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# ON THE OCCURRENCE OF "VACUUM STATES" FOR $2 \times 2$ QUASILINEAR HYPERBOLIC CONSERVATION LAWS*

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**Abstract** We show that the solution to the Cauchy problem of $2 \times 2$ nonlinear conservation laws, in general, may go out the strictly hyperbolic region of the system in a finite time, here the initial data are given in the strictly hyperbolic region. In other words, in general, we can't confine our attention to solve the Cauchy problem of $2 \times 2$ nonlinear conservation laws in strictly hyperbolic type. However, we can expect that it may be solved under the additional conditions (A) and (b).

**Key Words** "Vacuum states"; quasilinear hyperbolic conservation laws.
**Classification** 35L65.

## 1. Introduction

This paper is a development of the papers [1–5]. In [1–5], we considered the equations of isentropic gas dynamics in Lagrangian coordinates. We proved that the vacuum never occurs in an isentropic flow consisting of rarefaction waves even though the density may tend to zero as time tends to infinity. Now, we study the similar issue for a pair of quasilinear hyperbolic conservation laws.

The problem of the occurrence of the vacuum in isentropic flows has been a central issue in this field for some time, which was addressed by some authors, e.g. Liu and Smoller [6]. When the vacuum appears, the speeds of the characteristics of two families coincide with each other, the system is not strictly hyperbolic and waves behave in a singular way, causing serious analytical difficulty [7–9]. It is well known that the solution of Riemann problem for isentropic gas dynamics may contain the vacuum. One thereby tends to believe that if the initial data are not far from the vacuum, then it will occur at a later time. We showed that [1–3] the aforementioned solution of the Riemann problem indeed is the only circumstance where the vacuum can occur. Precisely, we showed that vacuum states cannot appear in rarefaction wave solutions of the equations unless the vacuum is present at time $t = 0^+$. It is well known that vacuum states can only appear due to the interaction of two rarefaction waves of different families, thus one tends to believe that they don't occur for the solutions in general. If it is proven to be true, then

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following DiPerna [10], we can get the existence theorem by compensated compactness theory, and avoid the serious analytical difficulties caused by the appearence of vacuum [7–9].

The similar issue is naturally addressed to a pair of quasilinear strictly hyperbolic conservation laws. This is a complicated problem. We can obtain a similar result under the additional assumptions (A) and (B) in Section 2. They hold for the equations of isentropic gas dynamics in Eulerian coordinates and that in Lagrangian coordinates. Once the assumption (A) or (B) is violated, no matter how narrowly, Examples 1 and 2 in Section 3 show that "vacuum states" may occur at a later time, where "vacuum state" means that, in the state, the speed of the characteristics of two families coincide with each other. But Example 3 shows that the condition (A) is not necessary for the nonoccurance of "vacuum".

**Remark** Examples 1 and 2 imply that the solution to the Cauchy problem of $2\times 2$ nonlinear conservation laws, in general, may go out the strictly hyperbolic region of the system in a finite time, here the initial data are given in the strictly hyperbolic region and bounded away from the boundary. In other words, in general, we can't confine our attention to solve the Cauchy problem of $2 \times 2$ nonlinear conservation laws in the strictly hyperbolic region of the system. However, under the assumptions (A) and (B), we can expect the (generalized) solution will stay in the strictly hyperbolic region of the system in any finite time, if the initial data are given in it. In other words, we can expect that the Cauchy problem of $2 \times 2$ nonlinear conservation laws may be solved in the strictly hyperbolic region of the system under the assumptions (A) and (B).

**Note** The result in Section 2 is much more general than that in the papars [1–5], and the proof is much simpler [11–13]. It is owing to the intent of the papers [1–5] that is to find some suitable frameworks to study the solutions with shocks.

## 2. Existence Theorem

We are concerned with the Cauchy problem of a pair of conservation laws

$$u_t + f(u,v)_x = 0, \quad v_t + g(u,v)_x = 0, \quad (t,x) \in R_+ \times R \tag{E$_1$}$$

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in R \tag{I$_1$}$$

here $f(u,v), g(u,v)$ are smooth in $D_0$, where $D_0$ is a region on $(u,v)$ plane, the system $(\mathrm{E})_1$ is strictly hyperbolic in a subregion $D_1 \subset D_0$, i.e. the Jocobian matrix of the system $(\mathrm{E})_1$ has two real and distinct eigenvalues $\lambda < \mu$ in $D_1$. Let $G =: \mu - \lambda$, then $0 < G < \infty$ in $D_1$.

