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GLOBAL RESOLVABILITY FOR QUASILINEAR HYPERBOLIC SYSTEMS*

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Abstract In this paper, we consider the globally smooth solutions of diagonalizable systems consisted of n-equations. We give a sufficient condition which guarantees the global existence of smooth solutions. The techniques used in this paper can be applied to study the globally smooth (or continuous) solutions diagonalizable nonstrict hyperbolic conversation laws.

Key Words Global resolvability; maximum principle; function transformation.
Classification 35L65, 35L45, 35L60.

1. Introduction

It is well-known that the classical solutions of Cauchy problems for quasilinear hyperbolic systems, generally speaking, exist only locally in time and will occur singularities in finite time, even if the initial data are sufficiently smooth and small ([1]–[3]). However, there are certain examples of globally defined classical solutions ([4]). Hence it is of interesting to determine the conditions which guarantee the existence of globally classical solutions.

Under the case of diagonalizable 2×2 systems, systematic results have been obtained ([5]–[8]). The diagonalizable systems consisted of n-equations ($n > 2$) were studied first by D. Hoff ([9]). By using an inequality given by Rozdestvenskii in [10], paper [11] also gets the same results as that of [9] under less restrictions on the initial data. The method used in [9] and [11] require that the systems under consideration are strictly hyperbolic.

In this paper, we also consider the globally smooth solutions of diagonalizable systems consisted of n-equations. We give a sufficient condition (2.6) and two of its more applicable cases (2.31), (2.32) which guarantee the global existence of smooth solutions. In our analysis, we do not require the systems considered are strictly hyperbolic, and when the systems considered are strictly hyperbolic, the result in [9] or [11] is direct

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corollary of our results. Our analysis also indicates that our function transformation is a generalization of the function transformation proposed by P. Lax in [3].

The techniques used in this paper can also be applied to study the globally smooth (or continuous) solutions to diagonalizable nonstrict hyperbolic conservation laws. As an example, we consider the globally smooth (or continuous) solutions to isentropic gas dynamics systems in Euler coordinates.

2. An Existence Theorem of Globally Smooth Solutions for the Diagonalizable Systems of n-Equations

Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial r^{(k)}}{\partial t} + \lambda^{(k)}(t,x,r)\dfrac{\partial r^{(k)}}{\partial x} = 0 \\ r^{(k)}|_{t=0} = r_0^{(k)}(x) \end{cases} \quad (k=1,2,\cdots,n) \tag{2.1}$$

where $r = \left(r^{(1)},\cdots,r^{(n)}\right)$, $t \in R_+$, $x \in R$, n-positive integer, and

$$\Omega = \left\{(t,x,r)\middle| |r^{(k)}| \leq M,\ 0 \leq t \leq T,\ |x| < \infty,\ k=1,2,\cdots,n\right\} \tag{2.2}$$

Suppose that

(i) $r_0^{(k)}(x) \in C^1(R)$, and

$$\left\|r_0^{(k)}\right\|_{C^1(R)} \leq M \tag{2.3}$$

(ii) $\lambda^{(k)}(t,x,r)$ are continuously differentiable functions on the domain Ω and

$$\left|\lambda^{(k)}\right| \leq N,\ 0 \leq \frac{\partial \lambda^{(k)}}{\partial x} \leq N,\ \left|\frac{\partial \lambda^{(k)}}{\partial r^{(j)}}\right| \leq N,\ j,k=1,2,\cdots,n \tag{2.4}$$

where M, N are positive constants, T is any given positive number.

We have the following theorem

Theorem 2.1 *Assume that the k-th characteristic field for the equation of the Cauchy problem* (2.1) *and corresponding initial data satisfy*

