ then $\sigma_{r+1} = C$, $\frac{1}{\sqrt{\varepsilon_{r+1}}} \le C\ln N$, $x_{j+1} - x_{j-1} \le CN^{-1}$ and, by Lemma 5.1, $B_r^L(x_{j-1}) \le B_r^L(\sigma_n - h_n^-) \le B_r^L(\sigma_r - h_r^-) \le CN^{-2}$. Thus Lemma 8.4 (82) and (71) give (88).
On the other hand, if $x_j \in J$, then $x_j \in \{\sigma_n, 1-\sigma_n, \ \ldots \ , 1-\sigma_1\}$. Thus, Lemma 8.4 (83) and (72) give (88).
(d3) By Lemma 5.1 with $r = n$, $B_n^L(x_{j-1}) \le B_n^L(\sigma_n - h_n^-) \le CN^{-2}$. Then Lemma 8.5 and (71) give (88). □

The following theorem gives the first order in time and essentially second order in space parameter-uniform error estimate.

Theorem 8.7. *Let assumptions* (3) - (9) *hold. Let $\vec{u}$ denote the exact solution of (1) and $\vec{U}$ the discrete solution of (59). Then*

$$||\vec{U} - \vec{u}|| \le C(M^{-1} + (N^{-1}\ln N)^2). \tag{89}$$

Proof. An application of the triangle inequality and the results of Theorems 8.1 and 8.6 lead immediately to the required result. □

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Department of Mathematics, Bishop Heber College, Tiruchirappalli-620017, Tamil Nadu, India.
E-mail: franklinvicto@gmail.com and sivambhcedu@gmail.com

Institute for Numerical Computation and Analysis, Dublin, Ireland.
E-mail: jm@incaireland.org

Department of Mathematics, Bishop Heber College, Tiruchirappalli-620017, Tamil Nadu, India.
E-mail: valarmathi07@gmail.com