Riemann invariants of $(\mathrm{E})_1$ $z = z(u,v)$, $w = w(u,v)$ give a bijective smooth mapping from $D_1$ onto a region $D$ on $(z,w)$ plane. Thus $0 < G < \infty$ in $D$. Let $C_0 = \{(z,w) \mid G(z,w) = 0\}$, $C_\infty = \{(z,w) \mid G(z,w) = \infty\}$, we assume $\delta D = C_0 \cup C_\infty$, where $\delta D$ denotes the boundary of $D$. The Riemann invariants diagonalize the principal part of the system $(\mathrm{E})_1$ as

$$z_t + \lambda(z,w) z_x = 0, \quad w_t + \mu(z,w) w_x = 0, \quad (t,x) \in R_+ \times R \tag{E}$$

and reduce the initial data $(\mathrm{I})_1$ to

$$z(0,x)=z_0(x),\quad w(0,x)=w_0(x),\quad x\in R \tag{I}$$

We assume that

$$0\le \lambda_z(z,w)<\infty,\quad 0\le \mu_w(z,w)<\infty^* \quad \text{in } D \tag{2.1}$$

The initial data are smooth and satisfy the condition

$$I\subset\subset D \tag{2.2}$$

where the bounded set $I=\{(z,w)\mid z=z_0(x),w=w_0(x),x\in R\}$.

**Theorem** *Under the aforementioned conditions, and the additional conditions*

$$\mu_w+\mu_z\ge 0,\quad \lambda_w+\lambda_z\ge 0 \quad in\ D \tag{A}$$

$$|G_z|<\infty,\quad |G_w|<\infty \quad in\ D \tag{B}$$

*if*

$$0\le z_0'(x)<N,\quad 0\le w_0'(x)<N,\quad x\in R \tag{M}$$

*where the constant $N$ can be arbitrarily large, then the Cauchy problem* (E), (I) *has a unique global bounded classical solution $z=z(t,x),w=w(t,x)$,*

$$(z(t,x),w(t,x))\subset Q,\qquad (t,x)\in R_+\times R \tag{2.3}$$

$$(z(t,x),w(t,x))\subset\subset D,\quad 0<t<T,\ x\in R \tag{2.4}$$

*where $T>0$ can be arbitrarily large, and $Q$ is a bounded rectangular $Q=\{(z,w)\mid z_*\le z\le z^*,w_*\le w\le w^*\}$, $z_*=\inf\limits_x z_0(x)$, $z^*=\sup\limits_x z_0(x)$, $w_*=\inf\limits_x w_0(x)$, $w^*=\sup\limits_x w_0(x)$. Moreover*

$$0\le z_x(t,x)<N,\quad 0\le w_x(t,x)<N,\quad (t,x)\in R_+\times R \tag{2.5}$$

**Proof** It is well known that the global classical existence theorem follows the local classical existence theorem and the *a priori* estimates (2.3), (2.4), (2.5). By (E), (2.3) is obvious. In order to get the *a priori* estimates (2.5), we differentiate (E) with respect to $x$, thus

$$\begin{cases} Z_t+\lambda Z_x+\lambda_z Z^2+\lambda_w WZ=0\\ W_t+\mu W_x+\mu_w W^2+\mu_z ZW=0\end{cases}$$

where $Z=:z_x, W=:w_x$. Let "dot" denote differentiation in the direction $\partial/\partial t+\lambda\partial/\partial x$, and "prime" denote differentiation in the direction $\partial/\partial t+\mu\partial/\partial x$, thus

$$\begin{aligned}\dot Z+\lambda_z Z^2+\lambda_w WZ=0\\ W'+\mu_w W^2+\mu_z ZW=0\end{aligned} \tag{2.6}$$

$^*$It is well known that the $\lambda$ (resp. $\mu$) characteristic field is genuinely nonlinear if and only if $\lambda_z\ne 0$ (resp. $\mu_w\ne 0$), and the field is linearly degenerate if and only if $\lambda_z\equiv 0$ (resp. $\mu_w\equiv 0$) (see [12, 14]).