(iii) $\dfrac{\partial \lambda^{(k)}}{\partial r^{(k)}} \geq 0,\ \dfrac{dr_0^{(k)}(x)}{dx} \geq 0,\ k=1,2,\cdots,n$

and conditions (i) *and* (ii) *are satisfied.*

If there exist n functions $p^{(k)}\left(r^{(1)},\cdots,r^{(n)}\right)$ $(k=1,2,\cdots,n)$ *satisfying*

$$p^{(k)}\left(r^{(1)},\cdots,r^{(n)}\right) \in C^1(R^n),\ p^{(k)}\left(r^{(1)},\cdots,r^{(n)}\right) > 0 \tag{2.5}$$

$$\frac{1}{n-1}p^{(k)}p^{(k)}\lambda_k^{(k)} + p^{(k)}p^{(j)}\lambda_j^{(j)} + p^{(j)}\left(p^{(k)}\left(\lambda^{(k)}-\lambda^{(j)}\right)\right)_{,j} \geq 0 \tag{2.6}$$

where $\lambda_j^{(k)} = \dfrac{\partial \lambda^{(k)}}{\partial r^{(j)}},\ p_j^{(k)} = \dfrac{\partial p^{(k)}}{\partial r^{(j)}},\ k,j=1,2,\cdots,n,$

then the Cauchy problem (2.1) admits a unique global smooth solution.

Proof According to the well-known result on the existence of local smooth solutions to the Cauchy problem for first order quasilinear hyperbolic system ([12]), for the diagonal system (2.1), there exists a constant t_1 such that on the domain

$$R(t_1) = \{(t,x) \mid 0 \le t \le t_1,\ |x| < +\infty\}$$

the Cauchy problem (2.1) possesses a unique smooth solution $r(t,x) = \left(r^{(1)}, \cdots, r^{(n)}\right)(t,x)$ provided that the initial datum $r_0(x) = \left(r_0^{(1)}, \cdots, r_0^{(n)}\right)(x)$ is a smooth function with bounded C^1-norm. Where t_1 depends only on the C_1-norm of the initial data. To complete the proof of Theorem 2.1, we only need to get the *a priori* estimates on $r^{(i)}, r_x^{(i)}$ $(i = 1, 2, \cdots, n)$.

The system of characteristic equations for the problem (2.1) is as follows

$$\begin{cases} \dfrac{dx^{(k)}\left(t,\beta^{(k)}\right)}{dt} = \lambda^{(k)}\left(t, x^{(k)}\left(t,\beta^{(k)}\right), r\left(t, x^{(k)}\left(t,\beta^{(k)}\right)\right)\right) \\ x^{(k)}\left(0,\beta^{(k)}\right) = \beta^{(k)} \end{cases} \tag{2.7$_1$}$$

$$\begin{cases} \dfrac{dr^{(k)}\left(t, x^{(k)}\left(t,\beta^{(k)}\right)\right)}{dt} = 0 \qquad k = 1, 2, \cdots, n \\ r^{(k)}\left(0,\beta^{(k)}\right) = r_0^{(k)}\left(\beta^{(k)}\right) \end{cases} \tag{2.7$_2$}$$

$(2.7)_1$, $(2.7)_2$ gives

$$r^{(k)}\left(t, x^{(k)}\left(t,\beta^{(k)}\right)\right) = r_0^{(k)}\left(\beta^{(k)}\right) \tag{2.8}$$

Differentiating $(2.7)_1$, $(2.7)_2$ on both side with respect to $\beta^{(k)}$ we get that

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial x^{(k)}}{\partial \beta^{(k)}}\right) = \left(\frac{\partial \lambda^{(k)}}{\partial x} + \sum_{j\ne k} \frac{\partial \lambda^{(k)}}{\partial r^{(j)}} \times \frac{\partial r^{(j)}}{\partial x^{(j)}}\right)\frac{\partial x^{(k)}}{\partial \beta^{(k)}} + \frac{\partial \lambda^{(k)}}{\partial r^{(k)}} \times \frac{\partial r^{(k)}}{\partial \beta^{(k)}} \\ &\frac{d}{dt}\left(\frac{\partial r^{(k)}}{\partial \beta^{(k)}}\right) = 0,\ \left.\frac{\partial x^{(k)}}{\partial \beta^{(k)}}\right|_{t=0} = 1,\ \left.\frac{\partial r^{(k)}}{\partial \beta^{(k)}}\right|_{t=0} = r_0^{(k)'}\left(\beta^{(k)}\right) \end{aligned} \tag{2.9}$$