Following Lax [11,12], we have

$$w_x(t,x) = \frac{w_0'(x)\exp\{-a(t,x)\}}{1 + w_0'(x_0)\int_0^t \mu_w(s,x)\exp\{-a(s,x)\}ds} \tag{2.7}$$

as long as the classical solution exists and is strictly contained in $D$, where $a(t,x)$ is a function satisfying $a_z = \mu_z/G$, $a(0,x) = 0$, the integration is along $\mu$-characteristic issued from $(0,x_0)$. By the monotone condition (M), we have *a priori* estimates $W(t,x) =: w_x(t,x) \geq 0$. Similary, $Z(t,x) \geq 0$. In order to complete *a priori* estimates (2.5), it is sufficient to prove $T = \infty$, where

$$T =: \sup\{t \mid n(t) < N\}, \quad n(t) =: \max_{R_t}\{Z(\tau,x), W(\tau,x)\}$$
$$R_t =: \{(\tau,x) \mid 0 \leq \tau \leq t, x \in R\}$$

We now prove it by contradiction. In fact, if $T < \infty$, then without loss of generality, we may assume that $W$ achieves $N$ at a point $(T,X)$, $W(T,X) = N > 0$. Since $T > 0$, there exists $\varepsilon > 0$, such that

$$W(t, x_2(t;T,X)) > 0 \quad \text{as } T - \varepsilon < t < T$$

where $x = x_2(t;T,X)$, $0 < t < T$, is the $\mu$-characteristic through $(T,X)$. According to the definition of $T$,

$$Z(t, x_2(t;T,X)) \leq n(t) < N = W(T,X) \quad \text{as } T - \varepsilon < t < T \tag{2.8}$$

On the other hand, solving (2.6), we have

$$\begin{aligned} W(T,X) =& W(t,x_2(t))\exp\left\{-\int_t^T \mu_w W(\tau,x_2(\tau))d\tau\right\} \\ &- \int_t^T \mu_z Z W(\tau,x_2(\tau))\exp\left\{-\int_t^T \mu_w W(s,x_2(s))ds\right\}d\tau \end{aligned} \tag{2.9}$$

Since

$$\begin{aligned} &-\int_t^T \mu_w W(\tau,x_2(\tau))\exp\left\{-\int_t^T \mu_w W(s,x_2(s))ds\right\}d\tau \\ &= \exp\left\{-\int_t^T \mu_w W(\eta,x_2(\eta))d\eta\right\} - 1 \end{aligned} \tag{2.10}$$

multiplying the both sides of (2.10) by $W(T,X)$, then

$$\begin{aligned} W(T,X) =& W(T,X)\exp\left\{-\int_t^T \mu_w W(\tau,x_2(\tau))d\tau\right\} \\ &+ \int_t^T \mu_w W(\tau,x_2(\tau))W(T,X))\exp\left\{-\int_t^T \mu_w W(s,x_2(s))ds\right\}d\tau \end{aligned} \tag{2.11}$$

Combining (2.9) with (2.11), we get

$$W(T,X) = W(t,x_2(t)) - \int_t^T [\mu_z Z(\tau,x_2(\tau)) + \mu_w(\tau,x_2(\tau))W(T,X)]\, W(\tau,x_2(\tau)) \cdot \exp\left\{-\int_\tau^t \mu_w W(s,x_2(s))ds\right\} d\tau$$

In view of (A) and $Z \geq 0$,

$$W(T,X) \leq W(t,x_2(t)) + \int_t^T [Z(\tau,x_2(\tau)) - W(T,X)]\mu_w W(\tau,x_2(\tau)) \cdot \exp\left[-\int_\tau^t \mu_w W(s,x_2(s))ds\right\} d\tau \tag{2.12}$$

Substituting (2.8) into (2.12), by (2.1), we have

$$W(T,X) \leq W(t,x_2(t)) \quad \text{as } T-\varepsilon < t < T$$

It contradicts the definition of $T$, hence $T = \infty$, i.e. (2.5) hold.