Therefore we obtain

$$\begin{aligned} \frac{\partial x^{(k)}}{\partial \beta^{(k)}} &= \exp\left\{\int_0^t \left(\frac{\partial \lambda^{(k)}}{\partial x} + \sum_{j\ne k} \frac{\partial \lambda^{(k)}}{\partial r^{(j)}} \times \frac{\partial r^{(j)}}{\partial x^{(j)}}\right) d\tau\right\} \\ &\cdot \left[1 + \int_0^t \frac{\partial \lambda^{(k)}}{\partial r^{(k)}} r_0^{(k)'}\left(\beta^{(k)}\right) \exp\left\{-\int_0^\tau \left(\frac{\partial \lambda^{(k)}}{\partial x} + \sum_{j\ne k} \frac{\partial \lambda^{(k)}}{\partial r^{(j)}} \times \frac{\partial r^{(j)}}{\partial x^{(j)}}\right) ds\right\} d\tau\right] \end{aligned} \tag{2.10}$$

From (2.8)–(2.10) and the assumption (iii), we have

$$\left|r^{(k)}\right| \le M,\ \frac{\partial r^{(k)}}{\partial x} \ge 0, \quad (k = 1, 2, \cdots, n) \tag{2.11}$$

Now we turn to estimate the upper bound of $\dfrac{\partial r^{(k)}}{\partial x}$.

Differentiating (2.1) on both sides in x, setting $w^{(k)} = \dfrac{\partial r^{(k)}}{\partial x}$, we have

$$\frac{\partial w^{(k)}}{\partial t} + \lambda^{(k)}(t,x,r)\frac{\partial w^{(k)}}{\partial x} + \sum_{j=1}^{n} \lambda_j^{(k)} w^{(j)} w^{(k)} + \frac{\partial \lambda^{(k)}}{\partial x} w^{(k)} = 0, \ (k = 1, 2, \cdots, n) \quad (2.12)$$

Making the transformation

$$w^{(k)} = p^{(k)}\left(r^{(1)}, \dots, r^{(n)}\right) z^{(k)}, \quad k = 1, 2, \cdots, n \quad (2.13)$$

we get

$$p^{(k)}\left(z_t^{(k)} + \lambda^{(k)} z_x^{(k)}\right) + \sum_{j=1}^{n}\Big(p_j^{(k)}\left(\lambda^{(k)} - \lambda^{(j)}\right)$$
$$+ \lambda_j^{(k)} p^{(k)}\Big) p^{(j)} z^{(k)} z^{(j)} + \frac{\partial \lambda^{(k)}}{\partial x} p^{(k)} z^{(k)} = 0 \quad (2.14)$$

Let $\alpha^{(k)} = \inf\limits_{r\in\Omega} p^{(k)}\left(r^{(1)}, \cdots, r^{(n)}\right)$, $\theta^{(k)} = \sup\limits_{r\in\Omega} p^{(k)}\left(r^{(1)}, \cdots, r^{(n)}\right)$. From (2.5), we have

$$0 < \alpha^{(k)} \le \theta^{(k)} < +\infty \quad (2.15)$$

Hence

$$0 \le z^{(k)}(0,x) = w^{(k)}(0,x)\left[p^{(k)}\left(r^{(1)}, \cdots, r^{(n)}\right)(0,x)\right]^{-1} \le M\left(\alpha^{(k)}\right)^{-1} \le C$$

where

$$C = \operatorname*{Max}_{1\le i\le n}\left\{M\left(\alpha^{(i)}\right)^{-1}\right\} > 0$$

Let

$$z^{(k)}(t,x) = \delta^{(k)}(t,x) + C + d\left(x^2 + c_1 L e^t\right)/L^2, \ k = 1, 2, \cdots, n \quad (2.16)$$

where $d = \operatorname*{Max}\limits_{1\le k\le n} \sup\left|z^{(k)}(t,x)\right|$, c_1, L are positive constants to be decided below. Then, $\delta^{(k)}(t,x)$ satisfy the following initial-boundary problem

$$\begin{cases} p^{(k)}\left(\delta_t^{(k)} + \lambda^{(k)}\delta_x^{(k)}\right) + p^{(k)}\left(c_1 L e^t + 2x\lambda^{(k)}\right) d/L^2 \\ \quad + \dfrac{\partial \lambda^{(k)}}{\partial x} p^{(k)} z^{(k)} + \sum\limits_{j\ne k}\Big(\dfrac{1}{n-1}\lambda_k^{(k)}\left(p^{(k)}\right)^2 + p^{(k)} p^{(j)} \lambda_j^{(j)} + p^{(j)}\Big(p^{(k)}(\lambda^{(k)} \\ \quad - \lambda^{(j)})_j\Big) z^{(k)} z^{(j)}\Big) + \sum\limits_{j\ne k} \dfrac{1}{n-1}\left(p^{(k)}\right)^2 \lambda_k^{(k)}\left(\delta^{(k)} - \delta^{(j)}\right) z^{(k)} = 0 \\ \delta^{(k)}(0,x) = z^{(k)}(0,x) - C - d\left(x^2 + c_1 L e^t\right)/L^2 < 0 \\ \delta^{(k)}(t,\pm L) = z^{(k)}(t,\pm L) - C - d\left(L^2 + c_1 L e^t\right)/L^2 < 0, \ k = 1, 2, \cdots, n \end{cases} \quad (2.17)$$

We conclude that $\delta^{(k)}(t,x)$ decided by (2.17) satisfy

$$\delta^{(k)}(t,x) < 0 \quad \text{on} \quad (t,x) \in (0,T)\times(-L,+L) \quad (2.18)$$

for any given $T > 0$.

Otherwise, let $\bar{t} = \sup\left\{t \mid \delta^{(k)}(t,x) < 0,\ k = 1, 2, \cdots, n, \forall x \in (-L, L)\right\}$, then, $0 < \bar{t} < +\infty$. We have from continuity that, without loss of generality, there exists $(\bar{t}, \bar{x})$ with $-L < \bar{x} < L$, such that $\delta^{(k)}(\bar{t},\bar{x}) = 0$, $\delta^{(j)}(\bar{t},\bar{x}) \le 0$ $(j \ne k)$ and $\delta_t^{(k)}(\bar{t},\bar{x}) \ge 0$, $\delta_x^{(k)}(\bar{t},\bar{x}) = 0$, namely

$$\left(\delta_t^{(k)} + \lambda^{(k)}\delta_x^{(k)}\right)\Big|_{(\bar{t},\bar{x})} \ge 0 \tag{2.19}$$

$$\left(\delta^{(k)} - \delta^{(j)}\right)\Big|_{(\bar{t},\bar{x})} \ge 0 \tag{2.20}$$

Now we choose c_1 sufficiently large satisfying

$$c_1 L + 2x\lambda^{(k)} > 0 \quad \text{on} \quad (0,t) \times (-L, L) \tag{2.21}$$

In virtue of (2,5), (2.6), (2.11)–(2.13), (2.19)–(2.21), we conclude that the left side of $(2.17)_1$ is strictly positive. This is a contradiction. (2.18) is proved.

(2.17), (2.19) gives

$$z^{(k)}(t_0, x_0) \le C + d\left(x_0^2 + c_1 L e^t\right)/L^2, \quad (t_0, x_0) \in (0,T) \times (-L, L) \tag{2.22}$$

we have

$$z^{(k)}(t,x) \le C \tag{2.23}$$

by letting $L \to +\infty$.

From (2.5), (2.13) and (2.23), we have

$$\frac{\partial r^{(k)}}{\partial x} = w^{(k)} \le C\theta^{(k)}$$

This completes the proof of Theorem 2.1.

If one of the characteristic fields, say $\lambda^{(k)}$, is linearly degenerate on Ω, then we have

Theorem 2.2 *Assume that one of the characteristic fields, say $\lambda^{(k)}$, is linearly degenerate on Ω and there exist n functions $p^{(i)}(r^{(1)}, \cdots, r^{(n)})$ $(i = 1, 2, \cdots, n)$ which satisfy* (2.5) *and*

$$p_j^{(i)}\left(\lambda^{(i)} - \lambda^{(j)}\right) + \lambda_j^{(i)} p^{(i)} \ge 0, \quad i, j = 1, 2, \cdots, n \tag{2.24}$$

then, under the conditions (i), (ii) *and*

(iii)′ $\dfrac{\partial \lambda^{(i)}}{\partial r^{(i)}} \ge 0$, $\dfrac{dr_0^{(j)}(x)}{dx} \ge 0$, $i = 1, 2, \cdots, n$; $j = 1, \cdots, k-1, k+1, \cdots, n$

the Cauchy problem (2.1) *admits a unique globally smooth solution.*

Proof From Theorem 2.1, we only need to establish *a priori* estimates on $z^{(i)}$, $i = 1, 2, \cdots, n$.