We now prove the *a priori* estimate (2.4). By (E), $w' = 0$, and

$$z' = z_t + \mu z_x = (z_t + \lambda z_x) + (\mu - \lambda) z_x = G z_x$$

thus

$$G' = G_z z' + G_w w' = G_z z_x G \tag{2.13}$$

If (2.4) fails, in view of (2.2), it is easy to prove there exists a point $(T,X)$, $0 < T < \infty$, such that $(z(T,X), w(T,X)) \in \delta D$, and $(z(t,x), w(t,x)) \subset D$ as $0 \leq t < T$, $x \in R$. If $(z(T,X), w(T,X)) \in C_0$, i.e. $G(T,X) = 0$, and $0 < G(t,x) < \infty$, as $0 \leq t < T$, $x \in R$. Solving (2.13), we have

$$0 = G(T,X) = G(0,x_2(0)) \exp\left\{\int_0^T G_z z_x(\tau,x_2(\tau))d\tau\right\}$$

where $x_2(t) =: x_2(t;T,X)$. By (B) and (2.5), $G(0,x_2(0)) = 0$. It contradicts (2.2). This contradiction implies $(z(T,X), w(T,X)) \overline{\in} C_0$. Similarly, $(z(T,X), w(T,X)) \overline{\in} C_\infty$. So (2.4) holds.

**Note** It is easy to check that the assumptions to the system (E) hold for the equations of isentropic gas dynamics in Eulerian coordinates and in Lagrangian coordinates under the ordinary assumptions to the state equation (see [15]).

## 3. Examples

The following example shows that once the assumption (B) is violated, "vacuum" may occur at a later time.

**Example 1.** We consider the system

$$u_t + \sigma(v)_x = 0, \quad v_t - u_x = 0 \tag{3.1}$$

where $\sigma(v) = v^\gamma$, $v > 0, \gamma > 1$. Thus $\sigma'(v) = \gamma v^{\gamma-1} > 0$. $\sigma''(v) = \gamma(\gamma-1)v^{\gamma-2} > 0$. The eigenvalues of the Jacobian matrix of (3.1) are $\lambda(v) = -\sqrt{\gamma}v^{\frac{\gamma-1}{2}}$ and $\mu(v) = -\lambda(v)$. The system is strictly hyperbolic in the region $D_1 =: \{(u,v) \mid v > 0, u \in R\}$. The Riemann invariants are taken as $z = -u - \Phi(v), w = -u + \Phi(v)$, where $\Phi(v) =: \int_0^v \mu(v)dv = Av^{\frac{\gamma+1}{2}}$, $A =: 2\sqrt{\gamma}/(\gamma+1)$. Then $u = -\frac{1}{2}(w+z)$, $\Phi = \frac{1}{2}(w-z)$, $\mu(v) = C(w-z)^{\frac{\gamma-1}{\gamma+1}}$, where $C = \sqrt{\gamma}(2A)^{\frac{\gamma-1}{\gamma+1}}$. The Riemann invariants give a smooth bijective mapping from $D_1$ to the region $D = \{(z,w) \mid w - z > 0\}$. Since $\mu_v = \frac{\gamma-1}{2}\mu v^{-1}$, $v_\Phi = \mu^{-1}$, then $\mu_w = \mu_v v_\Phi \Phi_w = \frac{\gamma-1}{4v} > 0$ and $\mu_z = \lambda\mu = -\lambda_z = -\mu_w$. The conditions (2.1) and (A) hold, but the condition (B) is violated, $G_w = -G_z = \frac{2C(\gamma-1)}{\gamma+1}(w-z)^{\frac{-2}{\gamma+1}} \to \infty$, as $w - z \to 0$. The initial data are taken as

$$u_0(x) = \begin{cases} LN & \text{as } x < -N \\ -Lx & \text{as } |x| \le N \\ -LN & \text{as } x > N \end{cases}$$
$$v_0(x) = v_0 \qquad \text{as } x \in R$$

where $L, N$ and $v_0$ are positive constants, then

$$\Phi(v(0,x)) = \Phi_0 =: Av_0^{\frac{\gamma+1}{2}} > 0$$

$$(z_0(x), w_0(x)) = \begin{cases} (-LN - \Phi_0, -LN + \Phi_0) & \text{as } x < -N \\ (Lx - \Phi_0, Lx + \Phi_0) & \text{as } |x| \le N \\ (LN - \Phi_0, LN + \Phi_0) & \text{as } x > N \end{cases}$$

The conditions (2.2) and (M) hold.