Similar to that of Theorem 2.1, we can obtain

$$z^{(i)} \ge 0, \ i = 1, 2, \cdots, k-1, k+1, \cdots, n \tag{2.25}$$

(2.14) yields

$$z_t^{(k)} + \lambda^{(k)} z_x^{(k)} + \frac{1}{p^{(k)}} \sum_{j\neq k} \Big(p_j^{(k)}\left(\lambda^{(k)} - \lambda^{(j)}\right) + \lambda_j^{(k)} p^{(k)}\Big) p^{(j)} z^{(j)} z^{(k)} + \frac{\partial \lambda^{(k)}}{\partial x} z^{(k)} = 0 \tag{2.26}$$

Hence

$$z^{(k)} = z_0^{(k)} \exp\left\{ - \int_0^t \left(\frac{1}{p^{(k)}} \sum_{j\neq k} \left(p_j^{(k)}\left(\lambda^{(k)} - \lambda^{(j)}\right) + \lambda_j^{(k)} p^{(k)}\right) p^{(j)} z^{(j)} + \frac{\partial \lambda^{(k)}}{\partial x}\right) d\tau \right\} \tag{2.27}$$

Combining (2.24), (2.25) with (2.27), we obtain

$$\left|z^{(k)}\right| \le \left|z_0^{(k)}\right| \le C \tag{2.28}$$

For $i \neq k$, we have

$$z_t^{(i)} + \lambda^{(i)} z_x^{(i)} + \frac{1}{p^{(i)}} \sum_{j\neq k} \left(p_j^{(i)}\left(\lambda^{(i)} - \lambda^{(j)}\right) + \lambda_j^{(i)} p^{(i)}\right) p^{(j)} z^{(j)} z^{(i)} + \frac{1}{p^{(i)}} \left(p_k^{(i)}\left(\lambda^{(i)} - \lambda^{(k)}\right) + \lambda_k^{(i)} p^{(i)}\right) p^{(k)} z^{(k)} z^{(i)} + \frac{\partial \lambda^{(i)}}{\partial x} z^{(i)} = 0, \quad i \neq k \tag{2.29}$$

then

$$\begin{aligned} z^{(i)} &= z_0^{(i)} \exp\left\{ - \int_0^t \left[\frac{1}{p^{(i)}} \sum_{j=1}^{n} \left(p_j^{(i)}\left(\lambda^{(i)} - \lambda^{(j)}\right) + \lambda_j^{(i)} p^{(i)}\right) p^{(j)} z^{(j)} + \frac{\partial \lambda^{(i)}}{\partial x}\right] d\tau \right\} \\ &\le z_0^{(i)} \exp\left\{ - \int_0^t \frac{p^{(k)}}{p^{(i)}} \left(p_k^{(i)}\left(\lambda^{(i)} - \lambda^{(k)}\right) + \lambda_k^{(i)} p^{(i)}\right) z^{(k)} d\tau \right\} \\ &\le z_0^{(i)} e^{-\overline{M} t} \qquad \text{(by (2.24), (2.25), (2.28) and (iii)}') \end{aligned} \tag{2.30}$$

where $\overline{M} = \sup\limits_{\Omega} \left(p^{(k)}\left(p_k^{(i)}\left(\lambda^{(i)} - \lambda^{(k)}\right) + \lambda_k^{(i)} p^{(i)}\right) / p^{(i)}\right) \cdot C$

Theorem 2.2 follows from (2.25), (2.23) and (2.30).