We now consider the determinate region, denoted by $B_N$, of the interval $(-N, N)$ on the initial line $t = 0$. It is easy to check

$$u(t,x) = -Lx, \quad v(t,x) = v_0 - Lt, \quad (t,x) \in B_N$$

is a solution in $B_N$. In view of the fact that $v(t,x)$ is independent of $x$,

$$\mu(t) =: \mu(v(t,x)) = \Phi'(v) = \Phi'(t)/v'(t) = -\Phi'(t)/L \text{ in } B_N \text{ thus}$$
$$x_2'(t) = -\Phi'(t)/L \quad \text{in } B_N \tag{3.2}$$

where $x_2(t)$ is a $\mu$-characteristic in $B_N$. Integrating (3.2) along $\mu$-characteristics, we have

$$x_2(t) = x_2(0) + L^{-1}[\Phi_0 - \Phi(t)] \quad \text{in } B_N$$

Similarly,

$$x_1(t) = x_1(0) - L^{-1}[\Phi_0 - \Phi(t)] \quad \text{in } B_N$$

The left (right) side of $B_N$ is the $\mu$-characteristic ($\lambda$-characteristic) issued from the initial point $(t,x) = (0,-N)$, $((t,x) = (0,N))$, which is denoted by $x_2^-(t), (x_1^+(t))$. $x_2^-(t) = -N + L^{-1}(\Phi_0 - \Phi(t))$, $x_1^+ = N - L^{-1}(\Phi_0 - \Phi(t))$. When $N > L^{-1}\Phi_0$, the upper side of $B_N$ is the horizontal segment: $b_0 = \{(t,x) \mid t = L^{-1}v_0, |x| \le N - L^{-1}\Phi_0\}$. On $b_0$, $v = \mu = \Phi = G = 0$. "Vacuum states" occur on $b_0$.

The solution is continuous in $P$ and consists of constant states and simple waves in $P\backslash B_N$, where $P =: \{(t,x) \mid 0 \le t < L^{-1}v_0, x \in R\}$.

**Note 1** It is quite interesting to study the following cases, when $\gamma = 1$, $0 < \gamma < 1$ and general $\sigma(v)$.

**Note 2** When $t > L^{-1}v_0$, the solution depends on the definition of $\sigma(v)$ in the region $v \le 0$.

The second example shows that once the assumption (A) is violated, no matter how narrowly, "vacuum states" may occur at a later time, and waves behave in a really singular way.

**Example 2** We consider the system

$$u_t - \left(\frac{1}{2}u^2 - p(v)\right)_x = 0, \quad v_t - (uv)_x = 0$$

where $p(v) = \gamma^{-1}v^{-\gamma}$, $v > 0$, $\gamma > 2$. The eigenvalues of the Jacobian matrix of the system are $\lambda = -u - h(v)$, $\mu = -u + h(v)$, where $h(v) = v^{-\frac{\gamma}{2}} > 0$. The Riemann invariants are taken as $z = u/2a - h/2b$, $w = u/2a + h/2b$, where $a = 2(2+\gamma)^{-1}$, $b = \gamma(2+\gamma)^{-1}$. Then $u = a(w+z)$, $h = b(w-z)$; $\lambda = -w + cz$, $\mu = cw - z$, where $0 < c =: b - a < 1$. $G =: \mu - \lambda = 2h(v) = 2b(w-z) > 0$ $(= 0)$ if and only if $w - z > 0$ $(= 0)$. $\lambda_w = \mu_z = -1$, $\lambda_z = \mu_w = c > 0$, $\lambda_z + \lambda_w = \mu_z + \mu_w = c - 1 < 0$. Thus the conditions (2.1) and (B) hold, and the condition (A) is violated. The initial data are taken as

$$u_0(x) = \begin{cases} -LN & \text{as } x \le -N \\ Lx & \text{as } |x| \le N \\ LN & \text{as } x \ge N \end{cases}$$

$$v_0(x) = v_0 \qquad \text{as } x \in R$$

where $L, N, v_0$ are positive constants, then

$$h(v(0,x)) = h_0 =: v_0^{-\frac{\gamma}{2}} > 0 \quad x \in R$$

$$(z_0(x), w_0(x)) = \begin{cases} \left(-AN - h_0(2b)^{-1}, -AN + h_0(2b)^{-1}\right) & \text{as } x \le -N \\ \left(Ax - h_0(2b)^{-1}, Ax + h_0(2b)^{-1}\right) & \text{as } |x| \le N \\ \left(AN - h_0(2b)^{-1}, AN + h_0(2b)^{-1}\right) & \text{as } x \ge N \end{cases}$$

where $A = L(2a)^{-1}$. The conditions (2.2) and (M) hold.

We now consider the determinate region, denoted by $B_N$, of the interval $(-N, N)$ on the initial line $t = 0$. It is easy to check

$$u(t,x) = Lx/(1-Lt), \quad v(t,x) = v_0/(1-Lt) \quad \text{in } B_N$$

is a solution in $B_N$. The equations of $\lambda$-characteristics and $\mu$-characteristics are

$$x_1(t) = \left[x_1(0) - \frac{2}{\gamma L}(h_0 - h(t))\right](1-Lt)^{-1} \to 0 \quad \text{as } t \to L^{-1}$$

$$x_2(t) = \left[x_2(0) + \frac{2}{\gamma L}(h_0 - h(t))\right](1-Lt)^{-1} \to 0 \quad \text{as } t \to L^{-1}$$

respectively. Along $\lambda$-characteristics and $\mu$-characteristics,

$$u(t,x_1(t)) = Lx_1(0) - 2\gamma^{-1}(h_0 - h(t)) \to Lx_1(0) - 2h_0\gamma^{-1} \quad \text{as } t \to L^{-1}$$

$$u(t,x_2(t)) = Lx_2(0) + 2\gamma^{-1}(h_0 - h(t)) - Lx_2(0) + 2h_0\gamma^{-1} \quad \text{as } t \to L^{-1}$$

$$\lambda(t,x_1(t)) = -u(t,x_1(t)) - h(t) \to -Lx_1(0) + 2h_0\gamma^{-1} \quad \text{as } t \to L^{-1}$$

$$\mu(t,x_2(t)) = -u(t,x_2(t)) + h(t) \to -Lx_2(0) - 2h_0\gamma^{-1} \quad \text{as } t \to L^{-1}$$

When $x_1(0) - x_2(0) = 4h_0\gamma^{-1}L^{-1}$,

$$\mu(t,x_2(t)) - \lambda(t,x_1(t)) \to 0 \quad \text{as } t \to L^{-1}$$

Hence, when $N > 2h_0\gamma^{-1}L^{-1}$, in the strip $\{(t,x) \mid 0 \le t \le L^{-1}, x \in R\}$, the solution is continuous and consists of constant states and simple wave outside the region $B_N$, but the point $P: (t,x) = (L^{-1}, 0)$ is a singular point. "Vacuum states" occur at $P$, and $z(L^{-1}, 0-) > z(L^{-1}, 0+), w(L^{-1}, 0-) > w(L^{-1}, 0+)$. It is quite interesting to study the solution in the upper half plane $t > L^{-1}$, and the cases $\gamma = 2,\ 0 < \gamma < 2$.