Corollary 2.3 *Addition to the conditions* (i), (ii) *and* (iii), *we assume that system* (2.1) *is strictly hyperbolic, i.e.*

$$\left|\lambda^{(k)} - \lambda^{(j)}\right| \ge \varepsilon > 0 \quad (j \neq k)$$

then Cauchy problem (2.1) *has a unique globally smooth solution.*

Furthermore, if one of the characteristic fields, say $\lambda^{(k)}$, *is linearly degenerate on* Ω, *then under the conditions* (i), (ii) *and* (iii)$'$, *the Cauchy problem* (2.1) *possesses a unique globally smooth solution.*

Proof Let $p^{(i)} = \exp\left(\sum_{j=1}^{n} l_{ij} r^{(j)}\right)$, $i = 1, 2, \cdots, n$, where

$$l_{ij} = \frac{1}{\varepsilon} a_{ij} \text{sign}\left(\lambda^{(i)} - \lambda^{(j)}\right), \quad a_{ij} = \sup\left|\lambda_j^{(i)}\right|$$

Hence

$$\begin{aligned} & p^{(i)} p^{(j)} \lambda_j^{(j)} + p^{(j)} \left(p^{(i)} \left(\lambda^{(i)} - \lambda^{(j)}\right)\right)_j \\ & = \exp\left(\sum_{j=1}^{n} l_{ij} r^{(j)} + \sum_{i=1}^{n} l_{ji} r^{(i)}\right) \left(l_{ij} \left(\lambda^{(i)} - \lambda^{(j)}\right) + \lambda_j^{(i)}\right) > 0 \end{aligned}$$

From Theorems 2.1, 2.2, we can immediately get Corollary 2.3.

Corollary 2.4 a) *Besides the conditions* (i), (ii) *and* (iii), *we assume further*

$$\frac{1}{n-1} \lambda_i^{(i)} + \lambda_j^{(i)} \geq 0, \quad i, j = 1, 2, \cdots, n \tag{2.31}$$

then Cauchy problem (2.1), (2.2) *admits a unique globally smooth solution.*

b) *If one of the characteristic fields, say* $\lambda^{(k)}$, *is linearly degenerate on* Ω *and*

$$\lambda_j^{(i)} \geq 0, \quad i, j = 1, 2, \cdots, n \tag{2.32}$$

then, under the conditions (i), (ii) *and* (iii)$'$, *the Cauchy problem* (2.1), (2.2) *admits a unique globally smooth solution.*

Proof Let $p^{(k)} = 1$, then (2.6), (2.24) reduce to (2.31), (2.32) respectively.

Remark 1 The function transformation, we used in Theorem 2.1, Theorem 2.2 is the generalization of the function transformation proposed by P. Lax in (3). In fact, when $n = 2$ and the systems under consideration are strictly hyperbolic, let $g\left(r^{(1)}, r^{(2)}\right)$, $h\left(r^{(1)}, r^{(2)}\right)$ be the Lax transformations, i.e.

$$\frac{\partial g}{\partial r^{(2)}} = \frac{\dfrac{\partial \lambda^{(1)}}{\partial r^{(2)}}}{\lambda^{(1)} - \lambda^{(2)}}$$

$$\frac{\partial h}{\partial r^{(1)}} = \frac{\dfrac{\partial \lambda^{(2)}}{\partial r^{(1)}}}{\lambda^{(2)} - \lambda^{(1)}}$$

and let $p^{(1)} = \exp(-g)$, $p^{(2)} = \exp(-h)$, we have

$$p_2^{(1)} \left(\lambda^{(1)} - \lambda^{(2)}\right) + \lambda_2^{(1)} p^{(1)} = 0$$

$$p_1^{(2)} \left(\lambda^{(2)} - \lambda^{(1)}\right) + \lambda_1^{(2)} p^{(2)} = 0$$

Hence (2.5), (2.6) (or (2.24)) is automatically satisfied for $n = 2$, and (2.14) reduces to

$$\begin{cases} z_t^{(1)} + \lambda^{(1)} z_x^{(1)} + \lambda_1^{(1)} \left(z^{(1)}\right)^2 p^{(1)} + \lambda_x^{(1)} z^{(1)} = 0 \\ z_t^{(2)} + \lambda^{(2)} z_x^{(2)} + \lambda_2^{(2)} p^{(2)} \left(z^{(2)}\right)^2 + \lambda_x^{(2)} z^{(2)} = 0 \end{cases} \tag{2.33}$$