The third example shows that the additional assumption (A) is not a necessary condition for the nonoccurance of "vacuum". The system describes the macroscopic behaviour of some bacterial populations, which are attracted by a chemical substrate [16, 17]

**Example 3** We consider the system

$$u_t - (uv)_x = 0, \quad v_t - u_x = 0 \tag{3.3}$$

the eigenvalues of the Jacobian matrix of the system and $\lambda(u) = \frac{1}{2}(-v - \Delta^{\frac{1}{2}}), \mu(u) = \frac{1}{2}(-v + \Delta^{\frac{1}{2}})$, where $\Delta = v^2 + 4u$. The system is strictly hyperbolic in the region $D_1 = \{(u,v) \mid u > 0\}$, $\lambda < 0 < \mu$ in $D_1$. Following Zheng [17], the Riemann invariants are taken as $z = \mu^{\frac{1}{2}}(3\lambda + \mu)$, $w = (-\lambda)^{\frac{1}{2}}(\lambda + 3\mu)$, which give a bijective smooth mapping from $D_1$ onto the region $D =: H \backslash D_+$, where $H =: \{(z,w) \mid z \in R, w \in R\}$, $D_+ =: \{(z,w) \mid z \ge 0, w \ge 0\}$. It is easy to check

$$\lambda_z = -\frac{4}{3}\lambda\mu^{\frac{1}{2}}\Delta^{-1} > 0, \quad \mu_w = \frac{4}{3}\mu(-\lambda)^{\frac{1}{2}}\Delta^{-1} > 0$$

and the assumptions to the system (E) in Section 2 hold for the system (3.3), but the additional condition (A) is violated, $\lambda_z + \lambda_w = 2/3(-\lambda)^{\frac{1}{2}}\Delta^{-1}(2u^{\frac{1}{2}} + v)$, $\mu_z + \mu_w = \frac{2}{3}\mu^{\frac{1}{2}}\Delta^{-1}(2u^{\frac{1}{2}} - v)$. Zheng proved that the above theorem in Section 2 holds for the system [17].

## References

[1] Lin L.W., On the vacuum states for the equations of isentropic gas dynamics, *J. Math. Anal. Appl.*, **121** (1987), 406–425.

[2] Lin L.W., Vacuum states and equidistribution of the random sequence for Glimm's scheme, *J. Math. Anal. Appl.*, **124**(1) (1987), 117–126.

[3] Lin L.W. and Hiseh F.P., Characteristics method in a certain macroscopic sense for Glimm's scheme, *Comm. Part. Diff. Eqs.* **15** (1990), 1179–1197.

[4] Lin L.W. and Yang T., Convergence of the Lax-Friedrichs' scheme for the equations of isentropic gas dynamics in Lagrangian coordinates (to appear in *Comm. Part. Diff. Eqs.*).

[5] Lin L.W. and Yang T., Convergence of the viscosity method for the equations of isentropic gas dynamics in Lagrangian coordinates (to appear).

[6] Liu T.P. and Smoller J.A., On the vacuum state for the isentropic gas dynamics equations, *Adv. Appli. Math.*, **1** (1980), 345–359.

[7] DiPerna R., Convergence of the viscosity method for isentropic gas dynamics, *Comm. Math. Phys.*, **91** (1983), 1–30.

[8] Ding X.X., Chen G.Q. and Luo P.Z., Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics, *Acta Math. Scien.* (1), **5** (1985), 415–432, (2), **5** (1985), 433–472.

[9] Chen G.Q., Convergence of the Lax-Friendrichs scheme for isentropic gas dynamics, *Acta Math. Scien.* (3), **6** (1986), 75–120.

[10] DiPerna R., Convergence of approximate solutions to convervation laws, *Arch. Rat. Mech. Anal.* **82** (1983), 27–70.

[11] Lax P.D., Development of singularities of solutions of nonlinear hyperbolic partial differential equations, *J. Math. Phys.*, **5** (1964), 611–613.

[12] Smoller J., Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, New York Inc., 1983.

[13] Li T.T. and Zhao Y.C., Vacuum problem of the systems of isentropic gas dynamics in one dimension, *Chinese Quar. J. Math.*, **1** (1986), 41–45.

[14] Lax P.D., Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.*, **10** (1957), 537–566.

[15] Courant R. and Friedrichs K.O., Supersonic Flow and Shock Waves, Wiley-Interscience, New York, 1948.

[16] Rascle M., The Riemann problem for a nonlinear non-strictly hyperbolic system arising in biology, *Comp. Maths. Appls.*, **11** (1985), 223–238.

[17] Zheng Y.S., Development of singularities of solutions of nonlinear non-strictly hyperbolic system arising in biology (to appear).