Then Corollary 2.3 follows immediately. If the system is linearly degenerate (i.e. $\lambda_i^{(i)} = 0,\ i = 1, 2$), we have

Corollary 2.5 *If the strictly hyperbolic system is linearly degenerate and* $\left|\lambda_x^{(i)}\right| \leq N$, *then for any smooth initial data with bounded* C^1 *norm, the conclusion of Corollary 2.3 holds.*

On the other hand, our results indicate that for $n > 2$, the function transformations analogous to the Lax transformation can no longer be existing. For there exist no function p satisfying

$$p_2\left(\lambda^{(1)} - \lambda^{(2)}\right) + \lambda_2^{(1)} p = 0, \quad p_3\left(\lambda^{(1)} - \lambda^{(3)}\right) + \lambda_3^{(1)} p = 0$$

Remark 2 Corollary 2.4 does not require that the system considered to be strictly hyperbolic. In Section 3, we will show how this result is used to consider the globally smooth (continuous) solution to nonstrictly hyperbolic systems.

3. An Example

In this section, we give an example to show how the results in Section 2 can be used to study the globally smooth (continuous) solution of the Cauchy problem of a special nonstrictly hyperbolic model: The isentropic gas dynamics system in Euler coordinates

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + \left(\rho u^2 + k^2 \rho^r\right)_x = 0 \end{cases} \tag{3.1}$$

$$(\rho, u)\mid_{t=0} = (\rho_0(x), u_0(x)) \tag{3.2}$$

where $k > 0$ is a constant, $r > 1$ is the adiabatic exponent.

The eigenvalues of (3.1) are

$$\lambda_1 = u - k r^{\frac{1}{2}} \rho^{\frac{r-1}{2}}, \quad \lambda_2 = u + k r^{\frac{1}{2}} \rho^{\frac{r-1}{2}} \tag{3.3}$$

Corresponding Riemann invariants are

$$w(\rho, u) = u + \frac{2kr^{\frac{1}{2}}}{r-1}\rho^{\frac{r-1}{2}}, \ z(\rho, u) = u - \frac{2kr^{\frac{1}{2}}}{r-1}\rho^{\frac{r-1}{2}} \tag{3.4}$$

respectively.

(3.4) and (3.3) yield

$$\lambda_1 = \frac{3-r}{4}w + \frac{r+1}{4}z, \quad \lambda_2 = \frac{r+1}{4}w + \frac{3-r}{4}z \tag{3.5}$$

$$\lambda_{2w} = \lambda_{1z} = \frac{1}{4}(r+1) > 0, \quad \lambda_{iw} + \lambda_{iz} = 1 > 0, \quad i = 1, 2 \tag{3.6}$$

Hence, from Corollary 2.4 the following Cauchy problem

$$\begin{cases} w_t + \lambda_2(w, z) w_x = 0 \\ z_t + \lambda_1(w, z) z_x = 0 \end{cases} \tag{3.7}$$

$$(w,z)\mid_{t=0}=(w_0(x),z_0(x))=(w(\rho_0(x),u_0(x)),z(\rho_0(x),u_0(x)) \quad (3.8)$$

possesses a unique globally smooth solution (w,z) provided that the initial data (w_0,z_0) are smooth functions with bounded C^1-norms and $w_{0x}\geq 0$, $z_{0x}\geq 0$.

Following the arguments in [13], [14], we can get the following theorem.

Theorem 3.1 *Suppose that* $(\rho_0,u_0)\in\Sigma=\{(\rho,u)\mid |w|\leq M, |z|\leq M,\ w-z\geq 0\}$ *and* $(w_0)(x),z_0(x))\in C^1(R)\times C^1(R)$ *with bounded* C^1*-norms. Furthermore* $w_{0x}\geq 0$, $z_{0x}\geq 0$. *Then we have*

1) *when* $1<r\leq 3$, *the Cauchy problem* (3.1), (3.2) *admits a globally smooth solution* $(\rho,u)\in\Sigma$.

2) *when* $r>3$, *the Cauchy problem* (3.1), (3.2) *possesses a globally Hölder continuous solution* $(\rho,u)\in\Sigma$.

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